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THE NATURAL OPERATORS $T^{(0,0)} \rightsquigarrow T^{(1,1)}T^{(r)}$

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Abstract. We study the problem of how a map $f : M \to \mathbb{R}$ on an n-manifold M induces canonically an affinor $A(f) : TT^{(r)}M \to TT^{(r)}M$ on the vector r-tangent bundle $T^{(r)}M = (J^r(M,\mathbb{R})_0)^*$ over M. This problem is reflected in the concept of natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$. For integers $r \geq 1$ and $n \geq 2$ we prove that the space of all such operators is a free $(r+1)^2$-dimensional module over $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$ and we construct explicitly a basis of this module.

0. In [2], Gancarzewicz and Kolář obtained a full classification of all natural (canonical) affinors $A : TT^{(r)}M \to TT^{(r)}M$ on the vector r-tangent bundle $T^{(r)}M = (J^r(M,\mathbb{R})_0)^*$ over an n-manifold M. They proved that for $r \geq 1$ any such A is a linear combination (with real coefficients) of the identity affinor $\mathrm{Id} : TT^{(r)}M \to TT^{(r)}M$ and the affinor $\delta : TT^{(r)}M \to TT^{(r)}M$ which is the composition

$$TT^{(r)}M \to T^{(r)}M \times_M TM \subset T^{(r)}M \times_M T^{(r)}M \cong VT^{(r)}M \subset TT^{(r)}M,$$

where the arrow is $(\pi^T, T\pi) : TT^{(r)}M \to T^{(r)}M \times_M TM$, $\pi^T : TT^{(r)}M \to T^{(r)}M$ is the tangent bundle projection, $\pi : T^{(r)}M \to M$ is the bundle projection, and the inclusion $TM \subset T^{(r)}M$ is the dualization of the jet projection $J^r(M,\mathbb{R})_0 \to J^1(M,\mathbb{R})_0$.

In this note we study the problem of how a map $f : M \to \mathbb{R}$ on an n-manifold M induces canonically an affinor $A(f) : TT^{(r)}M \to TT^{(r)}M$ on $T^{(r)}M$. This problem is reflected in the concept of natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$ in the sense of [3], where $T^{(r)} = (J^r(\cdot,\mathbb{R})_0)^*$ is the r-tangent vector bundle functor on manifolds. For integers $r \geq 1$ and $n \geq 2$ we prove that the space of all such operators is a free $(r+1)^2$-dimensional module over $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$ and we exhibit an explicit basis of this module.

Natural affinors $A : TT^{(r)}M \to TT^{(r)}M$ can be considered as "constant" natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$. So, we recover the above mentioned result of [2]. Other generalizations of [2] are presented in [6]–[8], [10].

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Natural affinors play an important role in differential geometry. For example, they can be used to study "generalized" torsions of a connection (see [4], [1]). It seems that a similar role can be played by affinors depending canonically on some geometric objects (functions, vector fields, connections, etc.). That motivates investigation of natural operators with values in affinors.

Throughout this note the usual coordinates on $\mathbb{R}^n$ are denoted by $x^1, \ldots, x^n$, and $\partial_i = \partial/\partial x^i$, $i = 1, \ldots, n$. All manifolds and maps are assumed to be of class C^∞.

1. In this section we recall how a map $f : M \to \mathbb{R}$ on an n-manifold M induces a map $F(f) : T^{(r)}M \to \mathbb{R}$. This problem is reflected in the concept of natural operators $F : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}T^{(r)}$.

EXAMPLE 1. Let $f : M \to \mathbb{R}$. Applying the functor $T^{(r)}$ we obtain $T^{(r)}f : T^{(r)}M \to T^{(r)}\mathbb{R}$. For a map $h \in \mathcal{C}^\infty(T^{(r)}\mathbb{R})$ we get $F^{(h)}(f) = h \circ T^{(r)}f : T^{(r)}M \to \mathbb{R}$. The correspondence $F^{(h)} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}T^{(r)}$ is a natural operator.

PROPOSITION 1 ([5]). *Every natural operator $F : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}T^{(r)}$ is of the form $F = F^{(h)}$ for some $h \in \mathcal{C}^\infty(T^{(r)}\mathbb{R})$.*

2. In this section we explain how a map $f : M \to \mathbb{R}$ on an n-manifold M induces canonically a vector field $V(f)$ on $T^{(r)}M$. This problem is reflected in the concept of natural operators $V : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$.

EXAMPLE 2. Let $f : M \to \mathbb{R}$. For any $s = 0, \ldots, r-1$ we define a vertical vector field $V^{[s]}(f)$ on $T^{(r)}M$ by

$$V^{[s]}(f)_\omega = (\omega, \omega^{[s,f]}) \in \{\omega\} \times T^{(r)}_x M \cong V_\omega T^{(r)}M, \qquad \omega \in T^{(r)}_x M,\ x \in M.$$

Here $\omega^{[s,f]} \in T^{(r)}_x M = (J^r_x(M,\mathbb{R})_0)^*$, $\omega^{[s,f]}(j^r_x(\gamma)) = \omega(j^r_x((f - f(x))^s \gamma))$, $\gamma : M \to \mathbb{R}$, $\gamma(x) = 0$. In particular $V^{[0]}(f)$ is the Liouville vector field. The correspondence $V^{[s]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}M$ is a natural operator.

The set of all natural operators $V : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$ is (in an obvious way) a module over the algebra of all natural operators $F : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}T^{(r)}$. So, by Proposition 1 it is a module over $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$.

The following fact will not be used in further considerations.

PROPOSITION 2. *For natural numbers $r \geq 2$ and $n \geq 3$ the natural operators $V^{[s]}$ for $s = 0, \ldots, r-1$ form a basis in the $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$-module of all natural operators $V : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$.*

Proof scheme. Let $V : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$ be a natural operator. For any $f : M \to \mathbb{R}$ we have $V(f) : T^*T^{(r)}M \to \mathbb{R}$, which is fiber linear. So, V can be considered as a natural operator $V : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(T^*T^{(r)})$ satisfying the obvious fiber linearity condition. Conversely, any natural operator $V : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(T^*T^{(r)})$ with the fiber linearity condition can be considered as a natural operator $V : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow TT^{(r)}$. So, to prove Proposition 2 it remains to "extract" the natural operators $V : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(T^*T^{(r)})$ satisfying the fiber linearity condition from all natural operators $V : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(T^*T^{(r)})$, described in [9]. We leave the details to the reader. ∎

3. In this section we show how a map $f : M \to \mathbb{R}$ on an n-manifold M induces canonically a linear transformation $A(f) : TT^{(r)}M \to T^{(r)}M$. The linearity means that $A(f)$ restricts to linear maps $A(f)_u : T_uT^{(r)}M \to T^{(r)}_xM$ for any $u \in T^{(r)}_xM$, $x \in M$. This problem is reflected in the concept of natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$.

Example 3. Consider the composition

$$\widetilde{\delta} : TT^{(r)}M \to TM \to T^{(r)}M$$

of the map $T\pi : TT^{(r)}M \to TM$ tangent to the bundle projection $\pi : T^{(r)}M \to M$ with the inclusion $TM \to T^{(r)}M$ given by the dualization of the jet projection $J^r(M,\mathbb{R})_0 \to J^1(M,\mathbb{R})_0$. Then $\widetilde{\delta}$ can be considered as the "constant" natural operator $\widetilde{\delta} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$.

Example 4. Let $s = 0,\dots,r-1$ and $k = 1,\dots,r$. Consider $h_k : T^{(r)}\mathbb{R} \to \mathbb{R}$ given by

$$h_k(\omega) = \langle \omega, j^r_x((t-x)^k)\rangle, \qquad \omega \in T^{(r)}_x\mathbb{R},\ x \in \mathbb{R},\ t = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}.$$

We define a natural operator $\widetilde{A}^{[s,k]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$ by

$$\widetilde{A}^{[s,k]}(f) = d(F^{(h_k)}(f)) \cdot (\mathrm{pr}_2 \circ V^{[s]}(f) \circ \pi^T), \qquad f : M \to \mathbb{R},\ M \in \mathrm{obj}(\mathcal{M}f_n).$$

Here $\pi^T : TT^{(r)}M \to T^{(r)}M$ is the tangent bundle projection and $\mathrm{pr}_2 : VT^{(r)}M \cong T^{(r)}M \times_M T^{(r)}M \to T^{(r)}M$ is the projection onto the second (essential) factor. The map $F^{(h_k)}(f) : T^{(r)}M \to \mathbb{R}$ is described in Example 1.

Example 5. Let $s = 0,\dots,r-1$. Let $\pi : T^{(r)}\mathbb{R} \to \mathbb{R}$ be the bundle projection. We define a natural operator $\widetilde{A}^{[s]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$ by

$$\widetilde{A}^{[s]}(f) = d(F^{(\pi)}(f)) \cdot (\mathrm{pr}_2 \circ V^{[s]}(f) \circ \pi^T), \qquad f : M \to \mathbb{R},\ M \in \mathrm{obj}(\mathcal{M}f_n).$$

Example 6. Let $q = 1,\dots,r-1$. For $f : M \to \mathbb{R}$ and $v \in (TT^{(r)}M)_x$ with $x \in M$ define $\omega^{(q,f,v)} \in T^{(r)}_xM$ by

$$\langle \omega^{(q,f,v)}, j^r_x(\gamma)\rangle = \langle d(F^{(h_1)}(\gamma)\circ \mathrm{pr}_2 \circ V^{[q]}(f)\circ \pi^T), v\rangle,$$
$$\gamma : M \to \mathbb{R},\ \gamma(x) = 0.$$

Here $h_1 : T^{(r)}\mathbb{R} \to \mathbb{R}$ is as in Example 4. By the assumption on q the value $\langle d(F^{(h_1)}(\gamma)\circ \mathrm{pr}_2 \circ V^{[q]}(f)\circ \pi^T), v\rangle$ depends linearly on $j^r_x(\gamma)$. Hence $\omega^{(q,f,v)} \in T^{(r)}_x M$ is well defined. We define $\widetilde{A}^{\langle q\rangle}(f) : TT^{(r)}M \to T^{(r)}M$ by

$$\widetilde{A}^{\langle q\rangle}(f)(v) = \omega^{(q,f,v)}, \quad v \in TT^{(r)}M,$$

and we obtain a natural operator $\widetilde{A}^{\langle q\rangle} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$.

Remark 1. For $q = 0$ the definition of $\omega^{(0,f,v)}$ as in Example 6 is not correct because the term defining $\omega^{(0,f,v)}$ is not determined by $j^r_x(\gamma)$.

The set of all natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$ is a module over the algebra of all natural operators $F : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}T^{(r)}$. So, by Proposition 1 it is a module over $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$.

The crucial point in our considerations is the following proposition.

Proposition 3. *For natural numbers $r \geq 1$ and $n \geq 2$ the $(r+1)^2 - 1$ natural operators $\widetilde{\delta}$, $\widetilde{A}^{\langle q\rangle}$, $\widetilde{A}^{[s]}$ and $\widetilde{A}^{[s,k]}$ for $q = 1,\dots,r-1$, $s = 0,\dots,r-1$ and $k = 1,\dots,r$ form a basis in the $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$-module of all natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$.*

Proof. Let $(j^r_0 x^\alpha)^* \in T^{(r)}_0\mathbb{R}^n$ for $\alpha \in (\mathbb{N}\cup\{0\})^n$ with $1 \leq |\alpha| \leq r$ be the usual basis. The coordinates of $v \in T^{(r)}_0\mathbb{R}^n$ with respect to the above basis will be denoted by $[v]_\alpha$, where the α are as above. The coordinates of $w \in \mathbb{R}^n$ with respect to the obvious basis will be denoted by $[w]_1,\dots,[w]_n$.

Fix a natural operator $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$. Clearly, A is determined by the values $\langle A(f)(y), j^r_0(\gamma)\rangle \in \mathbb{R}$ for any $f : \mathbb{R}^n \to \mathbb{R}$, any $\gamma : \mathbb{R}^n \to \mathbb{R}$ with $\gamma(0) = 0$ and any $y = (y_1, y_2, y_3) \in \mathbb{R}^n \times T^{(r)}_0\mathbb{R}^n \times T^{(r)}_0\mathbb{R}^n \cong (TT^{(r)}\mathbb{R}^n)_0$. Therefore, we will now study those values.

Since $n \geq 2$, by the rank theorem we can assume that $f = x^n + a$ and $\gamma = x^1$. By the naturality of A with respect to the homotheties $a_t = (t^1x^1,\dots,t^{n-1}x^{n-1},x^n)$ for $t = (t^1,\dots,t^{n-1}) \in \mathbb{R}^{n-1}_+$, we have

$$\langle A(x^n + a)(TT^{(r)}(a_t)(y)), j^r_0(x^1)\rangle = t^1\langle A(x^n + a)(y), j^r_0(x^1)\rangle$$

for any $t = (t^1,\dots,t^{n-1}) \in \mathbb{R}^{n-1}_+$. Then, using the homogeneous function theorem ([3]) and the fiber linearity of $A(x^n + a)$ we obtain

$$(1)\quad \langle A(x^n+a)(y), j_0^r(x^1)\rangle = \lambda(a,([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_1]_1 + \sum_{s=0}^{r-1}\mu^s(a,([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_2]_{(1,0,\dots,0,s)}[y_1]_n + \sum_{s=0}^{r-1}\nu^s(a,([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_3]_{(1,0,\dots,0,s)} + \sum_{s=0}^{r-1}\sum_{q=1}^{r}\varrho^{sq}(a,([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_2]_{(1,0,\dots,0,s)}[y_3]_{(0,\dots,0,q)}$$

for some maps $\lambda,\mu^s,\nu^s,\varrho^{sq} : \mathbb{R}\times\mathbb{R}^r \cong T^{(r)}\mathbb{R} \to \mathbb{R}$, where $a\in\mathbb{R}$ and $y=(y_1,y_2,y_3)\in\mathbb{R}^n\times T_0^{(r)}\mathbb{R}^n\times T_0^{(r)}\mathbb{R}^n$.

It is easy to verify that

$$(2)\quad \langle\widetilde{\delta}(x^n+a)(y), j_0^r(x^1)\rangle = [y_1]_1,$$

$$(3)\quad \langle\widetilde{A}^{[s]}(x^n+a)(y), j_0^r(x^1)\rangle = [y_2]_{(1,0,\dots,0,s)}[y_1]_n,$$

$$(4)\quad \langle\widetilde{A}^{[s,k]}(x^n+a)(y), j_0^r(x^1)\rangle = [y_2]_{(1,0,\dots,0,s)}[y_3]_{(0,\dots,0,k)},$$

$$(5)\quad \langle\widetilde{A}^{\langle q\rangle}(x^n+a)(y), j_0^r(x^1)\rangle = [y_3]_{(1,0,\dots,0,q)},$$

for $q=1,\dots,r-1$, $s=0,\dots,r-1$ and $k=1,\dots,r$, $a\in\mathbb{R}$ and $y=(y_1,y_2,y_3)\in\mathbb{R}^n\times T_0^{(r)}\mathbb{R}^n\times T_0^{(r)}\mathbb{R}^n$. So, replacing A by

$$A-\lambda\widetilde{\delta}-\sum_{s=0}^{r-1}\mu^s\widetilde{A}^{[s]}-\sum_{q=1}^{r-1}\nu^q\widetilde{A}^{\langle q\rangle}-\sum_{s=0}^{r-1}\sum_{q=1}^{r}\varrho^{sq}\widetilde{A}^{[s,q]}$$

and using (1) we can assume that

$$(6)\quad \langle A(x^n+a)(y), j_0^r(x^1)\rangle = \nu^0(a,([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_3]_{(1,0,\dots,0)}$$

for some map $\nu^0 : \mathbb{R}\times\mathbb{R}^r\to\mathbb{R}$, where $a\in\mathbb{R}$ and $y=(y_1,y_2,y_3)\in\mathbb{R}^n\times T_0^{(r)}\mathbb{R}^n\times T_0^{(r)}\mathbb{R}^n$.

We are going to show that $A=0$. Let $\varphi=(x^1+(x^1)^{r+1},x^2,\dots,x^n)$ near $0\in\mathbb{R}^n$. By the invariance of A with respect to φ we have

$$(7)\quad \langle A(x^n+a)(y), j_0^r(x^1)\rangle = \langle A(x^n+a)(TT^{(r)}(\varphi)(y), j_0^r(x^1)\rangle,$$

because φ preserves both the map x^n+a and $j_0^r(x^1)$. If $y=(e_1,y_2,0)$, $e_1=(1,0,\dots,0)\in\mathbb{R}^n$, then $TT^{(r)}(\varphi)(y)=(e_1,\widetilde{y}_2,\widetilde{y}_3)$ for some $\widetilde{y}_2,\widetilde{y}_3\in T_0^{(r)}\mathbb{R}^n$ with $[\widetilde{y}_2]_{(0,\dots,0,l)}=[y_2]_{(0,\dots,0,l)}$ for $l=0,\dots,r$ and $[\widetilde{y}_3]_{(1,0,\dots,0)}=\alpha[y_2]_{(r,0,\dots,0)}$ for some $\alpha\neq 0$. Now, by using (7) and (6) with y and $TT^{(r)}(\varphi)(y)$ we see that $\nu^0=0$, i.e. $\langle A(x^n+a)(y), j_0^r(x^1)\rangle=0$ for any $y=(y_1,y_2,y_3)\in\mathbb{R}^n\times T_0^{(r)}\mathbb{R}^n\times T_0^{(r)}\mathbb{R}^n$ and $a\in\mathbb{R}$. Consequently, $A=0$. ∎

4. In this section we study the problem of how a map $f : M \to \mathbb{R}$ on an n-manifold M induces canonically a linear transformation $A(f) : TT^{(r)}M \to TM$. The linearity means that $A(f)$ restricts to linear maps $A(f)_u : T_uT^{(r)}M \to T_xM$ for any $u \in T^{(r)}_x M$, $x \in M$. This problem is reflected in the concept of natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T)_{\mathrm{lin}}$.

Example 7. The tangent map $T\pi : TT^{(r)}M \to TM$ of the bundle projection $\pi : T^{(r)}M \to M$ can be considered as the "constant" natural operator $T\pi : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T)_{\mathrm{lin}}$.

Example 8. Let $k = 1, \dots, r$. Consider the natural operator $\widetilde{A}^{[r-1,k]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$ of Example 4. Given $f : M \to \mathbb{R}$ and $v \in (TT^{(r)}M)_x$ with $x \in M$ we have $\langle \widetilde{A}^{[r-1,k]}(f)(v), j^r_x(\gamma)\rangle = 0$ for any $\gamma : M \to \mathbb{R}$ with $j^1_x(\gamma) = 0$, i.e. $\widetilde{A}^{[r-1,k]}(f)(v) \in T_xM \subset T^{(r)}_x M$. So, by corestriction we produce a natural operator $B^{[k]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T)_{\mathrm{lin}}$ by

$$B^{[k]}(f) := \widetilde{A}^{[r-1,k]}(f) : TT^{(r)}M \to TM, \qquad f : M \to \mathbb{R}.$$

Example 9. Similarly, using the natural operator $\widetilde{A}^{[r-1]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$ of Example 5, by corestriction we produce a natural operator $B^{[]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T)_{\mathrm{lin}}$ by

$$B^{[]}(f) := \widetilde{A}^{[r-1]}(f) : TT^{(r)}M \to TM, \qquad f : M \to \mathbb{R}.$$

The set of all natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T)_{\mathrm{lin}}$ is a module over the algebra of all natural operators $F : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}T^{(r)}$. So, by Proposition 1 it is a module over $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$.

Proposition 4. *For natural numbers $r \geq 1$ and $n \geq 2$ the $r + 2$ natural operators $T\pi$, $B^{[]}$ and $B^{[k]}$ for $k = 1, \dots, r$ form a basis in the $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$-module of all natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T)_{\mathrm{lin}}$.*

Proof. Consider a natural operator $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T)_{\mathrm{lin}}$. Applying the natural inclusion $TM \subset T^{(r)}M$ we can consider A as a natural operator $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$. So, by Proposition 3 we have

$$A = \lambda\widetilde{\delta} + \sum_{s=0}^{r-1} \mu^s \widetilde{A}^{[s]} + \sum_{q=1}^{r-1} \nu^q \widetilde{A}^{\langle q\rangle} + \sum_{s=0}^{r-1}\sum_{k=1}^{r} \varrho^{sk} \widetilde{A}^{[s,k]}$$

for some smooth maps $\lambda, \mu^s, \nu^q, \varrho^{sk} : T^r\mathbb{R} = \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. Replacing A by

$A - \lambda T\pi - \mu^{r-1} B^{[]} - \sum_{k=1}^{r} \varrho^{r-1\ k} B^{[k]}$ we can assume

$$A = \sum_{s=0}^{r-2} \mu^s \widetilde{A}^{[s]} + \sum_{q=1}^{r-1} \nu^q \widetilde{A}^{\langle q\rangle} + \sum_{s=0}^{r-2}\sum_{k=1}^{r} \varrho^{sk} \widetilde{A}^{[s,k]}$$

for some smooth maps $\mu^s, \nu^q, \varrho^{sk} : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. Now, for $y = (y_1, y_2, y_3) \in \mathbb{R}^n \times T_0^{(r)}\mathbb{R}^n \times T_0^{(r)}\mathbb{R}^n$ and $a \in \mathbb{R}$ we have

$$\begin{aligned}(8)\qquad \langle A(x^n + a)(y), j_0^r(x^1)\rangle = &\sum_{s=0}^{r-2} \mu^s(a, ([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_1]_n [y_2]_{(1,0,\dots,0,s)} \\ &+ \sum_{q=1}^{r-1} \nu^q(a, ([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_3]_{(1,0,\dots,0,q)} \\ &+ \sum_{s=0}^{r-2}\sum_{k=1}^{r} \varrho^{sk}(a, ([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_3]_{(0,\dots,0,k)}[y_2]_{(1,0,\dots,0,s)}.\end{aligned}$$

Then for $r = 1$ we have $\langle A(x^n + a)(y), j_0^r(x^1)\rangle = 0$, i.e. $A = 0$.

Let now $r \geq 2$. As $A(x^n+a)(y) \in T_0\mathbb{R}^n$, we have $\langle A(x^n+a)(y), j_0^r((x^1)^2)\rangle = 0$, i.e.

$$\langle A(x^n + a)(y), j_0^r(x^1)\rangle = \langle A(x^n + a)(y), j_0^r(x^1 + (x^1)^2)\rangle$$

for any a and y as above. Using the invariance of A with respect to $\varphi = (x^1 + (x^1)^2, x^2, \dots, x^n)$ near $0 \in \mathbb{R}^n$ we find $\langle A(x^n + a)(y), j_0^r(x^1 + (x^1)^2)\rangle = \langle A(x^n + a)(T^{(r)}(\varphi)(y)), j_0^r(x^1)\rangle$. Thus

$$(9)\qquad \langle A(x^n + a)(y), j_0^r(x^1)\rangle = \langle A(x^n + a)(TT^{(r)}(\varphi)(y)), j_0^r(x^1)\rangle$$

for any y and a as above.

If $y = (e_n, y_2, y_3)$, $e_n = (0, \dots, 0, 1) \in \mathbb{R}^n$, then $TT^{(r)}(\varphi)(y) = (e_n, \widetilde{y}_2, \widetilde{y}_3)$ for some $\widetilde{y}_2, \widetilde{y}_3 \in T_0^{(r)}\mathbb{R}^n$ with $[\widetilde{y}_2]_{(0,\dots,0,l)} = [y_2]_{(0,\dots,0,l)}$ for $l = 0, \dots, r$, $[\widetilde{y}_2]_{(1,0,\dots,0,s)} = [y_2]_{(1,0,\dots,0,s)} + [y_2]_{(2,0,\dots,0,s)}$ for $s = 0, \dots, r-2$, $[\widetilde{y}_3]_{(0,\dots,0,k)} = [y_3]_{(0,\dots,0,k)}$ for $k = 1, \dots, r$, $[\widetilde{y}_3]_{(1,0,\dots,0,q)} = [y_3]_{(1,0,\dots,0,q)} + [y_3]_{(2,0,\dots,0,q)}$ for $q = 1, \dots, r-2$, and $[\widetilde{y}_3]_{(1,0,\dots,0,r-1)} = [y_3]_{(1,0,\dots,0,r-1)}$. Now, it is easy to verify (by using (9) and (8) with $TT^{(r)}(\varphi)(y)$ playing the role of y) that $\mu^s = 0$ for $s = 0, \dots, r-2$, $\nu^q = 0$ for $q = 1, \dots, r-2$, and $\varrho^{sk} = 0$ for $s = 0, \dots, r-2$ and $k = 1, \dots, r$. Therefore

$$(10)\qquad \langle A(x^n + a)(y), j_0^r(x^1)\rangle = \nu^{r-1}(a, ([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_3]_{(1,0,\dots,0,r-1)}$$

for any $y = (y_1, y_2, y_3) \in \mathbb{R}^n \times T_0^{(r)}\mathbb{R}^n \times T_0^{(r)}\mathbb{R}^n$ and $a \in \mathbb{R}$.

If $y = (e_1, y_2, y_3)$, $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$, then $TT^{(r)}(\varphi)(y) = (e_1, \widetilde{y}_2, \widetilde{y}_3)$ for some $\widetilde{y}_2, \widetilde{y}_3 \in T_0^{(r)}\mathbb{R}^n$ with $[\widetilde{y}_2]_{(0,\dots,0,l)} = [y_2]_{(0,\dots,0,l)}$ for $l = 0, \dots, r$ and $[\widetilde{y}_3]_{(1,0,\dots,0,r-1)} = [y_3]_{(1,0,\dots,0,r-1)} + \alpha[y_2]_{(1,0,\dots,r-1)}$ for some $\alpha \neq 0$. Now, by using (9) and (10) with $TT^{(r)}(\varphi)(y)$ playing the role of y we see that $\nu^{r-1} = 0$, i.e. $\langle A(x^n + a)(y), j_0^r(x^1)\rangle = 0$ for any $y = (y_1, y_2, y_3) \in \mathbb{R}^n \times T_0^{(r)}\mathbb{R}^n \times T_0^{(r)}\mathbb{R}^n$ and $a \in \mathbb{R}$. Consequently, $A = 0$. ∎

5. The purpose of the present paper is to study how a map $f : M \to \mathbb{R}$ on an n-manifold M induces canonically an affinor $A(f) : TT^{(r)}M \to TT^{(r)}M$ on $T^{(r)}M$. This problem is reflected in the concept of natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$.

EXAMPLE 10. The natural identity affinor $\mathrm{Id} : TT^{(r)} \to TT^{(r)}$ on $T^{(r)}$ can be considered as the "constant" natural operator $\mathrm{Id} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$.

EXAMPLE 11. The natural affinor $\delta : TT^{(r)} \to TT^{(r)}$ on $T^{(r)}$ can be considered as the "constant" natural operator $\delta : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$.

EXAMPLE 12. Let $s = 0, \ldots, r-1$ and $k = 1, \ldots, r$. We define a natural operator $A^{[s,k]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$ by

$$A^{[s,k]}(f)(v) = (\omega, \widetilde{A}^{[s,k]}(f)(v)) \in \{\omega\} \times T_x^{(r)}M \cong V_\omega T^{(r)}M \subset T_\omega T^{(r)}M$$

for $f : M \to \mathbb{R}$, $v \in T_\omega T^{(r)}M$, $\omega \in T_x^{(r)}M$, $x \in M$. We recall that $\widetilde{A}^{[s,k]}$ is the operator from Example 4.

EXAMPLE 13. Let $s = 0, \ldots, r-1$. We define a natural operator $A^{[s]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$ by

$$A^{[s]}(f)(v) = (\omega, \widetilde{A}^{[s]}(f)(v)) \in \{\omega\} \times T_x^{(r)}M$$

for $f : M \to \mathbb{R}$, $v \in T_\omega T^{(r)}M$, $\omega \in T_x^{(r)}M$, $x \in M$. We recall that $\widetilde{A}^{[s]}$ is the operator from Example 5.

EXAMPLE 14. Let $q = 1, \ldots, r-1$. We define a natural operator $A^{\langle q\rangle} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$ by

$$A^{\langle q\rangle}(f)(v) = (\omega, \widetilde{A}^{\langle q\rangle}(f)(v)) \in \{\omega\} \times T_x^{(r)}M$$

for $f : M \to \mathbb{R}$, $v \in T_\omega T^{(r)}M$, $\omega \in T_x^{(r)}M$, $x \in M$. We recall that $\widetilde{A}^{\langle q\rangle}$ is the operator from Example 6.

The set of all natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$ is a module over the algebra of all natural operators $F : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}T^{(r)}$. So, by Proposition 1 it is a module over $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$.

The main result of the present paper is the following classification theorem.

THEOREM 1. *For natural numbers $r \geq 1$ and $n \geq 2$ the $(r+1)^2$ natural operators* Id, δ, $A^{\langle q\rangle}$, $A^{[s]}$ *and* $A^{[s,k]}$ *for $q = 1, \ldots, r-1$, $s = 0, \ldots, r-1$ and*

$k = 1,\dots,r$ form a basis in the $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$-module of all natural operators $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$.

Proof. Let $A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$ be a natural operator. Applying $T\pi : TT^{(r)}M \to TM$ for any n-manifold M, we produce a natural operator $T\pi \circ A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T)_{\mathrm{lin}}$ such that $(T\pi \circ A)(f) = T\pi \circ A(f)$ for any $f : M \to \mathbb{R}$. By Proposition 4 we have

$$(11) \qquad T\pi \circ A = \alpha T\pi + \beta B^{[]} + \sum_{k=1}^{r} \gamma^k B^{[k]}$$

for some $\alpha, \beta, \gamma^k \in \mathcal{C}^\infty(T^{(r)}\mathbb{R})$. Replacing A by $A - \alpha\,\mathrm{Id}$ we can assume that $\alpha = 0$.

We are going to prove that $T\pi \circ A = 0$.

It is easy to verify that for any $a \in \mathbb{R}$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^n \times T^{(r)}_0\mathbb{R}^n \times T^{(r)}_0\mathbb{R}^n$ we have

$$\langle B^{[]}(x^n + a)(y), j^r_0(x^n)\rangle = [y_2]_{(0,\dots,0,r)}[y_1]_n,$$
$$\langle B^{[k]}(x^n + a)(y), j^r_0(x^n)\rangle = [y_2]_{(0,\dots,0,r)}[y_3]_{(0,\dots,0,k)}.$$

Then for any a and y as above we obtain

$$(12) \qquad \langle T\pi \circ A(x^n + a)(y), j^r_0(x^n)\rangle = \beta(a, ([y_2]_{(0,\dots,0,l)})^r_{l=1})[y_2]_{(0,\dots,0,r)}[y_1]_n + \sum_{k=1}^{r} \gamma^k(a, ([y_2]_{(0,\dots,0,l)})^r_{l=1})[y_2]_{(0,\dots,0,r)}[y_3]_{(0,\dots,0,k)}.$$

Therefore putting $A(x^n + a)(y) = \overline{y} = (\overline{y}_1, \overline{y}_2, \overline{y}_3) \in \mathbb{R}^n \times T^{(r)}_0\mathbb{R}^n \times T^{(r)}_0\mathbb{R}^n$ we have $\overline{y}_2 = y_2$ (because $A(x^n + a)$ is an affinor) and

$$(13) \qquad [\overline{y}_1]_n = \beta(a, ([y_2]_{(0,\dots,0,l)})^r_{l=1})[y_2]_{(0,\dots,0,r)}[y_1]_n + \sum_{k=1}^{r} \gamma^k(a, ([y_2]_{(0,\dots,0,l)})^r_{l=1})[y_2]_{(0,\dots,0,r)}[y_3]_{(0,\dots,0,k)}.$$

Using $\widetilde{A}^{[r-1]}$ from Example 5 we produce a natural operator $\widetilde{A}^{[r-1]} \circ A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$ defined by

$$(\widetilde{A}^{[r-1]} \circ A)(f) = \widetilde{A}^{[r-1]}(f) \circ A(f), \qquad f : M \to \mathbb{R}.$$

By Proposition 3 we have

$$(14) \qquad A^{[r-1]} \circ A = \overline{\lambda}\widetilde{\delta} + \sum_{s=0}^{r-1} \overline{\mu}^s \widetilde{A}^{[s]} + \sum_{q=1}^{r-1} \overline{\nu}^q \widetilde{A}^{\langle q\rangle} + \sum_{s=0}^{r-1}\sum_{k=1}^{r} \overline{\varrho}^{sk} \widetilde{A}^{[s,k]}$$

for some smooth maps $\overline{\lambda}, \overline{\mu}^s, \overline{\nu}^q, \overline{\varrho}^{sk} : T^{(r)}\mathbb{R} = \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. Then using (2)–(5) we have

$$(15)\quad \langle A^{[r-1]}(x^n + a) \circ A(x^n + a)(y), j_0^r(x^1)\rangle = \overline{\lambda}(a, ([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_1]_1 + \sum_{s=0}^{r-1} \overline{\mu}^s(a, ([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_2]_{(1,0,\dots,0,s)}[y_1]_n + \sum_{q=1}^{r-1} \overline{\nu}^q(a, ([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_3]_{(1,0,\dots,0,q)} + \sum_{s=0}^{r-1}\sum_{k=1}^{r} \overline{\varrho}^{sk}(a, ([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_2]_{(1,0,\dots,0,s)}[y_3]_{(0,\dots,0,k)}$$

for any a and y as above.

On the other hand, if $A(x^n + a)(y) = \overline{y} = (\overline{y}_1, \overline{y}_2, \overline{y}_3)$ then

$$\langle A^{[r-1]}(x^n + a) \circ A(x^n + a)(y), j_0^r(x^1)\rangle = [\overline{y}_2]_{(1,0,\dots,0,r-1)}[\overline{y}_1]_n$$

(see (3)). Hence by (13) we have

$$(15)\quad \langle A^{[r-1]}(x^n + a) \circ A(x^n + a)(y), j_0^r(x^1)\rangle = [\overline{y}_2]_{(1,0,\dots,0,r-1)}(\beta(a, ([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_2]_{(0,\dots,0,r)}[y_1]_n + \sum_{k=1}^{r} \gamma^k(a, ([y_2]_{(0,\dots,0,l)})_{l=1}^r)[y_2]_{(0,\dots,0,r)}[y_3]_{(0,\dots,0,k)})$$

for any a and y as above.

Using (15) and (16) we deduce that $\beta = 0$ and $\gamma^k = 0$, i.e. $T\pi \circ A = 0$.

From the last fact we deduce that $A(f) : TT^{(r)}M \to VT^{(r)}M \cong T^{(r)}M \times_M T^{(r)}M$ for any $f : M \to \mathbb{R}$. Using the projection $\mathrm{pr}_2 : VT^{(r)}M \to T^{(r)}M$ onto the second factor we produce a natural operator $\mathrm{pr}_2 \circ A : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow (TT^{(r)} \to T^{(r)})_{\mathrm{lin}}$ such that $(\mathrm{pr}_2 \circ A)(f) = \mathrm{pr}_2 \circ A(f)$ for any $f : M \to \mathbb{R}$. By Proposition 3 we have

$$\mathrm{pr}_2 \circ A = \lambda\widetilde{\delta} + \sum_{s=0}^{r-1} \mu^s \widetilde{A}^{[s]} + \sum_{q=1}^{r-1} \nu^q \widetilde{A}^{\langle q\rangle} + \sum_{s=0}^{r-1}\sum_{k=1}^{r} \varrho^{sk} \widetilde{A}^{[s,k]}$$

for some $\lambda, \mu^s, \nu^q, \varrho^{sk} \in \mathcal{C}^\infty(T^{(r)}\mathbb{R})$. Then

$$A = \lambda\delta + \sum_{s=0}^{r-1} \mu^s A^{[s]} + \sum_{q=1}^{r-1} \nu^q A^{\langle q\rangle} + \sum_{s=0}^{r-1}\sum_{k=1}^{r} \varrho^{sk} A^{[s,k]}. \ \blacksquare$$

REMARK 2. In [10] we studied the problem of how a vector field X on M induces an affinor $A(X)$ on $T^{(r)}M$. This problem is reflected in the concept of natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$. We proved that for $n \geq 3$

the vector space over $\mathbb{R}$ of all such operators is $(r+1)$-dimensional and we constructed explicitly a basis of this vector space.

6. To end this paper we explain how a map $f : M \to \mathbb{R}$ on an n-manifold M induces canonically a 1-form $\omega(f) \in \Omega^1(T^{(r)}M)$ on $T^{(r)}M$. This problem is reflected in the concept of natural operators $\omega : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{(r)}$.

EXAMPLE 15. Let $k = 1, \ldots, r$. We define a natural operator $\omega^{[k]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{(r)}$ by

$$\omega^{[k]}(f) = d(F^{(h_k)}(f)), \qquad f : M \to \mathbb{R}.$$

The map $h_k : T^{(r)}\mathbb{R} \to \mathbb{R}$ is defined in Example 4.

EXAMPLE 16. Let $\pi : T^{(r)}\mathbb{R} \to \mathbb{R}$ be the bundle projection. We define a natural operator $\omega^{[]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{(r)}$ by

$$\omega^{[]}(f) = d(F^{(\pi)}(f)), \qquad f : M \to \mathbb{R}.$$

The set of all natural operators $\omega : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{(r)}$ is a module over the algebra of all natural operators $F : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}T^{(r)}$. So, by Proposition 1 it is a module over $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$.

PROPOSITION 5. *For natural numbers $r \ge 1$ and $n \ge 1$ the $r+1$ natural operators $\omega^{[]}$ and $\omega^{[k]}$ for $k = 1, \ldots, r$ form a basis in the $\mathcal{C}^\infty(T^{(r)}\mathbb{R})$-module of all natural operators $\omega : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{(r)}$.*

Proof. Consider a natural operator $\omega : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{(r)}$. Let $a \in \mathbb{R}$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^n \times T^{(r)}_0\mathbb{R}^n \times T^{(r)}_0\mathbb{R}^n$. The naturality of ω with respect to $a_t = (x^1, tx^2, \ldots, tx^n)$ for $t \neq 0$ implies that $\omega(x^1 + a)(T^{(r)}(a_t)(y)) = \omega(x^1 + a)(y)$. Then using the fiber linearity of $\omega(x^1 + a)$ we easily deduce that

$$\omega(x^1 + a)(y) = \alpha(a, ([y_2]_{(l,0,\ldots,0)})_{l=1}^r)[y_1]_1 + \sum_{k=1}^r \beta^k(a, ([y_2]_{(l,0,\ldots,0)})_{l=1}^r)[y_3]_{(k,0,\ldots,0)}$$

for some maps $\alpha, \beta^k : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. Since ω is determined by the values $\omega(x^1+a)(y)$ for all a and y as above, it follows that $\omega = \alpha\omega^{[]} + \sum_{k=1}^r \beta^k\omega^{[k]}$. ∎

REMARK 3. (a) Proposition 5 for $r \ge 2$ is also a corollary of Theorem 1. More precisely, any natural operator $\omega : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{(r)}$ defines a natural operator $\omega \otimes V^{[0]} : T^{(0,0)}_{|\mathcal{M}f_n} \rightsquigarrow T^{(1,1)}T^{(r)}$ by

$$(\omega \otimes V^{[0]})(f) = \omega(f) \otimes V^{[0]}, \qquad f : M \to \mathbb{R},$$

where $V^{[0]}$ is the Liouville vector field on $T^{(r)}M$ (see Example 2). Using Theorem 1 we can express $\omega \otimes V^{[0]}$ as a linear combination. Then it is easy to study ω and to complete the proof of Proposition 5.

(b) Proposition 5 is also a corollary of the result of [5]. Indeed, $\omega(f) : TT^{(r)}M \to \mathbb{R}$ is a fiber linear map for any $f : M \to \mathbb{R}$. Hence ω can be considered as a natural operator $\omega : T^{(0,0)}_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(TT^{(r)})$ satisfying the obvious fiber linearity condition. So, to prove Proposition 5 it is sufficient to "extract" the natural operators $\omega : T^{(0,0)}_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(TT^{(r)})$ satisfying the fiber linearity condition from all natural operators $F : T^{(0,0)}_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}(TT^{(r)})$, classified in [5]. This is an easy exercise on the homogeneous function theorem.

REFERENCES

[1] M. Doupovec and J. Kurek, *Torsions of connections on higher order tangent bundles*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 55 (2001), 15–22.

[2] J. Gancarzewicz and I. Kolář, *Natural affinors on the extended r-th order tangent bundles*, Rend. Circ. Mat. Palermo Suppl. 30 (1993), 95–100.

[3] I. Kolář, P. W. Michor and J. Slovák, *Natural Operations in Differential Geometry*, Springer, Berlin, 1993.

[4] I. Kolář and M. Modugno, *Torsions of connections on some natural bundles*, Differential Geom. Appl. 2 (1992), 1–16.

[5] W. M. Mikulski, *Natural transformations transforming functions and vector fields to functions on some natural bundles*, Math. Bohemica 117 (1992), 217–223.

[6] —, *The natural affinors on $\otimes^k T^{(r)}$*, Note Mat. 19 (1999), 269–274.

[7] —, *The natural transformations $TT^{(r)} \to TT^{(r)}$*, Arch. Math. (Brno) 36 (2000), 71–75.

[8] —, *The natural transformations $T^*T^{(r)} \to T^*T^{(r)}$*, Note Mat. 20 (2000/2001), 81–87.

[9] —, *Natural operators lifting functions to cotangent bundles of linear higher order tangent bundles*, Rend. Circ. Mat. Palermo Suppl. 43 (1996), 199–206.

[10] —, *Natural transformations transforming vector fields into affinors on the extended r-th order tangent bundles*, Arch. Math. (Brno) 29 (1993), 59–70.

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