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A NOTE ON INTERSECTIONS OF NON-HAAR NULL SETS

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Abstract. We show that in every Polish, abelian, non-locally compact group G there exist non-Haar null sets A and B such that the set $\{g \in G;\ (g+A)\cap B \text{ is non-Haar null}\}$ is empty. This answers a question posed by Christensen.

Let G be a Polish abelian group. A universally measurable set $A \subset G$ is said to be a *Haar null set* if there exists a probability Borel measure on G such that $\mu(g+A) = 0$ for every $g \in G$. This family was introduced by Christensen [C] to have an analogy of Lebesgue null sets also in non-locally compact groups. Haar null sets were used to study differentiation properties of Lipschitz function on separable Banach spaces (cf. [C, p. 121]). Christensen proved the following result.

THEOREM 1 (Christensen, [C, p. 115]). *Let $A, B \subset G$ be two universally measurable sets. Then the set*

$$F(A,B) = \{g \in G;\ (g+A)\cap B \text{ is not Haar null}\}$$

is open.

Christensen posed the following question in [C]: Let $A, B \subset G$ be two universally measurable non-Haar null subsets of a Polish abelian group G and let G be non-locally compact. Can $F(A,B)$ be empty?

This problem was answered positively by Dougherty [D] in $\mathbb{R}^{\mathbb{N}}$ and by Matoušková and Zajíček [MZ] in c_0. The main aim of this paper is to present an observation that Solecki's method [S] of construction of non-Haar null sets gives a positive answer to the question in every Polish abelian non-locally compact group.

We will need the following theorems; the first one can be found, for example, in [DS, p. 90].

THEOREM 2. *Let G be a Polish abelian group. There exists an equivalent complete metric ϱ on G, which is invariant* (*i.e.* $\forall g,a,b \in G : \varrho(a,b) = \varrho(g+a, g+b)$).

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THEOREM 3 (Christensen, [C, p. 119]). *Let G be a Polish abelian non-locally compact group. Then every compact subset of G is Haar null.*

Let G be a Polish abelian group. The symbol $B(\varepsilon)$ denotes the open ball with center at 0 and radius ε. If $X \subset G$, then $B(X,\varepsilon)$ denotes the set $\{y \in G;\ \operatorname{dist}(y,X) < \varepsilon\}$. We start with the following lemma.

LEMMA. *Let G be a Polish, abelian, non-locally compact group with a complete invariant metric ϱ. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any two finite sets $F_1, F_2 \subset G$ there exists $g \in B(\varepsilon)$ such that $\operatorname{dist}(F_1, g + F_2) \geq \delta$.*

Proof. Since the group G is not locally compact, there exist an infinite set $D \subset B(\varepsilon)$ and $\delta > 0$ such that $\varrho(d,d') > 2\delta$ whenever $d, d' \in D$, $d \neq d'$. Suppose that $\operatorname{dist}(F_1, g+F_2) < \delta$ for every $g \in D$. Since F_1 and F_2 are finite, there exist $f_1 \in F_1$, $f_2 \in F_2$ and $g_1, g_2 \in D$, $g_1 \neq g_2$, such that $\varrho(f_1, g_1 + f_2) \leq \delta$ and $\varrho(f_1, g_2 + f_2) \leq \delta$. This implies $\varrho(g_1, g_2) = \varrho(g_1 + f_2, g_2 + f_2) \leq \varrho(g_1 + f_2, f_1) + \varrho(g_2 + f_2, f_1) \leq \delta + \delta = 2\delta$, which is a contradiction. ∎

THEOREM 4. *Let G be a Polish, abelian, non-locally compact group. Then there exist closed non-Haar null sets $A, B \subset G$ such that $(g + A) \cap B$ is compact for every $g \in G$. Consequently, $F(A,B)$ is empty.*

Proof. According to Theorem 1 there exists an equivalent complete invariant metric on G. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence that is dense in G.

Fix a sequence $\{Q_k\}_{k=1}^{\infty}$ of finite sets such that $Q_k \subset Q_{k+1}$ for every $k \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} Q_k$ is dense in G.

Fix $\varepsilon > 0$ and take $\delta > 0$ such that 3δ satisfies the conclusion of the Lemma for ε. We find inductively sequences $\{g_k\}_{k=1}^{\infty}$, $\{\widetilde{g}_k\}_{k=1}^{\infty}$ such that

- $\forall k \in \mathbb{N} : g_k, \widetilde{g}_k \in B(\varepsilon)$,
- $\forall k \in \mathbb{N} : \operatorname{dist}(g_k + Q_k, \bigcup_{i<k}(g_i + Q_i)) \geq 3\delta$,
- $\forall k \in \mathbb{N} : \operatorname{dist}(\widetilde{g}_k + Q_k, \bigcup_{i<k}(\widetilde{g}_i + Q_i)) \geq 3\delta$,
- $\forall n \in \mathbb{N}\ \forall i,j \in \mathbb{N},\ i,j \geq n : \operatorname{dist}(s_n + g_i + Q_i, \widetilde{g}_j + Q_j) \geq 3\delta$.

Put $g_1 = 0$. Put $F_1 = (g_1 + Q_1) \cup (s_1 + g_1 + Q_1)$ and $F_2 = Q_1$. Now the Lemma gives $\widetilde{g}_1 \in B(\varepsilon)$. Suppose that we have defined $g_1, \dots, g_{k-1}$, $\widetilde{g}_1, \dots, \widetilde{g}_{k-1} \in B(\varepsilon)$. Put

$$F_1 = \bigcup_{j<k}(g_j + Q_j) \cup \bigcup_{l \leq k} \bigcup_{j<k} (-s_l + \widetilde{g}_j + Q_j),$$
$$F_2 = Q_k.$$

Applying the Lemma we obtain $g_k \in B(\varepsilon)$ such that $\operatorname{dist}(F_1, g_k + F_2) \geq 3\delta$.

Now put

$$F_1 = \bigcup_{l\le k}\bigcup_{j\le k}(s_l + g_j + Q_j) \cup \bigcup_{j<k}(\widetilde{g}_j + Q_j),$$
$$F_2 = Q_k.$$

The Lemma gives $\widetilde{g}_k \in B(\varepsilon)$ such that $\operatorname{dist}(F_1, \widetilde{g}_k + F_2) \ge 3\delta$. This finishes the construction of our sequences.

Fix sequences $\{\varepsilon_m\}_{m=1}^{\infty}$ and $\{\delta_m\}_{m=1}^{\infty}$ such that $\sum_{i>m}\varepsilon_i < \delta_m/2$ and $3\delta_m$ satisfies the conclusion of the Lemma for ε_m.

Using the above construction we obtain $g_k^m, \widetilde{g}_k^m \in G$, $k, m \in \mathbb{N}$, such that for every $m \in \mathbb{N}$ we have

(i) $\forall k \in \mathbb{N} : g_k^m, \widetilde{g}_k^m \in B(\varepsilon_m)$,
(ii) $\forall k \in \mathbb{N} : \operatorname{dist}(g_k^m + Q_k, \bigcup_{i<k}(g_i^m + Q_i)) \ge 3\delta_m$,
(iii) $\forall k \in \mathbb{N} : \operatorname{dist}(\widetilde{g}_k^m + Q_k, \bigcup_{i<k}(\widetilde{g}_i^m + Q_i)) \ge 3\delta_m$,
(iv) $\forall n \in \mathbb{N}\ \forall i, j \in \mathbb{N}$, $i, j \ge n : \operatorname{dist}(s_n + g_i^m + Q_i, \widetilde{g}_j^m + Q_j) \ge 3\delta_m$.

Now we define the desired sets:

$$A = \bigcap_{m=1}^{\infty}\bigcup_{k=1}^{\infty}\overline{B(g_k^m + Q_k, \delta_m)}, \qquad B = \bigcap_{m=1}^{\infty}\bigcup_{k=1}^{\infty}\overline{B(\widetilde{g}_k^m + Q_k, \delta_m)}.$$

Conditions (ii) and (iii) give that both A and B are closed. We show that for every compact set K there exists $g \in G$ with $g + K \subset A$. This easily implies that A is not a Haar null set. The same argument works for B. Let $K \subset G$ be a compact set. There exists $n_1 \in \mathbb{N}$ such that $K \subset B(Q_{n_1}, \delta_1/2)$. Thus $g_{n_1}^1 + K \subset B(g_{n_1}^1 + Q_{n_1}, \delta_1/2)$. Now suppose that we have defined $n_1, \dots, n_{k-1}$. There exists $n_k \in \mathbb{N}$ such that

$$g_{n_1}^1 + g_{n_2}^2 + \ldots + g_{n_{k-1}}^{k-1} + K \subset B(Q_{n_k}, \delta_k/2).$$

Thus

$$(\star) \qquad g_{n_1}^1 + g_{n_2}^2 + \ldots + g_{n_k}^k + K \subset B(g_{n_k}^k + Q_{n_k}, \delta_k/2).$$

The sequence $\{\sum_{j=1}^m g_{n_j}^j\}_{m=1}^{\infty}$ is convergent because of our choice of ε_j's and condition (i). Put $g = \sum_{j=1}^{\infty} g_{n_j}^j$. We have

$$g + K \subset B(g_{n_k}^k + Q_{n_k}, \delta_k) \quad \text{for every } k \in \mathbb{N},$$

since

$$\sup_{y\in g+K}\operatorname{dist}(g_{n_1}^1 + \ldots + g_{n_k}^k + K, y) \le \sum_{j=k+1}^{\infty}\operatorname{dist}(0, g_{n_j}^j) \le \sum_{j=k+1}^{\infty}\varepsilon_j < \delta_k/2$$

and $(\star)$ holds. Thus we can conclude that $g + K \subset A$, and we have shown that A is not Haar null.

Let $s \in G$ be arbitrary. We will show that the set $(s + A) \cap B$ is compact. Since A and B are closed it is sufficient to prove that our set is totally bounded. Choose $\eta > 0$. It is easy to see that $\lim \delta_m = 0$. We choose $m, n \in \mathbb{N}$ so large that $\delta_m < \eta$ and $\varrho(s, s_n) < \delta_m/2$. Using condition (iv) we obtain

$$\begin{aligned}(s+A) \cap B &\subset \Big(s_n + \bigcup_{k=1}^{\infty} \overline{B(g_k^m + Q_k, 3\delta_m/2)}\Big) \cap \Big(\bigcup_{k=1}^{\infty} \overline{B(\widetilde{g}_k^m + Q_k, \delta_m)}\Big) \\ &= \Big(\bigcup_{k=1}^{\infty} \overline{B(s_n + g_k^m + Q_k, 3\delta_m/2)}\Big) \cap \Big(\bigcup_{k=1}^{\infty} \overline{B(\widetilde{g}_k^m + Q_k, \delta_m)}\Big) \\ &\subset \Big(\bigcup_{k=1}^{n-1} \overline{B(s_n + g_k^m + Q_k, 3\delta_m/2)}\Big) \cup \Big(\bigcup_{k=1}^{n-1} \overline{B(\widetilde{g}_k^m + Q_k, \delta_m)}\Big).\end{aligned}$$

The last union can be covered by finitely many closed balls with radii $3\delta_m/2 < 2\eta$. Thus we can find a finite 2η-net of the set $(s+A) \cap B$ for each $\eta > 0$ and therefore our set is compact. ■

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