AN m-CONVEX $B_0$-ALGEBRA
WITH ALL LEFT BUT NOT ALL RIGHT IDEALS CLOSED

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Abstract. We construct an example as announced in the title. We also indicate all right, left and two-sided ideals in this example.

An $F$-algebra is a complete metric real or complex topological algebra. It is called a $B_0$-algebra if it is, moreover, locally convex. The topology of such a $B_0$-algebra $A$ can be given by means of an increasing sequence of seminorms

$$|x|_1 \leq |x|_2 \leq \cdots, \quad x \in A.$$  

In the particular case when the seminorms (1) can be chosen in such a way that

$$|xy|_i \leq |x|_i|y|_i \quad \text{for all } x, y \in A, \ i = 1, 2, \ldots,$$

$A$ is called $m$-convex. The class of $m$-convex $B_0$-algebras is perhaps the best understood class of topological algebras, strictly larger than the class of Banach algebras. For more information on this class the reader is referred to [4], [8], [9] or [11].

The question whether there exists an $F$-algebra with all left but not all right ideals closed was posed in [13]. Apart from its own interest, this question is connected with the study of Noetherian topological algebras. Such a study was initiated by Grauert and Remmert, who proved that a Noetherian commutative Banach algebra is necessarily finite-dimensional ([6, Chapter 1, Remark 2 in the Appendix to §5]). This result was extended to the non-commutative case by Sinclair and Tullo [10], who proved that a left Noetherian Banach algebra is also finite-dimensional (and so it is right Noetherian as well). Ferreira and Tomassini observed that there exist Noetherian infinite-dimensional commutative $m$-convex $B_0$-algebras, but in such algebras all ideals must be closed ([5, Theorem 4.6]). A more general result has been obtained independently by Choukri and El Kinani [2] and the au-
thor [12]: a commutative $F$-algebra has all ideals closed if and only if it is Noetherian. Finally, Choukri, El Kinani and Oudadess [3] have shown that an $m$-convex $B_0$-algebra has all left ideals closed if and only if it is left Noetherian. Our present result shows that, in contrast to Banach algebras, the closedness of left ideals in $m$-convex $B_{0r}$-algebras does not imply that the right ones are also closed. Thus the result in [3] cannot be improved in this direction. It is still an open question whether this result can be extended to the class of all $F$-algebras. It is only known ([13]) that an $F$-algebra has all one-sided ideals closed if and only if it is both left and right Noetherian.

Our example is constructed as follows. Denote by $A_0$ the (real or complex) algebra of all formal power series

$$a = \sum_{n=0}^{\infty} \alpha_n(a) t^n$$

provided with the formal (Cauchy) multiplication of power series, and the locally convex topology given by the seminorms

$$|a|_n = \sum_{n=0}^{n} |\alpha_n(a)|.$$  \hfill (3)

Under this topology $A_0$ is a commutative $m$-convex $B_{0r}$-algebra, i.e. it is complete and the seminorms (3) satisfy relations (1) and (2). $A_0$ treated as an $F$-space was considered by Banach in his treatise [1] and is there denoted by $(s)$. It is a unital algebra; its unity, denoted by $e$, is the “constant” power series with $\alpha_0(e) = 1$. An element $a$ in $A_0$ is invertible if and only if $\alpha_0(a) \neq 0$, and every element $a \neq 0$ in $A_0$ can be written in the form

$$a = t^n b^{-1}$$

for some non-negative integer $n$ and invertible $b \in A_0$ (we put $t^0 = e$). Consequently, every non-zero ideal in $A_0$ is of the form

$$I = t^n A_0, \quad n \geq 1,$$

and so it is closed. All the properties of $A_0$ listed above are well known and easy to see (cf. e.g. [5]).

Our example is an algebra of the form

$$A = A_0 + A_0 w,$$

where $w$ is a new element and the multiplication in $A$ is obtained from multiplication in $A_0$ and the relations $w^2 = 0$ and $wt = 0$ (this idea comes from J. Dieudonné, who constructed in this way a ring which is left but not right Noetherian, see [7, Example 1.26, p. 21]). Each element of $A$ is of the
Form $x = a + bw$, $a, b \in A_0$, so that $A_0$ can be treated as a subalgebra of $A$. The formula for multiplication in $A$ is

$$
(a + bw)(c + dw) = ac + (ad + \alpha_0(c)b)w, \ a, b, c, d \in A_0.
$$

We define the topology in $A$ by the seminorms

$$
\|a + bw\|_n = |a|_n + |b|_n, \quad n = 1, 2, \ldots.
$$

Using (2), (6) and the obvious relation $|\alpha_0(c)| \leq |c|_n$ for all $n$, we obtain

$$
\|(a + bw)(c + dw)\|_n = |ac|_n + |ad + \alpha_0(c)b|_n
\leq |a|_n|c|_n + |a|_n|d|_n + |\alpha_0(c)| |b|_n
\leq (|a|_n + |b|_n(|c|_n + |d|_n) = \|(a + bw)\|_n\|c + dw\|_n
$$

for all $n$. Since $A$ is complete as a direct sum of two complete subspaces $A_0$ and $A_0w$, it is an $m$-convex $B_0$-algebra. Observe that by (6) the unity $e$ of $A_0$, treated as an element of $A$ ($e = e + 0w$) is a unity in it, and that $A_0w$ is a (closed) subalgebra of $A$ with zero multiplication. Observe also that an element $x = a + bw$ in $A$ is invertible if and only if $a$ is invertible, or $\alpha_0(a) \neq 0$. Indeed, the necessity of this condition follows from (6). On the other hand, if $a$ is invertible, then writing $x = a(e + a^{-1}bw)$ we immediately see that

$$
x^{-1} = (e - a^{-1}bw)a^{-1} = a^{-1} + \alpha(a^{-1})a^{-1}bw.
$$

It can be easily verified that all left-invertible or right-invertible elements in $A$ are invertible.

We can now formulate our main result.

**Proposition 1.** All left ideals in $A$ are closed and some of its right ideals are non-closed.

**Proof.** Observe first that by (6), for every vector subspace $X$ of $A_0$, the subset $Xw \subset A$ is a right ideal in $A$ and, as a subspace, it is topologically isomorphic to $X$. Choosing $X$ to be non-closed (e.g. all finite linear combinations of powers $t^n$, $n = 0, 1, \ldots$) we obtain a non-closed right ideal in $A$. We shall show below (Proposition 3) that these are the only non-closed right ideals in $A$.

It remains to show that all left ideals in $A$ are closed. Let $I$ be a proper left ideal, i.e. $\{0\} \neq I \neq A$. Since the constant term of any element in $I$ is zero, the formula (6) implies $A_0wI = \{0\}$, so that

$$
AI = A_0I \subset I.
$$

In the proof we shall consider four cases.

**Case (1).** Consider first the situation when $t^n \in I$ for some natural $n$ and denote by $n_0$ the least integer with this property. Assume moreover that
there is no non-negative integer m such that $t^mw \in I$. We will show that in this case,

$$I = At^{m_0} = t^{m_0}A_0$$

and so, by the remark after formula (5), I is closed. In fact, if some $x = a^{-1}t^n + b^{-1}t^mw$ is in I (by (4) every element in A is of this form), then

$$t^{m_0}x = a^{-1}t^{m_0+n} + b^{-1}t^{m_0+m}w \in I,$$

and the difference

$$t^{m_0}x - a^{-1}t^mt^{m_0} = b^{-1}t^{m_0+m}w$$

is in I. Thus $t^{m_0+m}w \in I$, which is a contradiction.

**Case (2).** Assume that $t^{m_0}w$ is in I, $m_0$ is the least non-negative integer with this property, and $t^n$ is in I for no natural n. In this case, similarly to case (1),

$$I = At^{m_0}w = A_0t^{m_0}w$$

and so it is closed.

**Case (3).** Assume that $t^{m_0}$ and $t^{m_0}w$ are in I and $m_0, n_0$ are the least integers with this property ($n_0 \geq 1, m_0 \geq 0$). In this case

$$I_0 = At^{m_0} + At^{m_0}w \subset I.$$  

We claim that the ideal $I_0$ is closed. In fact, using (6), we can replace A by $A_0$ in the above formula. If a sequence $(x_i) = (a_it^{m_0} + b_it^{m_0}w), a_i, b_i \in A_0,$ of elements of $I_0$ is convergent, then, by (7), the sequences $(a_it^{m_0})$ and $(b_it^{m_0})$ are convergent in $A_0$ to some elements $a_0$ and $b_0.$ Since the ideals (5) are closed in $A_0, a_0 = a_0't^{m_0}$ and $b_0 = b_0't^{m_0}.$ Thus the sequence $(x_i)$ tends to $x_0 = a_0 + b_0w,$ which is in $I_0$ and our claim follows. Moreover $I_0$ has a finite codimension in A, equal to $m_0 + n_0.$ Consequently, any larger subspace, in particular the ideal I, is also closed. Indeed, consider the quotient map $q$ of A onto the finite-dimensional space $A/I_0$. Then $q(I)$ is a closed subspace of $A/I_0$ since all vector subspaces of a finite-dimensional topological vector space are closed. Thus I is closed as the inverse image $q^{-1}(q(I))$ of a closed set under the continuous map $q$.

**Case (4).** Finally, assume that no element of the form $t^n$ or $t^mw$ is in I. Since I is proper, there is an element $x = a^{-1}t^{m_0} + b^{-1}t^mw \in I$ with the least possible $n_0$, and an element $y = c^{-1}t^n + d^{-1}t^mw \in I$ with the least possible $m_0$, $n_0 \geq 1, m_0 \geq 0, a, b, c, d \in A_0.$ If $m > m_0$ and $n > n_0$ then $x + y$ is in I and it is of the form $a_1^{-1}t^{m_0} + b_1^{-1}t^{m_0}w$. Thus we can assume that I contains an element $x_0$ of the form $t^{m_0} + a^{-1}t^nw$ and for any element $y = b^{-1}t^n + c^{-1}t^mw \in I$, we have $n \geq n_0$ and $m \geq m_0.$
Put $I_1 = A x_0 \subset I$; we shall show that $I = I_1$. All elements in $I_1$ are of the form
\[
(b^{-1}t^k + c^{-1}t^m w)(t^n_0 + a^{-1}t^{m_0} w) = b^{-1}(t^{n_0+k} + a^{-1}t^{m_0+k} w)
\]
and, in particular, all elements of the form
\[
y = t^{n_0+k} + a^{-1}t^{m_0+k} w, \quad k = 0, 1, \ldots,
\]
are in the ideal $I_1$. If $I \neq I_1$, then $I$ contains an element $z_0 = t^{n_0+k_1} + b^{-1}t^{m_0+k_2} w$, $k_1, k_2 \geq 0$, and either $k_1 \neq k_2$, or $b \neq a$. In the first situation, setting in (9) $k = k_1$, we observe that $y - z_0 = (a^{-1}t^{m_0+k_1} - b^{-1}t^{m_0+k_2}) w$ is a non-zero element in $I$, and it is of the form $c^{-1}t^m w$, where $m = \min\{m_0+k_1, m_0+k_2\}$. This contradicts the assumption that no $t^m w$ is in $I$.

Thus $k_1 = k_2$ and so $y - z_0 = (b^{-1} - a^{-1})t^{m_0+k_1} w$. Since this is non-zero, we have $b^{-1} - a^{-1} = c^{-1}p$ for some invertible $c$ in $A_0$ and some natural $p$. Thus $c^{-1}t^{m_0+k_1+p} w$ is in $I$ and again we obtain the same contradiction, proving the equality $I = I_1$.

It remains to show that the ideal $I = I_1 = A x_0$ is closed (using (8) we can write $I = A_0 x_0$). To this end it is sufficient to show that if a sequence $(z_n x_0)$ of elements of $I$ ($z_n \in A_0$) is convergent to, say, $y$, then the sequence $(z_n)$ is convergent to some $z_0$, so that $y = z_0 x_0$ is in $I$. Since $A$ is complete, we have to show that $(z_n)$ is a Cauchy sequence. Similarly to (8), we can prove that $A x_0 = A_0 x_0$, so that we can assume that all $z_i$ are in $A_0$. By (3) and (7) we have
\[
|z x_0|_i = |z(t^{n_0} + a^{-1}t^{m_0} w)|_i = |zt^{n_0} i| + |za^{-1}t^{m_0} w|_i \geq |zt^{n_0} i| = |z|_i+n_0
\]
for all $z$ in $A_0$. This formula implies that whenever
\[
z_p x_0 - z_q x_0 = (z_p - z_q) x_0 \to 0 \quad \text{for } p, q \to \infty,
\]
then, as well, $z_p - z_q \to 0$. Thus $(z_i)$ is a Cauchy sequence and we are done. The conclusion follows.

In the above proof we have “almost” described all left ideals in $A$. To finish the job let us formulate the following

**Proposition 2.** Let $I$ be a proper left ideal in $A$. Then $I$ is of one of the following forms: $I = A t^{n_0}$, $n_0 \geq 1$; $I = A t^{m_0} w$, $m_0 \geq 0$; $I = A x_0$, $x_0 = t^{n_0} + a^{-1}t^{m_0} w$, $n_0 > 0$; $I = A t^{m_0} + A t^{m_0} w + A x_0$, where $x_0 = 0$ or $x_0 = t^{n_0-k_0} + a^{-1}t^{m_0-s_0} w$ with $0 < k_0 \leq \min\{n_0-1, m_0\}$, $a \in A_0$.

**Proof.** Most of the above statement is obtained within the proof of Proposition 1. We only have to consider the case when $I_0 = A t^{n_0} + A t^{m_0} w$ is properly contained in a left ideal $I$ and it is a maximal subideal of $I$ of this form. Following the proof of Proposition 1, we can assume that there is some element $x_0 = t^{m_1} + t^{m_1} a^{-1} w \in I$, $0 \leq m_1 < m_0$, such that $I$ contains no element of this form with exponents $n_2, m_2$ such that $n_2 < n_1$ or $m_2 < m_1$. 


First we show that \( n_0 - n_1 = m_0 - m_1 \). Suppose that, say, \( n_0 - n_1 > m_0 - m_1 \). Since \( t^{m_0-m_1}x_0 \) and \( t^{m_0}a_0^{-1}w \) belong to \( I \), their difference \( t^{m_0-m_1+n_1} \) is also in \( I \) and \( m_0 - m_1 + n_1 < n_0 \), contrary to the definition of \( n_0 \). Similarly we prove that \( n_0 - n_1 < m_0 - m_1 \) is impossible.

Setting \( k_0 = n_0 - n_1 = m_0 - m_1 \) we can now write \( x_0 = t^{n_0-k_0} + t^{n_0-k_0}a_0^{-1}w \) so that \( I \) contains all elements of the form \( x = t^k x_0 = t^{n_0-k_0+k} + t^{m_0-k_0+k}a_0^{-1}w \), \( 0 \leq k < k_0 \). The proof that no other elements of the form \( t^n + t^m b^{-1}w \) with \( n < n_0 \) or \( m < m_0 \) are in \( I \) is performed exactly in the same way as in the proof of Proposition 1 when considering the case \( I = Ax_0 \). We have \( n_0 > n_0 - k_0 \geq 1 \) and \( m_0 > m_0 - k_0 \geq 0 \), which implies the condition \( 0 < k_0 \leq \min\{n_0, m_0 - 1\} \). The conclusion follows.

We shall now describe all right ideals in \( A \).

**Proposition 3.** Let \( I \) be a proper right ideal in \( A \). Then either \( I \) is a vector subspace of \( A_0w \), and conversely all such non-zero subspaces are right ideals, or

\[
I = x_0 A + X_0 w,
\]

where \( x_0 \) is a non-invertible element of \( A \) and \( X_0 w \) is a finite-dimensional subspace of \( A_0w \). Moreover, all ideals (10) are closed and of finite codimension.

**Proof.** We already know (Proposition 1) that all vector subspaces of \( A_0w \) are right ideals in \( A \). Let now \( I \) be a proper ideal in \( A \) which is not of this form, and so it has an element \( x \) which contains a term \( t^{n_1} \). We have \( n_1 \geq 1 \) and we can assume that \( n_1 \) is the least natural number with this property. If \( I \) contains \( t^{n_1} \), we put \( n_0 = n_1 \), and then \( I \) contains the right ideal

\[
I_0 = t^{n_0} A = t^{n_0} A_0 + t^{n_0} A_0 w.
\]

In this case we put \( x_0 = t^{n_0} \). If some element

\[
x = a^{-1} t^m + b^{-1} t^n w
\]

is in \( I \setminus I_0 \), then \( m \geq n_0 \) and so

\[
y = b^{-1} t^n w \in I \setminus I_0.
\]

Consequently, \( n < n_0 \). Write \( y = \sum_{i=n}^{\infty} \alpha_i t^i w = (\sum_{i=n}^{n_0-1} + \sum_{i=n_0}^{\infty}) \alpha_i t^i w = y_1 + y_2 \). Since \( y_2 \) is in \( I_0 \), \( y_1 \) is in \( I \) and so in the set

\[
X_0 w = I \cap \text{span}\{t^k w : 0 \leq k < n_0\}.
\]

Since all elements in \( I \setminus I_0 \) are of the form (12) or (13), we have \( I \subset X_0 w + I_0 \). The opposite inclusion is obvious, so that (10) holds in the case considered.

Consider now the case when no \( t^{n_1} \) belongs to \( I \). Consider the elements \( x \) in \( I \) of the form

\[
x = t^{n_1} + a^{-1} t^m w
\]
with the minimal exponent \( n_1 \). We have \( xt = t^{n_1+1} \in I \), and in this case we put \( n_0 = n_1 + 1 \), so that the ideal \( I_0 \) given by (11) is contained in \( I \). Observe that the exponent \( m \) in (15) must be smaller than \( n_0 - 1 \), otherwise we can write \( x = t^{n_0-1}(e + a^{-1}t^{m-n_0+1}w) \) and multiplying it from the right by the inverse of \( e + a^{-1}t^{m-n_0+1}w \) we obtain \( t^{n_0-1} \in I \), which is a contradiction. Taking in (15) the largest possible \( m = m_0 \) we fix from now on an element \( x_0 = t^{n_0-1} + a^{-1}t^{m_0}w \), \( m_0 < n_0 - 1 \), obtained in this way. Since \( x_0t = t^{n_0} \), we have

(16) \( I_0 \subset x_0A \).

We now prove (10). Clearly, the ideal given by this formula, with \( X_0w \) defined by (14), is contained in \( I \), so that we have to obtain the converse inclusion. The elements in \( I \) of the form \( a^{-1}tk \) are in \( I_0 \), as also are the elements of the form \( a^{-1}tmw \) with \( m \geq n_0 \). An element \( x = a^{-1}tmw \in I \), \( m < n_0 \), can be written, as above, \( x = \sum_{i=1}^{n_0-1} \alpha_i t^i + \sum_{i=n_0}^{\infty} \alpha_i t^i w = y_1 + y_2 \), with \( y_1 \in X_0w \) and \( y_2 \in I_0 \), so that it is again in the ideal of the form (10). The only remaining elements in \( I \setminus (I_0 \cup X_0w) \) are of the form \( x = a^{-1}tn_1 + b^{-1}tmw \) with \( m \leq m_0 \). We have

\[ xa - x_0 = t^m(a_0(a)b - t^{m_0-m}a_0^{-1})w \in I, \]

so that, as before, \( xa - x_0 = y_1 + y_2 \) with \( y_1 \in X_0w \) and \( y_2 \in I_0 \). Thus, by (16), \( x \in x_0a^{-1} + X_0w + I_0 \subset x_0A + X_0w \) and the formula (10) holds. Since the ideals (10) contain ideals (11) which are closed and of finite codimension, they are closed and of finite codimension. The conclusion follows.

Comparing Propositions 2 and 3 and taking into account formulas (11) and (16) we obtain the following

**Corollary.** Every two-sided ideal in \( A \) is either of the form

\[ I = t^{m_0}A_0w, \quad m_0 \geq 0, \]

or of the form

\[ I = t^{n_0}A_0 + t^{m_0}A_0w, \quad n_0 \geq 1, \quad 0 \leq m_0 \leq n_0. \]

Finally, notice that the algebra \( A \) has exactly one multiplicative-linear functional \( F \) given by the formula

\[ F(a + bw) = \alpha_0(a), \quad a, b \in A_0; \]

its kernel is a maximal two-sided ideal which is also maximal as a right and as a left ideal. Moreover, every right or left ideal in \( A \) is contained in this ideal. Consequently, it is the radical of \( A \).

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