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PRODUCT PRESERVING BUNDLE FUNCTORS ON FIBERED FIBERED MANIFOLDS

BY

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Abstract. We investigate the category of product preserving bundle functors defined on the category of fibered fibered manifolds. We show a bijective correspondence between this category and a certain category of commutative diagrams on product preserving bundle functors defined on the category $\mathcal{M}f$ of smooth manifolds. By an application of the theory of Weil functors, the latter category is considered as a category of commutative diagrams on Weil algebras. We also mention the relation with natural transformations between product preserving bundle functors on the category of fibered manifolds.

0. Introduction. Recently it has been clarified that product preserving or fiber product preserving bundle functors on some admissible categories on manifolds represent a unifying technique for studying a large class of geometric problems. That is why such functors have been classified by many authors ([3], [7], [4], [6] and [11]).

The first papers on this subject were written about 1986 independently by Kainz and Michor [4], Eck [3] and Luciano [7]. These results describe all product preserving bundle functors defined on the category $\mathcal{M}f$ of smooth manifolds with smooth maps in terms of Weil algebras. They give a bijection between product preserving bundle functors on $\mathcal{M}f$ and Weil algebras and a bijection between natural transformations on such bundle functors and algebra homomorphisms between the corresponding Weil algebras.

Fibered manifolds are surjective submersions $\pi : Y \to \underline{Y}$ between manifolds. In 1996, the first author described all product preserving bundle functors F on the category $\mathcal{FM}$ of fibered manifolds in terms of Weil algebras A_0, A_1 representing couples of product preserving bundle functors on $\mathcal{M}f$ determined by F and of Weil algebra homomorphisms $A_1 \to A_0$ (see [10]; cf. also [9] and [2]).

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Fibered fibered manifolds are fibered surjective submersions $\pi : Y \to \underline{Y}$ between fibered manifolds. They form a category $\mathcal{F}^2\mathcal{M}$, which is over manifolds. It is endowed with products and is admissible in the sense of [5]. Fibered fibered manifolds appear naturally in differential geometry if we consider transverse natural bundles in the sense of R. Wolak [13]. In the present paper, we describe all product preserving bundle functors F on the category $\mathcal{F}^2\mathcal{M}$ in terms of morphisms $(A_1 \to A_2) \to (A_3 \to A_4)$ between Weil algebra homomorphisms. The main result gives a bijective correspondence between product preserving bundle functors in question and morphisms between Weil algebra homomorphisms. On the level of morphisms, this bijection assigns to a natural transformation between two product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ a morphism between morphisms over Weil algebra homomorphisms.

Recently, natural operators lifting projectable tensor fields of type $(1, q)$ to product preserving bundle functors on $\mathcal{FM}$ and jets of fibered manifold morphisms have been studied ([1], [2] and [12]). In those papers, the results of [10] have been applied. It seems that the results of the present paper will allow one to obtain similar results to those of [1], [2] and [12].

All manifolds considered are assumed to be of class C^∞, finite-dimensional and without boundaries. All maps between manifods are assumed to be (C^∞) smooth.

I. The category of fibered fibered manifolds. We start by recalling the concept of a fibered fibered manifold (see e.g. [8]). For fibered manifolds Y and $\underline{Y}$, a surjective submersion $\pi : Y \to \underline{Y}$ is said to be a *fibered fibered manifold* if it is fibered and transforms submersively fibers of Y onto fibers of $\underline{Y}$. For another fibered fibered manifold $\pi' : Y' \to \underline{Y}'$ we define a morphism $f : Y \to Y'$ to be a smooth fibered map (between fibered manifolds Y and Y') for which there is the so called base map $\underline{f} : \underline{Y} \to \underline{Y}'$ which is a fibered map (between the fibered manifolds $\underline{Y}$ and $\underline{Y}'$). In symbols, $\pi' \circ f = \underline{f} \circ \pi$. Thus all fibered fibered manifolds form a category which will be denoted by $\mathcal{F}^2\mathcal{M}$. The category $\mathcal{F}^2\mathcal{M}$ is endowed with products and it is local and admissible in the sense of [5].

In what follows, let $pp\mathcal{BF}_{\mathcal{F}^2\mathcal{M}}$ denote the category of all product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ with their natural transformations.

II. The commutative diagrams. In the present section, we introduce a certain category $\mathcal{K}_0$ of commutative diagrams over product preserving bundle functors defined on $\mathcal{M}f$. For $p = 1, \dots, 4$, let $\overset{(p)}{G} : \mathcal{M}f \to \mathcal{FM}$ be such bundle functors. Put $S = \{(1, 2), (1, 3), (2, 4), (3, 4)\}$. Consider commutative diagrams

$$(*)\qquad\qquad \Theta:\quad \begin{array}{ccccc} & & \overset{(2)}{G} & & \\ & \overset{12}{\Theta}\nearrow & & \searrow\overset{24}{\Theta} & \\ \overset{(1)}{G} & & & & \overset{(4)}{G} \\ & \overset{13}{\Theta}\searrow & & \nearrow\overset{34}{\Theta} & \\ & & \overset{(3)}{G} & & \end{array}$$

where all $\overset{pq}{\Theta} : \overset{(p)}{G} \to \overset{(q)}{G}$ for $(p,q) \in S$ are natural transformations. The objects of $\mathcal{K}_0$ are such commutative diagrams, i.e. we put $\mathrm{Obj}(\mathcal{K}_0) = \mathrm{Diag}(pp\mathcal{BF}_{\mathcal{M}f})$. Consider another object of $\mathcal{K}_0$, say $\Theta' = (\overset{pq}{\Theta}{}')_{(p,q)\in S}$ with vertices $\overset{(p)}{G}{}'$. Then $\Phi : \Theta \to \Theta' = (\overset{(p)}{\Phi} : \overset{(p)}{G} \to \overset{(p)}{G}{}')_{p=1,\dots,4}$ is said to be a morphism if the diagrams

$$\begin{array}{ccc} \overset{(p)}{G} & \xrightarrow{\overset{(p)}{\Phi}} & \overset{(p)}{G}{}' \\ {\scriptstyle \overset{pq}{\Theta}}\downarrow & & \downarrow{\scriptstyle \overset{pq}{\Theta}{}'} \\ \overset{(q)}{G} & \xrightarrow{\overset{(q)}{\Phi}} & \overset{(q)}{G}{}' \end{array}$$

commute for all $(p,q) \in S$. The fact that $\Phi : \Theta \to \Theta'$ is a morphism is easy to verify.

III. From product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ to diagrams in $\mathcal{K}_0$. Let $F : \mathcal{F}^2\mathcal{M} \to \mathcal{FM}$ be a product preserving bundle functor, i.e. a $pp\mathcal{BF}_{\mathcal{F}^2\mathcal{M}}$-object. We are going to define a $\mathcal{K}_0$-object Θ^F associated to F in terms of four product preserving bundle functors $\mathcal{M}f \to \mathcal{F}^2\mathcal{M}$. They are defined as follows:

$$\begin{aligned} j_1(M) &= (i_1(M) \xrightarrow{\mathrm{id}_M} i_1(M)), & j_1(f) &= f : j_1(M) \to j_1(N),\\ j_2(M) &= (i_1(M) \xrightarrow{\mathrm{pt}_M} \mathrm{Pt}), & j_2(f) &= f : j_2(M) \to j_2(N),\\ j_3(M) &= (i_2(M) \xrightarrow{\mathrm{id}_M} i_2(M)), & j_3(f) &= f : j_3(M) \to j_3(N),\\ j_4(M) &= (i_2(M) \xrightarrow{\mathrm{pt}_M} \mathrm{Pt}), & j_4(f) &= f : j_4(M) \to j_4(N), \end{aligned}$$

where M, N are manifolds, $f : M \to N$ is a smooth map. Further, $i_1(M) = (M \xrightarrow{\mathrm{id}_M} M)$, $i_2(M) = (M \to \mathrm{pt})$ and $\mathrm{Pt} = (\mathrm{pt} \to \mathrm{pt})$ are $\mathcal{FM}$-objects for pt being the one-point manifold. We have the commutative diagram

$$\begin{array}{ccccc} & & j_2 & & \\ & \overset{12}{\tau}\nearrow & & \searrow\overset{24}{\tau} & \\ j_1 & & & & j_4 \\ & \overset{13}{\tau}\searrow & & \nearrow\overset{34}{\tau} & \\ & & j_3 & & \end{array}$$

of natural transformations $\overset{pq}{\tau} = (\overset{pq}{\tau})_M = \mathrm{id}_M : j_p(M) \to j_q(M)$ for any $(p, q) \in S$. Applying $F : \mathcal{F}^2\mathcal{M} \to \mathcal{FM}$, we obtain product preserving bundle functors $\overset{(p)}{G} = F \circ j_p : \mathcal{M}f \to \mathcal{FM}$ $(p = 1, \ldots, 4)$ and the $\mathcal{K}_0$-object

$$\Theta^F : \quad \begin{array}{ccccc} & & \overset{(2)}{G}{}^F & & \\ & \overset{12}{\Theta}{}^F \nearrow & & \searrow \overset{24}{\Theta}{}^F & \\ \overset{(1)}{G}{}^F & & & & \overset{(4)}{G}{}^F \\ & \overset{13}{\Theta}{}^F \searrow & & \nearrow \overset{34}{\Theta}{}^F & \\ & & \overset{(3)}{G}{}^F & & \end{array}$$

where $\overset{pq}{\Theta}{}^F_M = F(\overset{pq}{\tau}_M) : \overset{(p)}{G}{}^F(M) \to \overset{(q)}{G}{}^F(M)$ for $(p, q) \in S$.

If $F' : \mathcal{F}^2\mathcal{M} \to \mathcal{FM}$ is another product preserving bundle functor and $\eta : F \to F'$ is a natural transformation, then we have a $\mathcal{K}_0$-morphism $\Phi^\eta : \Theta^F \to \Theta^{F'}$ defined by

$$\overset{(p)}{\Phi}{}^F_M = \eta_{j_p(M)}, \qquad M \in \mathrm{Obj}(\mathcal{M}f),\ p = 1, \ldots, 4,$$

Further, if η is an isomorphism, then so is Φ^η. Summing up, we have constructed a functor $\mathcal{U} : pp\mathcal{BF}_{\mathcal{F}^2\mathcal{M}} \to \mathcal{K}_0$ giving the correspondence $F \mapsto \Theta^F$ and $\eta \mapsto \Phi^\eta$.

IV. From diagrams in $\mathcal{K}_0$ to product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$. In this section, we assign a product preserving bundle functor on $\mathcal{F}^2\mathcal{M}$ to a given $\mathcal{K}_0$-object. Let Θ be a $\mathcal{K}_0$-object of the form $(*)$. We define a product preserving bundle functor $F^\Theta : \mathcal{F}^2\mathcal{M} \to \mathcal{FM}$ as follows. Consider an $\mathcal{F}^2\mathcal{M}$-object $Y = (\pi : Y \to \underline{Y})$ where $Y = (p : Y \to M)$ and $\underline{Y} = (\underline{p} : \underline{Y} \to \underline{M})$ are fibered manifolds and $\pi : Y \to \underline{Y}$ is a fibered surjective submersion covering $\underline{\pi} : M \to \underline{M}$. So we have the commutative diagram

$$(**) \qquad \begin{array}{ccc} Y & \xrightarrow{\ \pi\ } & \underline{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \underline{p}} \\ M & \xrightarrow{\ \underline{\pi}\ } & \underline{M} \end{array}$$

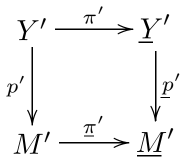
We define a fibered manifold over Y by

$$\begin{aligned} F^\Theta Y = \{(x_1, x_2, x_3, x_4) \in \overset{(1)}{G}(\underline{M}) \times \overset{(2)}{G}(M) \times \overset{(3)}{G}(\underline{Y}) \times \overset{(4)}{G}(Y);\\ \overset{12}{\Theta}_{\underline{M}}(x_1) = \overset{(2)}{G}(\underline{\pi})(x_2), \quad \overset{13}{\Theta}_{\underline{M}}(x_1) = \overset{(3)}{G}(\underline{p})(x_3),\\ \overset{24}{\Theta}_M(x_2) = \overset{(4)}{G}(p)(x_4), \quad \overset{34}{\Theta}_{\underline{Y}}(x_3) = \overset{(4)}{G}(\pi)(x_4)\}. \end{aligned}$$

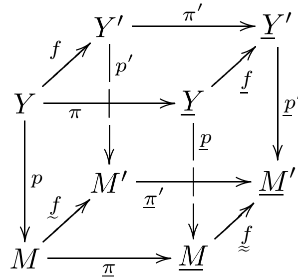
LEMMA 1. *$F^\Theta Y$ is a non-empty regular submanifold in $\overset{(1)}{G}(\underline{M}) \times \overset{(2)}{G}(M) \times \overset{(3)}{G}(\underline{Y}) \times \overset{(4)}{G}(Y)$.*

Proof. The mappings $\overset{(2)}{G}(\underline{\pi}) : \overset{(2)}{G}(M) \to \overset{(2)}{G}(\underline{M})$ and $\overset{(3)}{G}(\underline{p}) : \overset{(3)}{G}(\underline{Y}) \to \overset{(3)}{G}(\underline{M})$ are surjective submersions since $\underline{\pi}$ and $\underline{p}$ are. We show that $(\overset{(4)}{G}(p), \overset{(4)}{G}(\pi)) : \overset{(4)}{G}(Y) \to \overset{(4)}{G}(M) \times_{\overset{(4)}{G}(\underline{M})} \overset{(4)}{G}(\underline{Y})$ is also a surjective submersion. This follows from the fact that $\overset{(4)}{G}(p,\pi) : \overset{(4)}{G}(Y) \to \overset{(4)}{G}(M \times_{\underline{M}} \underline{Y})$ is a surjective submersion since $(p,\pi) : Y \to M \times_{\underline{M}} \underline{Y}$ is; moreover, $\overset{(4)}{G}$ preserves products and the obvious map $\overset{(4)}{G}(M \times_{\underline{M}} \underline{Y}) \to \overset{(4)}{G}(M) \times_{\overset{(4)}{G}(\underline{M})} \overset{(4)}{G}(\underline{Y})$ is a diffeomorphism. ∎

Let $\pi' : Y' \to \underline{Y}'$ be another $\mathcal{F}^2\mathcal{M}$-object with the corresponding commutative diagram



Further, let $f : Y \to Y'$ be an $\mathcal{F}^2\mathcal{M}$-morphism giving the commutative cubic diagram



LEMMA 2. *The mapping* $\overset{(1)}{G}(\underset{\approx}{f}) \times \overset{(2)}{G}(\underset{\sim}{f}) \times \overset{(3)}{G}(\underline{f}) \times \overset{(4)}{G}(f) : \overset{(1)}{G}(\underline{M}) \times \overset{(2)}{G}(M) \times \overset{(3)}{G}(\underline{Y}) \times \overset{(4)}{G}(Y) \to \overset{(1)}{G}(\underline{M}') \times \overset{(2)}{G}(M') \times \overset{(3)}{G}(\underline{Y}') \times \overset{(4)}{G}(Y')$ *sends* $F^{\Theta}Y$ *to* $F^{\Theta}Y'$.

Proof. The proof is standard, by a direct verification of all identities in the definition of F^{Θ}. ∎

For an $\mathcal{F}^2\mathcal{M}$-morphism $f : Y \to Y'$ we define $F^{\Theta}f : F^{\Theta}Y \to F^{\Theta}Y'$ as the restriction and corestriction of $\overset{(1)}{G}(\underset{\approx}{f}) \times \overset{(2)}{G}(\underset{\sim}{f}) \times \overset{(3)}{G}(\underline{f}) \times \overset{(4)}{G}(f)$. Hence we have a product preserving bundle functor $F^{\Theta} : \mathcal{F}^2\mathcal{M} \to \mathcal{F}\mathcal{M}$.

Let Θ' be another $\mathcal{K}_0$-object and $\Phi = \overset{(p)}{\Phi} : \Theta \to \Theta'$ be a $\mathcal{K}_0$-morphism. The following lemma defines a natural transformation $F^{\Theta} \to F^{\Theta'}$.

LEMMA 3. *Let Y be an $\mathcal{F}^2\mathcal{M}$-object giving the commutative diagram* $(**)$. *Then the map* $\overset{(1)}{\Phi}_{\underline{M}} \times \overset{(2)}{\Phi}_M \times \overset{(3)}{\Phi}_{\underline{Y}} \times \overset{(4)}{\Phi}_Y : \overset{(1)}{G}(\underline{M}) \times \overset{(2)}{G}(M) \times \overset{(3)}{G}(\underline{Y}) \times \overset{(4)}{G}(Y) \to \overset{(1)}{G}{}'(\underline{M}) \times \overset{(2)}{G}{}'(M) \times \overset{(3)}{G}{}'(\underline{Y}) \times \overset{(4)}{G}{}'(Y)$ *sends* $F^{\Theta} \to F^{\Theta'}$.

Proof. The proof is standard. ■

Define $\eta^{\Phi}_Y : F^{\Theta}Y \to F^{\Theta'}Y$ as the corresponding restriction and corestriction of $\overset{(1)}{\Phi}_{\underline{M}} \times \overset{(2)}{\Phi}_M \times \overset{(3)}{\Phi}_{\underline{Y}} \times \overset{(4)}{\Phi}_Y$. By Lemma 3, $\eta^{\Phi} = (\eta^{\Phi}_Y : F^{\Theta}Y \to F^{\Theta'}Y)$ is a natural transformation. Further, if Φ is an isomorphism then so is η^{Φ}. Summing up we have constructed a functor $\mathcal{W} : \mathcal{K}_0 \to pp\mathcal{BF}_{\mathcal{F}^2\mathcal{M}}$ mapping $\Theta \mapsto F^{\Theta}$ and $\Phi \mapsto \eta^{\Phi}$.

REMARK. The constructions from Sections III and IV almost work in the situation when the assumption that F preserves products is omitted. This assumption is used only in Lemma 1 to prove that $\overset{(4)}{G}$ preserves products. The validity of Lemma 1 without the product preserving assumption is at present an open problem.

V. An isomorphism $F \simeq F^{\Theta^F}$. Let $F : \mathcal{F}^2\mathcal{M} \to \mathcal{FM}$ be a product preserving bundle functor. We have the corresponding $\mathcal{K}_0$-object Θ^F and the product preserving bundle functor $F^{\Theta^F} : \mathcal{F}^2\mathcal{M} \to \mathcal{FM}$. We are going to construct a natural transformation $\vartheta : F \to F^{\Theta^F}$ as follows. Let Y be an $\mathcal{F}^2\mathcal{M}$-object defining the commutative diagram $(**)$. Define $\widetilde{\vartheta}_Y : FY \to \overset{(1)}{G}{}^F(\underline{M}) \times \overset{(2)}{G}{}^F(M) \times \overset{(3)}{G}{}^F(\underline{Y}) \times \overset{(4)}{G}{}^F(Y)$ by

$$\widetilde{\vartheta}_Y = (F(\underline{p} \circ \pi), F(p), F(\pi), F(\mathrm{id}_Y))$$

where we use the $\mathcal{F}^2\mathcal{M}$-morphisms $\underline{p} \circ \pi : Y \to j_1(\underline{M})$, $p : Y \to j_2(M)$, $\pi : Y \to j_3(\underline{Y})$ and $\mathrm{id}_Y : Y \to j_4(Y)$. We have the following lemma which can be verified directly.

LEMMA 4. $\mathrm{Im}\,\widetilde{\vartheta}_Y \subset F^{\Theta^F}Y$.

Lemma 4 enables us to define the map $\vartheta_Y : FY \to F^{\Theta^F}Y$ as the obvious restriction of $\widetilde{\vartheta}_Y$ for any $\mathcal{F}^2\mathcal{M}$-object Y. It is easy to verify that ϑ is a natural transformation. The following assertion states that ϑ is an isomorphism.

PROPOSITION 1. *Let $F : \mathcal{F}^2\mathcal{M} \to \mathcal{FM}$ be a product preserving bundle functor. Then $\vartheta : F \to F^{\Theta^F}$ is an isomorphism.*

Proof. We have to show that ϑ_Y is a diffeomorphism for any $\mathcal{F}^2\mathcal{M}$-object. Since Y is locally a multi-product of $j_1(\mathbb{R}), j_2(\mathbb{R}), j_3(\mathbb{R}), j_4(\mathbb{R})$ and F preserves products, it suffices to show that $\vartheta_{j_p(\mathbb{R})}$ is a diffeomorphism for $p = 1, \ldots, 4$. To verify this, observe that the projection of $F^{\Theta^F}(j_p(\mathbb{R}))$ onto the pth factor gives a diffeomorphism. The composition of this diffeomorphism with $\vartheta_{j_p(\mathbb{R})}$ yields the identity map on $F(j_p(\mathbb{R}))$, which proves the assertion for $\vartheta_{j_p(\mathbb{R})}$. ■

VI. An isomorphism $\Theta \simeq \Theta^{F^\Theta}$. Let Θ be a $\mathcal{K}_0$-object of the form $(*)$ and $F^\Theta : \mathcal{F}^2\mathcal{M} \to \mathcal{FM}$ be the corresponding product preserving bundle functor. Further, consider the corresponding $\mathcal{K}_0$-object Θ^{F^Θ} and denote it by Θ^*. We write Θ^* in the form

$$\begin{array}{ccccc} & & \overset{(2)}{G}{}^* & & \\ & \overset{12}{\Theta}{}^* \nearrow & & \searrow \overset{24}{\Theta}{}^* & \\ \overset{(1)}{G}{}^* & & & & \overset{(4)}{G}{}^* \\ & \overset{13}{\Theta}{}^* \searrow & & \nearrow \overset{34}{\Theta}{}^* & \\ & & \overset{(3)}{G}{}^* & & \end{array}$$

and for any manifold we have

$$\begin{aligned}
&\overset{(1)}{G}{}^*(M) = F^\Theta(j_1(M)) \subset \overset{(1)}{G}(M) \times \overset{(2)}{G}(M) \times \overset{(3)}{G}(M) \times \overset{(4)}{G}(M),\\
&\overset{(2)}{G}{}^*(M) \subset \mathrm{pt} \times \overset{(2)}{G}(M) \times \mathrm{pt} \times \overset{(4)}{G}(M),\\
&\overset{(3)}{G}{}^*(M) \subset \mathrm{pt} \times \mathrm{pt} \times \overset{(3)}{G}(M) \times \overset{(4)}{G}(M),\\
&\overset{(4)}{G}{}^*(M) \subset \mathrm{pt} \times \mathrm{pt} \times \mathrm{pt} \times \overset{(4)}{G}(M).
\end{aligned}$$

We define $\overset{(p)}{\Psi}_M : \overset{(p)}{G}{}^*(M) \to \overset{(p)}{G}(M)$ as the restriction of the corresponding projection, i.e. $\overset{(p)}{\Psi}_M = \mathrm{pr}_{p|\overset{(p)}{G}{}^*(M)}$ for the pth projection $\mathrm{pr}_p : \overset{(1)}{G}(M) \times \overset{(2)}{G}(M) \times \overset{(3)}{G}(M) \times \overset{(4)}{G}(M) \to \overset{(p)}{G}(M)$. The correspondence $\overset{(p)}{\Psi} : \overset{(p)}{G}{}^* \to \overset{(p)}{G}$ is a natural transformation and the sequence $\Psi = \{\overset{(p)}{\Psi}\} : \Theta^* \to \Theta$ is a $\mathcal{K}_0$-morphism. The verification is standard but rather long; so also is the proof of the following assertion, and therefore they are omitted.

PROPOSITION 2. *If Θ is a $\mathcal{K}_0$-object then $\Psi : \Theta^* \to \Theta$ is an isomorphism.*

VII. Object classification theorem. In the following sections we are going to give the main results in the form of several classification theorems. The first theorem reads:

THEOREM 1. *The correspondence $\mathcal{U} : F \mapsto \Theta^F$ described in Section* III *induces a bijective correspondence between the equivalence classes of product preserving bundle functors F defined on $\mathcal{F}^2\mathcal{M}$ and the equivalence classes of diagrams Θ of the form $(*)$ with product preserving bundle functors $\overset{(p)}{G}$ defined on $\mathcal{M}f$. The inverse bijection is induced by the correspondence $\mathcal{W} : \Theta \mapsto F^\Theta$.*

Proof. The correspondence $[F] \mapsto [F^{\Theta}]$ is well defined since if $\eta : F \to F'$ is an isomorphism then so is $\Phi^{\eta} : \Theta^{F} \to \Theta^{F'}$. Analogously, the correspondence $[\Theta] \mapsto [F^{\Theta}]$ is well defined since if $\Phi : \Theta \to \Theta'$ is an isomorphism then so is $\eta^{\Phi} : F^{\Theta} \to F^{\Theta'}$.

Further, Proposition 1 implies that $[F] = [F^{\Theta^{F}}]$ for any product preserving F defined on $\mathcal{F}^2\mathcal{M}$ and Proposition 2 yields $[\Theta] = [\Theta^{F^{\Theta}}]$, which completes the proof. ∎

VIII. Morphism classification theorem. We start with an assertion giving a bijective correspondence between natural transformations of product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ and $\mathcal{K}_0$-morphisms.

PROPOSITION 3. *Let F and F' be product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ and $\Phi : \Theta^{F} \to \Theta^{F'}$ be a $\mathcal{K}_0$-morphism between the corresponding $\mathcal{K}_0$-objects. Let $\eta^{[\Phi]} : F \to F'$ be a natural transformation given by the composition*

$$F \xrightarrow[\simeq]{\vartheta} F^{\Theta^{F}} \xrightarrow{\eta^{\Phi}} F^{\Theta^{F'}} \xrightarrow[\simeq]{\vartheta^{-1}} F'$$

where ϑ is the isomorphism from Section V *for F and F'. Then $\widetilde{\eta} = \eta^{[\Phi]}$ is the unique natural transformation $F \to F'$ satisfying $\Phi^{\widetilde{\eta}} = \Phi$ where $\Phi^{\widetilde{\eta}}$ is described in Section* III *and η^{Φ} in Section* IV.

Proof. It is sufficient to check this property for the $\mathcal{F}^2\mathcal{M}$-objects $Y = j_1(\mathbb{R})$, $j_2(\mathbb{R})$, $j_3(\mathbb{R})$ and $j_4(\mathbb{R})$. ∎

We can now state the main result of this section.

THEOREM 2. *Let F and F' be product preserving bundle functors defined on the category $\mathcal{F}^2\mathcal{M}$. Then the correspondence $\mathcal{U} : \eta \mapsto \Phi^{\eta}$ from Section* III *sending natural transformations $F \to F'$ to morphisms $\Theta^{F} \to \Theta^{F'}$ is bijective. The inverse correspondence defined by $\Phi \mapsto \eta^{[\Phi]}$ is the correspondence from Proposition* 3.

IX. Relations with the Weil theory. Taking into account the classical result from [5] characterizing product preserving bundle functors on $\mathcal{M}f$ as Weil functors, we can reformulate our results in terms of Weil algebras. Thus Theorems 1 and 2 are reformulated in the following couple of theorems:

THEOREM 1′. *There is a bijective correspondence between the equivalence classes of product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ and the equivalence classes of commutative diagrams*

$$(***)\qquad \begin{array}{ccccc} & & A_2 & & \\ & \overset{12}{\varphi}\nearrow & & \searrow\overset{24}{\varphi} & \\ A_1 & & & & A_4 \\ & \overset{13}{\varphi}\searrow & & \nearrow\overset{34}{\varphi} & \\ & & A_3 & & \end{array}$$

consisting of Weil algebra homomorphisms.

Theorem 2′. *Given two product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ there is a bijection between natural transformations over them and morphisms of the corresponding commutative diagrams of Weil algebra homomorphisms.*

X. Relations with natural transformations between product preserving bundle functors on fibered manifolds. Consider a diagram $(***)$. By the theory of Weil bundles and the results from [10] and [9] we have two product preserving bundle functors $T^{\overset{12}{\varphi}}$ and $T^{\overset{34}{\varphi}}$ and a natural transformation $\varrho^{(\overset{13}{\varphi},\overset{23}{\varphi})} : T^{\overset{12}{\varphi}} \to T^{\overset{34}{\varphi}}$. Conversely, having a natural transformation $\varrho : G \to G'$ between product preserving bundle functors on $\mathcal{FM}$ we have a commutative diagram of Weil algebra homomorphisms. We can reformulate Theorems 1′ and 2′ as follows:

Theorem 1″. *There is a bijective correspondence between the equivalence classes of product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$ and the equivalence classes of natural transformations between product preserving bundle functors on $\mathcal{FM}$.*

Theorem 2″. *Given two product preserving bundle functors on $\mathcal{F}^2\mathcal{M}$, there is a bijection between natural transformations over them and morphisms of the corresponding natural transformations between product preserving bundle functors on $\mathcal{FM}$.*

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