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ON WEAK TYPE INEQUALITIES FOR RARE MAXIMAL FUNCTIONS IN $\mathbb{R}^n$

BY

A. M. STOKOLOS (Chicago, IL)

Dedicated to the memory of Miguel de Guzmán

Abstract. The study of one-dimensional rare maximal functions was started in [4, 5]. The main result in [5] was obtained with the help of some general procedure. The goal of the present article is to adapt the procedure (we call it "dyadic crystallization") to the multidimensional setting and to demonstrate that rare maximal functions have properties not better than the Strong Maximal Function.

The well-known Jessen–Marcinkiewicz–Zygmund theorem [6] states that the differentiation basis of all n-dimensional intervals differentiates a.e. the integrals of all functions from $L(\log^+ L)^{n-1}$. The importance of this theorem is discussed, for example, in [3, 7]. Miguel de Guzmán [2, 3] found the quantitative version of the theorem by proving the following weak type estimate for the corresponding maximal function Mf (the so-called "Strong Maximal Function"):

$$|\{x : Mf(x) > \lambda\}| \lesssim \int \frac{|f(x)|}{\lambda}\left(1 + \log^+ \frac{|f(x)|}{\lambda}\right)^{n-1} dx \tag{1}$$

where $\lesssim$ denotes inequality with a constant depending only on dimension.

This is the best possible estimate as can be easily seen from the following example which is a multidimensional dyadic version of the well-known Bohr construction (see Note 1 in [1]). Let Q be the unit cube, m be an arbitrary positive integer, $\alpha \equiv (i_1, \dots, i_n)$ be such that $i_1 + \cdots + i_n = m$ and

$$I_\alpha \equiv [0, 2^{i_1}] \times \cdots \times [0, 2^{i_n}].$$

Then it is clear that $|I_\alpha| = 2^m$, $Q \subset I_\alpha$ and $|Q \cap I_\alpha| = 2^{-m}|I_\alpha|$. Hence

$$X_m \equiv \bigcup_{i_1+\cdots+i_n=m} I_\alpha \subset \{x : M\chi_Q(x) \ge 2^{-m}\}.$$

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All the I_α are pairwise incomparable n-dimensional intervals, hence

$$|\{x : M\chi_Q(x) \geq 2^{-m}\}| \geq |X_m| \gtrsim \sum_{i_1+\cdots+i_n=m} |I_\alpha| = 2^m \sum_{i_1+\cdots+i_n=m} 1.$$

Since $\{(i_1,\dots,i_n) : i_1+\cdots+i_n = m\} = \{(i_1,\dots,i_{n-1}, m-(i_1+\cdots+i_{n-1})) : i_1+\cdots+i_{n-1} \leq m\}$, we have

$$\#\{(i_1,\dots,i_n) : i_1+\cdots+i_n = m\} = \#\{(i_1,\dots,i_{n-1}) : i_1+\cdots+i_{n-1} \leq m\}.$$

On the other hand, $\{i_1 \leq m/(n-1),\dots,i_{n-1} \leq m/(n-1)\} \subset \{i_1+\cdots+i_{n-1} \leq m\}$. Altogether, this gives

$$\sum_{i_1+\cdots+i_n=m} 1 \geq \sum_{i_1\leq m/(n-1),\dots,i_{n-1}\leq m/(n-1)} 1 \gtrsim m^{n-1}. \tag{2}$$

Thus

$$|\{x : M\chi_Q(x) \geq 2^{-m}\}| \gtrsim m^{n-1}2^m|Q| \gtrsim \int \frac{\chi_Q}{2^{-m}}\left(1+\log^+\frac{\chi_Q}{2^{-m}}\right)^{n-1} dx \tag{3}$$

and (1) cannot be improved.

Now, if one tries to repeat the same procedure for the basis whose side lengths are not just all dyadic, but more sparse, e.g. like 2^{-n^2}, then the above procedure does not work because the desired inequality

$$\sum_{i_1^2+\cdots+i_{n-1}^2\leq m} 1 \gtrsim m^{n-1}$$

is false. In fact,

$$\sum_{i_1^2+\cdots+i_{n-1}^2\leq m} 1 \sim m^{(n-1)/2}.$$

To circumvent this difficulty, we will use a procedure we call *dyadic crystallization.* We hope that it will be a good complement to the basic harmonic analysis procedures like *linearization*, *dualization*, etc. The reader can observe the result of application of the procedure by comparing Figures 1 and 2 below and get a justification of its name.

The one-dimensional crystallization runs as follows. Given a number m and a sequence of integers $k_0 < \cdots < k_j < \cdots$ (or equivalently, a sequence of intervals of lengths $2^{k_0},\dots,2^{k_j},\dots$) let

$$Y_j \equiv \{x \in [0,2^{k_m}] : r_{k_i}(x) = 1 \text{ for all } i = j,\dots,m\}$$

where $r_i(x) = \operatorname{sign}\sin(\pi 2^{-i}x)$ are the standard Rademacher functions extended to the whole real line. The sets Y_j consist of disjoint dyadic intervals of length 2^{k_j} and have the following easily verified properties:

$$Y_j \subset Y_{j+1}, \qquad |Y_{j+1}| = 2|Y_j|.$$

Now, in general assume that we are given a number $m \geq 1$ and a sparse family of n-dimensional intervals whose projections on the sth axis have lengths $2^{k_0(s)}, \dots, 2^{k_j(s)}, \dots$.

Then one can form dyadic crystals Y_j^s in each variable with the properties

$$Y_j^s \subset Y_{j+1}^s, \qquad |Y_{j+1}^s| = 2|Y_j^s|$$

for $s = 1, \dots, n$. We have

$$Y_0^s \subset Y_j^s, \qquad |Y_j^s| = 2^j |Y_0^s|. \tag{4}$$

For each fixed $\alpha = (i_1, \dots, i_n)$ such that $i_1 + \dots + i_n = m$ we define

$$Y_\alpha \equiv Y_{i_1}^1 \times \dots \times Y_{i_n}^n.$$

Since each $Y_{i_s}^s$ is a union of disjoint dyadic intervals of length $2^{k_{i_s}(s)}$, it is clear that each Y_α is a union of disjoint congruent n-dimensional dyadic intervals I whose side lengths are of the type $2^{k_{i_1}(1)}, \dots, 2^{k_{i_n}(n)}$.

Now, we construct a *crystal*

$$X_m \equiv \bigcup_{i_1 + \dots + i_n = m} Y_\alpha.$$

We call X_m a crystal because it has a self-similar structure. Applying the crystallization to the set of rectangles in Figure 1 we will get the crystal in Figure 2. In other words, the set in Figure 2 "sprouts" from the set in Figure 1 using the crystallization procedure.

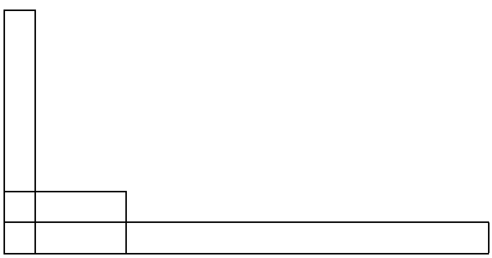

Fig. 1

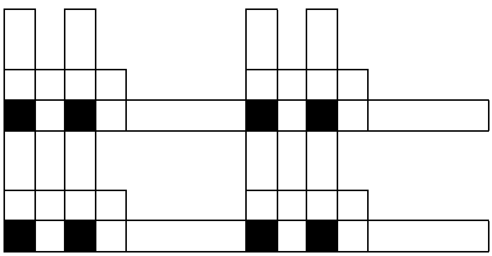

Fig. 2

The figures present the two-dimensional case with $m = 2$. There are 4 "high" congruent rectangles I which form a set $Y_{(0,2)}$, 4 "middle" rectangles which form a set $Y_{(1,1)}$, and 2 "low" rectangles which form a set $Y_{(2,0)}$.

Let $Q = Y_0^1 \times \cdots \times Y_0^n$ (in the figure these are the black rectangles). It is clear that $Q \subset Y_\alpha$ and the crucial property is the following:

$$\frac{|I \cap Q|}{|I|} = \frac{|Y_\alpha \cap Q|}{|Y_\alpha|} \tag{5}$$

where I are congruent n-dimensional intervals forming Y_α.

Since $Q \subset Y_\alpha$, (4) yields

$$\frac{|Y_\alpha \cap Q|}{|Y_\alpha|} = \frac{|Q|}{|Y_\alpha|} = \frac{|Y_0^1|}{|Y_{i_1}^1|} \cdots \frac{|Y_0^n|}{|Y_{i_n}^n|} = 2^{-i_1} \cdots 2^{-i_n} = 2^{-m}. \tag{6}$$

This together with (5) implies that $X_m \subset \{x : M\chi_Q(x) \geq 2^{-m}\}$, and by (6) and (2),

$$\begin{aligned} |\{x : M\chi_Q(x) \geq 2^{-m}\}| \geq |X_m| &\gtrsim \sum_{i_1+\cdots+i_n=m} |Y_\alpha| = 2^m|Q| \sum_{i_1+\cdots+i_n=m} 1 \\ &\gtrsim m^{n-1}2^m|Q|. \end{aligned} \tag{7}$$

Hence, (7) implies (3).

The above considerations prove the following theorem.

Theorem. *Let R_s, $s = 1, \ldots, n$, be arbitrary infinite sets of integers. Let Mf be a maximal function with respect to n-dimensional intervals whose sth side length can be any number 2^k with $k \in R_s$, regardless of what the other side lengths are. Then for any $0 < \lambda < 1$ there is a measurable bounded set Q such that*

$$|\{x : M\chi_Q(x) \geq \lambda\}| \gtrsim \int \frac{\chi_Q}{\lambda}\left(1 + \log^+ \frac{\chi_Q}{\lambda}\right)^{n-1}.$$

This theorem demonstrates that *the rarefaction of the side length of the intervals does not improve the properties of the corresponding maximal function.*

Indeed, if

$$|\{x : Mf(x) > \lambda\}| \lesssim \int \varphi\left(\frac{|f(x)|}{\lambda}\right) dx$$

then

$$|\{x : M\chi_Q(x) \geq 2^{-m}\}| \lesssim \varphi(2^m)|Q|.$$

Comparing this with (7) yields $\varphi(2^m) \gtrsim m^{n-1}2^m$.

This gives us a better understanding of the behavior of translation invariant subbases of the basis of all multidimensional intervals. The general situation is still very unclear and only a few partial results are known so far.

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REFERENCES

[1] C. Carathéodory, *Vorlesungen über reelle Funktionen*, 3rd (corrected) ed., Chelsea, New York, 1968.
[2] M. de Guzmán, *An inequality for the Hardy–Littlewood maximal operator with respect to the product of differentiation bases*, Studia Math. 42 (1972), 265–286.
[3] —, *Differentiation of Integrals in* R^n, Lecture Notes in Math. 481, Springer, 1975.
[4] P. A. Hagelstein, *A note on rare maximal functions*, Colloq. Math. 95 (2003), 49–51.
[5] K. Hare and A. Stokolos, *On weak type inequalities for rare maximal functions*, ibid. 83 (2000), 173–182.
[6] B. Jessen, J. Marcinkiewicz, and A. Zygmund, *Note on the differentiability of multiple integrals*, Fund. Math. 25 (1935), 217–234.
[7] A. Zygmund, *Trigonometric Series*, Vol. 2, Cambridge Univ. Press, 1958.

Department of Mathematical Sciences
DePaul University
Chicago, IL 60614, U.S.A.
E-mail: astokolo@math.depaul.edu

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