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AN m-CONVEX B_0-ALGEBRA WITH ALL LEFT BUT NOT ALL RIGHT IDEALS CLOSED

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Abstract. We construct an example as announced in the title. We also indicate all right, left and two-sided ideals in this example.

An *F-algebra* is a complete metric real or complex topological algebra. It is called a *B_0-algebra* if it is, moreover, locally convex. The topology of such a B_0-algebra A can be given by means of an increasing sequence of seminorms

$$|x|_1 \le |x|_2 \le \cdots, \quad x \in A. \tag{1}$$

In the particular case when the seminorms (1) can be chosen in such a way that

$$|xy|_i \le |x|_i |y|_i \quad \text{for all } x, y \in A,\ i = 1, 2, \ldots, \tag{2}$$

A is called *m-convex*. The class of m-convex B_0-algebras is perhaps the best understood class of topological algebras, strictly larger than the class of Banach algebras. For more information on this class the reader is referred to [4], [8], [9] or [11].

The question whether there exists an F-algebra with all left but not all right ideals closed was posed in [13]. Apart from its own interest, this question is connected with the study of Noetherian topological algebras. Such a study was initiated by Grauert and Remmert, who proved that a Noetherian commutative Banach algebra is necessarily finite-dimensional ([6, Chapter I, Remark 2 in the Appendix to §5]). This result was extended to the non-commutative case by Sinclair and Tullo [10], who proved that a left Noetherian Banach algebra is also finite-dimensional (and so it is right Noetherian as well). Ferreira and Tomassini observed that there exist Noetherian infinite-dimensional commutative m-convex B_0-algebras, but in such algebras all ideals must be closed ([5, Theorem 4.6]). A more general result has been obtained independently by Choukri and El Kinani [2] and the au-

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thor [12]: a commutative F-algebra has all ideals closed if and only if it is Noetherian. Finally, Choukri, El Kinani and Oudadess [3] have shown that an m-convex B_0-algebra has all left ideals closed if and only if it is left Noetherian. Our present result shows that, in contrast to Banach algebras, the closedness of left ideals in m-convex B_0-algebras does not imply that the right ones are also closed. Thus the result in [3] cannot be improved in this direction. It is still an open question whether this result can be extended to the class of all F-algebras. It is only known ([13]) that an F-algebra has all one-sided ideals closed if and only if it is both left and right Noetherian.

Our example is constructed as follows. Denote by A_0 the (real or complex) algebra of all formal power series

$$a = \sum_{n=0}^{\infty} \alpha_n(a)t^n$$

provided with the formal (Cauchy) multiplication of power series, and the locally convex topology given by the seminorms

$$|a|_n = \sum_{n=0}^{n} |\alpha_n(a)|. \tag{3}$$

Under this topology A_0 is a commutative m-convex B_0-algebra, i.e. it is complete and the seminorms (3) satisfy relations (1) and (2). A_0 treated as an F-space was considered by Banach in his treatise [1] and is there denoted by (s). It is a unital algebra; its unity, denoted by e, is the "constant" power series with $\alpha_0(e) = 1$. An element a in A_0 is invertible if and only if $\alpha_0(a) \neq 0$, and every element $a \neq 0$ in A_0 can be written in the form

$$a = t^n b^{-1} \tag{4}$$

for some non-negative integer n and invertible $b \in A_0$ (we put $t^0 = e$). Consequently, every non-zero ideal in A_0 is of the form

$$I = t^n A_0, \quad n \geq 1, \tag{5}$$

and so it is closed. All the properties of A_0 listed above are well known and easy to see (cf. e.g. [5]).

Our example is an algebra of the form

$$A = A_0 + A_0 w,$$

where w is a new element and the multiplication in A is obtained from multiplication in A_0 and the relations $w^2 = 0$ and $wt = 0$ (this idea comes from J. Dieudonné, who constructed in this way a ring which is left but not right Noetherian, see [7, Example 1.26, p. 21]). Each element of A is of the form $x = a + bw$, $a, b \in A_0$, so that A_0 can be treated as a subalgebra of A. The formula for multiplication in A is

$$(a + bw)(c + dw) = ac + (ad + \alpha_0(c)b)w, \quad a, b, c, d \in A_0. \tag{6}$$

We define the topology in A by the seminorms

$$\|a + bw\|_n = |a|_n + |b|_n, \qquad n = 1, 2, \ldots. \tag{7}$$

Using (2), (6) and the obvious relation $|\alpha_0(c)| \le |c|_n$ for all n, we obtain

$$\begin{aligned}\|(a + bw)(c + dw)\|_n &= |ac|_n + |ad + \alpha_0(c)b|_n \\ &\le |a|_n|c|_n + |a|_n|d|_n + |\alpha_0(c)|\,|b|_n \\ &\le (|a|_n + |b|_n)(|c|_n + |d|_n) = \|a + bw\|_n\|c + dw\|_n\end{aligned}$$

for all n. Since A is complete as a direct sum of two complete subspaces A_0 and A_0w, it is an m-convex B_0-algebra. Observe that by (6) the unity e of A_0, treated as an element of A ($e = e + 0w$) is a unity in it, and that A_0w is a (closed) subalgebra of A with zero multiplication. Observe also that an element $x = a + bw$ in A is invertible if and only if a is invertible, or $\alpha_0(a) \ne 0$. Indeed, the necessity of this condition follows from (6). On the other hand, if a is invertible, then writing $x = a(e + a^{-1}bw)$ we immediately see that

$$x^{-1} = (e - a^{-1}bw)a^{-1} = a^{-1} + \alpha(a^{-1})a^{-1}bw.$$

It can be easily verified that all left-invertible or right-invertible elements in A are invertible.

We can now formulate our main result.

PROPOSITION 1. *All left ideals in A are closed and some of its right ideals are non-closed.*

Proof. Observe first that by (6), for every vector subspace X of A_0, the subset $Xw \subset A$ is a right ideal in A and, as a subspace, it is topologically isomorphic to X. Choosing X to be non-closed (e.g. all finite linear combinations of powers t^n, $n = 0, 1, \ldots$) we obtain a non-closed right ideal in A. We shall show below (Proposition 3) that these are the only non-closed right ideals in A.

It remains to show that all left ideals in A are closed. Let I be a proper left ideal, i.e. $\{0\} \ne I \ne A$. Since the constant term of any element in I is zero, the formula (6) implies $A_0wI = \{0\}$, so that

$$AI = A_0I \subset I. \tag{8}$$

In the proof we shall consider four cases.

CASE (1). Consider first the situation when $t^n \in I$ for some natural n and denote by n_0 the least integer with this property. Assume moreover that there is no non-negative integer m such that $t^m w \in I$. We will show that in this case,

$$I = At^{n_0} = t^{n_0} A_0$$

and so, by the remark after formula (5), I is closed. In fact, if some $x = a^{-1}t^n + b^{-1}t^m w$ is in I (by (4) every element in A is of this form), then

$$t^{n_0} x = a^{-1}t^{n_0+n} + b^{-1}t^{n_0+m} w \in I,$$

and the difference

$$t^{n_0} x - a^{-1}t^n t^{n_0} = b^{-1}t^{n_0+m} w$$

is in I. Thus $t^{n_0+m} w \in I$, which is a contradiction.

Case (2). Assume that $t^{m_0} w$ is in I, m_0 is the least non-negative integer with this property, and t^n is in I for no natural n. In this case, similarly to case (1),

$$I = At^{m_0} w = A_0 t^{m_0} w$$

and so it is closed.

Case (3). Assume that t^{n_0} and $t^{m_0} w$ are in I and m_0, n_0 are the least integers with this property ($n_0 \geq 1, m_0 \geq 0$). In this case

$$I_0 = At^{n_0} + At^{m_0} w \subset I.$$

We claim that the ideal I_0 is closed. In fact, using (6), we can replace A by A_0 in the above formula. If a sequence $(x_i) = (a_i t^{n_0} + b_i t^{m_0} w)$, $a_i, b_i \in A_0$, of elements of I_0 is convergent, then, by (7), the sequences $(a_i t^{n_0})$ and $(b_i t^{m_0})$ are convergent in A_0 to some elements a_0 and b_0. Since the ideals (5) are closed in A_0, $a_0 = a'_0 t^{n_0}$ and $b_0 = b'_0 t^{m_0}$. Thus the sequence (x_i) tends to $x_0 = a_0 + b_0 w$, which is in I_0 and our claim follows. Moreover I_0 has a finite codimension in A, equal to $m_0 + n_0$. Consequently, any larger subspace, in particular the ideal I, is also closed. Indeed, consider the quotient map q of A onto the finite-dimensional space A/I_0. Then $q(I)$ is a closed subspace of A/I_0 since all vector subspaces of a finite-dimensional topological vector space are closed. Thus I is closed as the inverse image $q^{-1}(q(I))$ of a closed set under the continuous map q.

Case (4). Finally, assume that no element of the form t^n or $t^m w$ is in I. Since I is proper, there is an element $x = a^{-1}t^{n_0} + b^{-1}t^m w \in I$ with the least possible n_0, and an element $y = c^{-1}t^n + d^{-1}t^{m_0} w$ with the least possible m_0, $n_0 \geq 1$, $m_0 \geq 0, a, b, c, d \in A_0$. If $m > m_0$ and $n > n_0$ then $x + y$ is in I and it is of the form $a_1^{-1}t^{n_0} + b_1^{-1}t^{m_0} w$. Thus we can assume that I contains an element x_0 of the form $t^{n_0} + a^{-1}t^{m_0} w$ and for any element $y = b^{-1}t^n + c^{-1}t^m w$ in I, we have $n \geq n_0$ and $m \geq m_0$.

Put $I_1 = Ax_0 \subset I$; we shall show that $I = I_1$. All elements in I_1 are of the form

$$(b^{-1}t^k + c^{-1}t^m w)(t^{n_0} + a^{-1}t^{m_0}w) = b^{-1}(t^{n_0+k} + a^{-1}t^{m_0+k}w)$$

and, in particular, all elements of the form

$$y = t^{n_0+k} + a^{-1}t^{m_0+k}w, \quad k = 0, 1, \ldots, \tag{9}$$

are in the ideal I_1. If $I \neq I_1$, then I contains an element $z_0 = t^{n_0+k_1} + b^{-1}t^{m_0+k_2}w$, $k_1, k_2 \geq 0$, and either $k_1 \neq k_2$, or $b \neq a$. In the first situation, setting in (9) $k = k_1$, we observe that $y - z_0 = (a^{-1}t^{m_0+k_1} - b^{-1}t^{m_0+k_2})w$ is a non-zero element in I, and it is of the form $c^{-1}t^m w$, where $m = \min\{m_0 + k_1, m_0 + k_2\}$. This contradicts the assumption that no $t^m w$ is in I. Thus $k_1 = k_2$ and so $y - z_0 = (b^{-1} - a^{-1})t^{m_0+k_1}w$. Since this is non-zero, we have $b^{-1} - a^{-1} = c^{-1}t^p$ for some invertible c in A_0 and some natural p. Thus $c^{-1}t^{m_0+k_1+p}w$ is in I and again we obtain the same contradiction, proving the equality $I = I_1$.

It remains to show that the ideal $I = I_1 = Ax_0$ is closed (using (8) we can write $I = A_0x_0$). To this end it is sufficient to show that if a sequence (z_nx_0) of elements of I ($z_n \in A_0$) is convergent to, say, y, then the sequence (z_n) is convergent to some z_0, so that $y = z_0x_0$ is in I. Since A is complete, we have to show that (z_n) is a Cauchy sequence. Similarly to (8), we can prove that $Ax_0 = A_0x_0$, so that we can assume that all z_i are in A_0. By (3) and (7) we have

$$|zx_0|_i = |z(t^{n_0} + a^{-1}t^{m_0}w)|_i = |zt^{n_0}|_i + |za^{-1}t^{m_0}w|_i \geq |zt^{n_0}|_i = |z|_{i+n_0}$$

for all z in A_0. This formula implies that whenever

$$z_px_0 - z_qx_0 = (z_p - z_q)x_0 \to 0 \quad \text{for } p, q \to \infty,$$

then, as well, $z_p - z_q \to 0$. Thus (z_i) is a Cauchy sequence and we are done. The conclusion follows.

In the above proof we have "almost" described all left ideals in A. To finish the job let us formulate the following

PROPOSITION 2. *Let I be a proper left ideal in A. Then I is of one of the following forms*: $I = At^{n_0}$, $n_0 \geq 1$; $I = At^{m_0}w$, $m_0 \geq 0$; $I = Ax_0$, $x_0 = t^{n_0} + a^{-1}t^{m_0}w$, $n_0 > 0$; $I = At^{n_0} + At^{m_0}w + Ax_0$, *where* $x_0 = 0$ *or* $x_0 = t^{n_0-k_0} + a_0^{-1}t^{m_0-s_0}w$ *with* $0 < k_0 \leq \min\{n_0 - 1, m_0\}$, $a \in A_0$.

Proof. Most of the above statement is obtained within the proof of Proposition 1. We only have to consider the case when $I_0 = At^{n_0} + At^{m_0}w$ is properly contained in a left ideal I and it is a maximal subideal of I of this form. Following the proof of Proposition 1, we can assume that there is some element $x_0 = t^{n_1} + t^{m_1}a_0^{-1}w \in I$, $0 \leq m_1 < m_0$, such that I contains no element of this form with exponents n_2, m_2 such that $n_2 < n_1$ or $m_2 < m_1$.

First we show that $n_0 - n_1 = m_0 - m_1$. Suppose that, say, $n_0 - n_1 > m_0 - m_1$. Since $t^{m_0-m_1}x_0$ and $t^{m_0}a_0^{-1}w$ belong to I, their difference $t^{m_0-m_1+n_1}$ is also in I and $m_0 - m_1 + n_1 < n_0$, contrary to the definition of n_0. Similarly we prove that $n_0 - n_1 < m_0 - m_1$ is impossible.

Setting $k_0 = n_0 - n_1 = m_0 - m_1$ we can now write $x_0 = t^{n_0-k_0} + t^{m_0-k_0}a_0^{-1}w$ so that I contains all elements of the form $x = t^k x_0 = t^{n_0-k_0+k} + t^{m_0-k_0+k}a_0^{-1}w$, $0 \le k < k_0$. The proof that no other elements of the form $t^n + t^m b^{-1}w$ with $n < n_0$ or $m < m_0$ are in I is performed exactly in the same way as in the proof of Proposition 1 when considering the case $I = Ax_0$. We have $n_0 > n_0 - k_0 \ge 1$ and $m_0 > m_0 - k_0 \ge 0$, which implies the condition $0 < k_0 \le \min\{n_0, m_0 - 1\}$. The conclusion follows.

We shall now describe all right ideals in A.

PROPOSITION 3. *Let I be a proper right ideal in A. Then either I is a vector subspace of A_0w, and conversely all such non-zero subspaces are right ideals*, *or*

$$I = x_0A + X_0w, \tag{10}$$

where x_0 is a non-invertible element of A and X_0w is a finite-dimensional subspace of A_0w. Moreover, *all ideals* (10) *are closed and of finite codimension.*

Proof. We already know (Proposition 1) that all vector subspaces of A_0w are right ideals in A. Let now I be a proper ideal in A which is not of this form, and so it has an element x which contains a term t^{n_1}. We have $n_1 \ge 1$ and we can assume that n_1 is the least natural number with this property. If I contains t^{n_1}, we put $n_0 = n_1$, and then I contains the right ideal

$$I_0 = t^{n_0}A = t^{n_0}A_0 + t^{n_0}A_0w. \tag{11}$$

In this case we put $x_0 = t^{n_0}$. If some element

$$x = a^{-1}t^m + b^{-1}t^n w \tag{12}$$

is in $I \setminus I_0$, then $m \ge n_0$ and so

$$y = b^{-1}t^n w \in I \setminus I_0. \tag{13}$$

Consequently, $n < n_0$. Write $y = \sum_{i=n}^{\infty} \alpha_i t^i w = (\sum_{i=n}^{n_0-1} + \sum_{i=n_0}^{\infty})\alpha_i t^i w = y_1 + y_2$. Since y_2 is in I_0, y_1 is in I and so in the set

$$X_0w = I \cap \operatorname{span}\{t^k w : 0 \le k < n_0\}. \tag{14}$$

Since all elements in $I \setminus I_0$ are of the form (12) or (13), we have $I \subset X_0w + I_0$. The opposite inclusion is obvious, so that (10) holds in the case considered.

Consider now the case when no t^{n_1} belongs to I. Consider the elements x in I of the form

$$x = t^{n_1} + a^{-1}t^m w \tag{15}$$

with the minimal exponent n_1. We have $xt = t^{n_1+1} \in I$, and in this case we put $n_0 = n_1 + 1$, so that the ideal I_0 given by (11) is contained in I. Observe that the exponent m in (15) must be smaller than $n_0 - 1$, otherwise we can write $x = t^{n_0-1}(e + a^{-1}t^{m-n_0+1}w)$ and multiplying it from the right by the inverse of $e + a^{-1}t^{m-n_0+1}w$ we obtain $t^{n_0-1} \in I$, which is a contradiction. Taking in (15) the largest possible $m = m_0$ we fix from now on an element $x_0 = t^{n_0-1} + a_0^{-1}t^{m_0}w$, $m_0 < n_0 - 1$, obtained in this way. Since $x_0 t = t^{n_0}$, we have

$$I_0 \subset x_0 A. \tag{16}$$

We now prove (10). Clearly, the ideal given by this formula, with $X_0 w$ defined by (14), is contained in I, so that we have to obtain the converse inclusion. The elements in I of the form $a^{-1}t^k$ are in I_0, as also are the elements of the form $a^{-1}t^m w$ with $m \geq n_0$. An element $x = a^{-1}t^m w \in I$, $m < n_0$, can be written, as above, $x = \sum_{i=m}^{n_0-1} \alpha_i t^i w + \sum_{i=n_0}^{\infty} \alpha_i t^i w = y_1 + y_2$, with $y_1 \in X_0 w$ and $y_2 \in I_0$, so that it is again in the ideal of the form (10). The only remaining elements in $I \setminus (I_0 \cup X_0 w)$ are of the form $x = a^{-1}t^{n_1} + b^{-1}t^m w$ with $m \leq m_0$. We have

$$xa - x_0 = t^m(\alpha_0(a)b^{-1} - t^{m_0-m}a_0^{-1})w \in I,$$

so that, as before, $xa - x_0 = y_1 + y_2$ with $y_1 \in X_0 w$ and $y_2 \in I_0$. Thus, by (16), $x \in x_0 a^{-1} + X_0 w + I_0 \subset x_0 A + X_0 w$ and the formula (10) holds. Since the ideals (10) contain ideals (11) which are closed and of finite codimension, they are closed and of finite codimension. The conclusion follows.

Comparing Propositions 2 and 3 and taking into account formulas (11) and (16) we obtain the following

COROLLARY. *Every two-sided ideal in A is either of the form*

$$I = t^{m_0} A_0 w, \qquad m_0 \geq 0,$$

or of the form

$$I = t^{n_0} A_0 + t^{m_0} A_0 w, \qquad n_0 \geq 1,\ 0 \leq m_0 \leq n_0.$$

Finally, notice that the algebra A has exactly one multiplicative-linear functional F given by the formula

$$F(a + bw) = \alpha_0(a), \qquad a, b \in A_0;$$

its kernel is a maximal two-sided ideal which is also maximal as a right and as a left ideal. Moreover, every right or left ideal in A is contained in this ideal. Consequently, it is the radical of A.

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