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LOW DIMENSIONAL HOMOTOPY GROUPS OF SUSPENSIONS OF THE HAWAIIAN EARRING

BY

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Abstract. We study the $(n+1)$st homotopy groups and the shape groups of the $(n-1)$-fold reduced and unreduced suspensions of the Hawaiian earring.

1. Introduction. We study the $(n+1)$st homotopy group of the $(n-1)$-fold reduced and unreduced suspensions of the Hawaiian earring $\mathbb{H}_1$. The *Hawaiian earring* $\mathbb{H}_1$ is the compact subset of the plane defined by

$$\mathbb{H}_1 = \bigcup_{j=1}^{\infty} \{(x,y) \mid (x-1/j)^2 + y^2 = 1/j^2\}$$

with base point $\mathbf{o} = \mathbf{o}_1 = (0,0)$. The $(n-1)$-fold reduced suspension $\mathbb{H}_n = \widetilde{\Sigma}_{\mathbf{o}}^{n-1}\mathbb{H}_1$ is a compact metric space whose underlying set is the one-point union of countably many n-dimensional spheres at the base point $\mathbf{o}_n$, and is called the *n-dimensional Hawaiian earring*. The singular homology of the space $\mathbb{H}_n$ is complicated ([1], [8]), and this paper is an attempt to understand the low dimensional homotopy groups of $\mathbb{H}_n$. The space $\mathbb{H}_n$ is $(n-1)$-connected and it is shown in [9] that for each $n \geq 2$, $\mathrm{H}_n(\mathbb{H}_n) \cong \pi_n(\mathbb{H}_n) \cong \mathbb{Z}^{\omega}$, the countable product of the integers. So the next step is to understand $\mathrm{H}_{n+1}(\mathbb{H}_n)$ and $\pi_{n+1}(\mathbb{H}_n)$. On the other hand, the singular homology of the $(n-1)$-fold unreduced suspension $\widehat{\mathbb{H}}_n = \Sigma_{\mathbf{o}}^{n-1}\mathbb{H}_1$ is easily seen to be as follows:

$$\widetilde{\mathrm{H}}_q(\widehat{\mathbb{H}}_n) \cong \begin{cases} \mathrm{H}_1(\mathbb{H}_1) & \text{if } q = n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{H}_1(\mathbb{H}_1)$ has been computed in [8]. The space $\widehat{\mathbb{H}}_n$ is $(n-1)$-connected and $\pi_n(\widehat{\mathbb{H}}_n) \cong \mathrm{H}_n(\widehat{\mathbb{H}}_n)$, and again the next nontrivial homotopy group to be computed is $\pi_{n+1}(\widehat{\mathbb{H}}_n)$.

Notice that $\mathbb{H}_n$ and $\widehat{\mathbb{H}}_n$ have the same shape type but do not have the same homotopy type by [1] and the above. In the present paper, the countable product of the n-dimensional spheres is denoted by S^n_∞. If we fix a base point $* \in S^n$ of the n-sphere, then the space $\mathbb{H}_n$ is naturally embedded in

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S^n_∞ as follows:

$$\mathbb{H}_n \approx \bigcup_{i=1}^{\infty}\{(x_j) \mid x_j = * \text{ for each } j \neq i\}.$$

In this paper we show the following results. The q-dimensional shape group (or Čech homotopy group) of a space X is denoted by $\check{\pi}_q(X)$.

(a) $\pi_{n+1}(\mathbb{H}_n) \cong \pi_{n+1}(S^n)^\omega \oplus \pi_{n+2}(S^n_\infty, \mathbb{H}_n)$ for each $n \geq 2$,
(b) $\pi_{n+1}(\widehat{\mathbb{H}}_n) \cong (\mathbb{Z}/2\mathbb{Z})^\omega \oplus \bigoplus_c(\mathbb{Z}/2\mathbb{Z})$ for each $n \geq 3$,
(c) $\check{\pi}_{n+1}(\mathbb{H}_n) \cong \check{\pi}_{n+1}(\widehat{\mathbb{H}}_n) \cong (\mathbb{Z}/2\mathbb{Z})^\omega$ for $n \geq 3$ and $\mathbb{Z}^\omega$ for $n = 2$.

Some comments on these results are in order. (a) is an infinite product analogue of the well known isomorphism for finite products. We have not succeeded in an explicit computation of the relative homotopy group $\pi_{n+2}(S^n_\infty, \mathbb{H}_n)$, which turns out to be isomorphic to the relative singular homology group $\mathrm{H}_{n+2}(S^n_\infty, \mathbb{H}_n)$. The computation in [1] shows that $\pi_4(S^2_\infty, \mathbb{H}_2)$ is nonzero. In (b), we have not obtained an explicit structure of $\pi_3(\widehat{\mathbb{H}}_2)$. With the help of [7] and [4] (see also [14]), the group can be represented as the kernel of a certain homomorphism (see the beginning of Section 4). However, the representation does not reveal the explicit structure of the group. Also the group is isomorphic to $\mathit{\Gamma}(\mathrm{H}_1(\mathbb{H}_1))$, where $\mathit{\Gamma}(A)$ denotes the Whitehead quadratic group of an abelian group A ([13]). Here we obtain an exact sequence containing $\mathit{\Gamma}(\mathrm{H}_1(X))$ for an arbitrary one-dimensional separable metric space X. However, the author has not succeeded in making an explicit computation of $\pi_3(\widehat{\mathbb{H}}_2)$. The computation (c) depends on the Hilton–Milnor theorem [12].

2. $\pi_{n+1}(\mathbb{H}_n)$ and $\check{\pi}_{n+1}(\mathbb{H}_n)$. For a countable collection $\{(X_i, x_i)\}$ of pointed compacta, let $\widetilde{\bigvee}_i X_i = \bigcup_i\{(y_j) \mid y_j = x_j \text{ for each } j \neq i\} \subset \prod_i X_i$. The base point $(x_i)_i$ is denoted by x_∞. The one-point union $\bigvee_{j=1}^k X_j$ is embedded in $\widetilde{\bigvee}_i X_i$ in the obvious way. Under this notation, $\mathbb{H}_n \approx \widetilde{\bigvee}_i S^n_i \subset \prod_i S^n_i = S^n_\infty$. In what follows, $\mathbb{H}_n$ is identified with $\widetilde{\bigvee}_i S^n_i$. The projection of $\prod_j X_j$ onto the ith factor X_i is denoted by p_i. The homomorphism $\varphi : \pi_q(\prod_i X_i) \to \prod_i \pi_q(X_i)$ defined by $\varphi(\alpha) = ((p_i)_\sharp(\alpha))_i$ for $\alpha \in \pi_q(\prod_i X_i)$ is an isomorphism. Our first result is stated as follows.

THEOREM 2.1. *For each $q \geq 2$, there exists a split exact sequence*

$$0 \to \pi_{q+1}\Big(\prod_i X_i, \widetilde{\bigvee_i} X_i\Big) \xrightarrow{\partial} \pi_q\Big(\widetilde{\bigvee_i} X_i\Big) \xrightarrow{i_\sharp} \pi_q\Big(\prod_i X_i\Big) \to 0$$

where $i_\sharp$ is induced by the inclusion $i : \widetilde{\bigvee}_i X_i \to \prod_i X_i$ and ∂ is the boundary homomorphism of the homotopy long exact sequence.

Proof. First we define $\lambda : \prod_i \pi_q(X_i) \to \pi_q(\widetilde{\bigvee}_i X_i)$ under the following notation and convention.

NOTATION AND CONVENTION. Let $a_j = 1 - 1/j$, $j \geq 1$, and notice that

$$\bigcup_{j=1}^{\infty} (I^{q-1} \times [a_j, a_{j+1}]) = I^{q-1} \times [0,1).$$

For simplicity, a map $f_j : (I^q, \partial I^q) \to (X_j, x_j)$ is identified with the map

$$f_j \circ (\mathrm{id} \times s_j) : (I^{q-1} \times [a_j, a_{j+1}], \partial(I^{q-1} \times [a_j, a_{j+1}])) \to (X_j, x_j),$$

where $s_j(t) = (t - a_j)/(a_{j+1} - a_j), t \in [a_j, a_{j+1}]$. Also we assume that $\widetilde{\bigvee}_i X_i$ has a metric so that $\operatorname{diam} X_i < 1/2^i$ for each i, and in particular $\operatorname{diam} \widetilde{\bigvee}_{j \geq i} X_j \to 0$ as $i \to \infty$.

For each $(\alpha_i) \in \prod_i \pi_q(X_i)$ and each i, take a map $f_i : (I^q, \partial I^q) \to (X_i, x_i)$ which represents the element α_i. Define the map $l : I^q \to \widetilde{\bigvee}_i X_i$ by $l | I^{q-1} \times [a_j, a_{j+1}] = f_j$ for $j \geq 1$ and $l | I^{q-1} \times \{1\} = x_\infty$ (recall the above convention). The crucial fact here is that $\operatorname{diam} f_j(I^q) \to 0$ as $j \to \infty$, and this guarantees that $l : I^q \to \widetilde{\bigvee}_i X_i$ is continuous. Let $\lambda((\alpha_i)) = [l] \in \pi_q(\widetilde{\bigvee}_i X_i)$.

CLAIM 1. *$\lambda((\alpha_i))$ is well defined, that is, the homotopy class $[l]$ does not depend on the choice of the maps (f_i).*

Proof of Claim 1. Suppose that $f_i \simeq g_i$ rel. ∂I^q and fix a homotopy $H_i : I^q \times [0,1] \to X_i$ such that $H_i(x, 0) = f_i(x)$ and $H_i(x, 1) = g_i(x)$ for $x \in I^q$, and $H_i(x, t) = x_\infty$ for all $x \in \partial I^q$ and $t \in [0,1]$. Let $f_\infty, g_\infty : (I^q, \partial I^q) \to (\widetilde{\bigvee}_i X_i, x_\infty)$ be the maps defined by $f_\infty | I^{q-1} \times [a_j, a_{j+1}] = f_j$ (recall the above convention) and $f_\infty | I^{q-1} \times \{1\} \equiv x_\infty$ etc. By the same reason as above, f_∞ and g_∞ are continuous. Again the fact that $\operatorname{diam} H_i(I^q \times [0,1]) \to 0$ as $i \to \infty$ guarantees that the map $H_\infty : I^q \times [0,1] \to \widetilde{\bigvee}_i X_i$ defined by $H_\infty | (I^{q-1} \times [a_j, a_{j+1}]) \times [0,1] = H_i$ (recall the above convention) and $H_\infty | (I^{q-1} \times \{1\}) \times [0,1] \equiv x_\infty$ is a continuous homotopy rel. ∂I^q from f_∞ to g_∞. This completes the proof.

CLAIM 2. *λ is a homomorphism.*

Proof. Take two sequences $\{f_i : (I^q, \partial I^q) \to (X_i, x_i)\}$ and $\{g_i : (I^q, \partial I^q) \to (X_i, x_i)\}$ of maps. We need to prove the equality

$$\lambda(([f_i] + [g_i])) = \lambda(([f_i])) + \lambda(([g_i])).$$

The element on the left hand side is represented by a map $h : (I^q, \partial I^q) \to (\widetilde{\bigvee}_i X_i, x_\infty)$ defined as follows. Let $b_i = (a_i + a_{i+1})/2$. Define the map h by $h | I^{q-1} \times [a_i, b_i] = f_i$ and $h | I^{q-1} \times [b_i, a_{i+1}] = g_i$ for each $i \geq 1$ and $h | I^{q-1} \times \{1\} \equiv x_\infty$. On the other hand, the element on the right hand side of the equality is represented by a map $k : (I^q, \partial I^q) \to (\widetilde{\bigvee}_i X_i, x_\infty)$

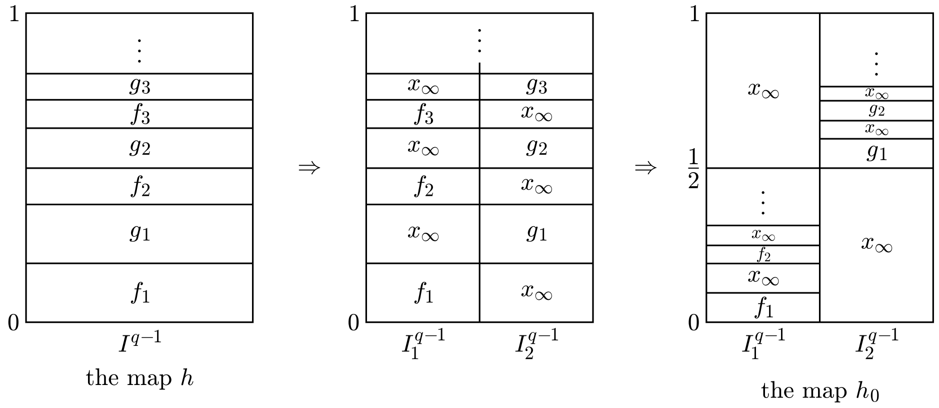


Fig. 1

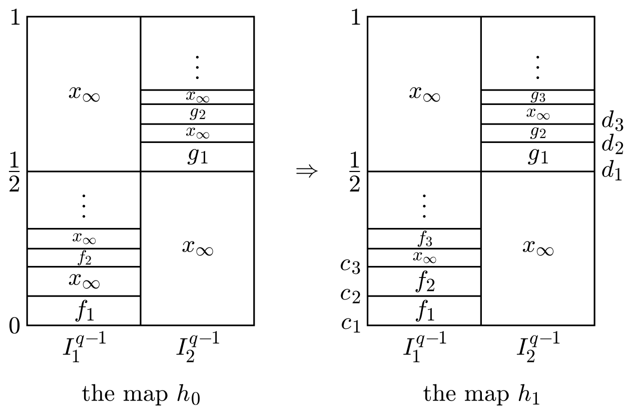


Fig. 2

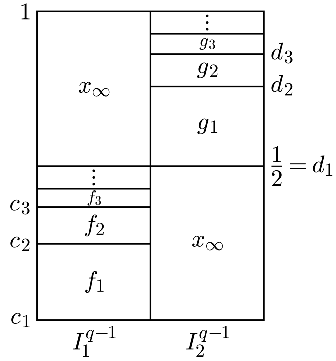
defined as follows. Let $c_i = a_i/2$ and $d_i = (a_i+1)/2$. Define the map k by $k|I^{q-1} \times [c_i, c_{i+1}] = f_i$, $k|I^{q-1} \times [d_i, d_{i+1}] = g_i$ and $k|I^{q-1} \times \{1/2\} \equiv k|I^{q-1} \times \{1\} \equiv x_\infty$. We need to prove that h and k are homotopic rel. ∂I^q. The following proof is motivated by the proof of the fact that the homotopy group of dimension at least 2 is abelian.

The map h is homotopic rel. ∂I^q to a map h_0 illustrated in Fig. 1. Here I_1^{q-1} and I_2^{q-1} denote the subsets of I^q defined by $I_1^{q-1} = I^{q-2} \times [0, 1/2]$ and $I_2^{q-1} = I^{q-2} \times [1/2, 1]$. Fig. 2 illustrates a map h_1 which is homotopic rel. ∂I^q to h_0 via a homotopy $H_0 : I^q \times [0,1] \to \widetilde{\bigvee}_i X_i$ such that $\operatorname{diam} H_0(\{z\} \times [0,1]) \le \operatorname{diam}(\bigvee_{j=1}^2 X_j) \le 1$ for each $z \in I^q$. The map h_1 is homotoped to a map h_2 such that

(2.1) $\quad h_2|I^{q-1} \times ([c_1, c_3] \cup [d_1, d_3]) = h_1|I^{q-1} \times ([c_1, c_3] \cup [d_1, d_3])$,

(2.2) $\quad h_2|I_1^{q-1} \times [c_3, c_4] = f_2$, $h_2|I_1^{q-1} \times [d_3, d_4]) = g_3$,

(2.3) $\quad h_2|I_2^{q-1} \times [0, 1/2] \equiv h_2|I_1^{q-1} \times [1/2, 1] \equiv x_\infty$.



the map h_∞

Fig. 3

A homotopy H_1 rel. ∂I^q from h_1 to h_2 may be chosen so that $H_1(z,t) = h_1(z,t)$ for each $(z,t) \in I^{q-1} \times ([c_1, c_3] \cup [d_1, d_3])$ and $\operatorname{diam} H_1(\{z\} \times [0,1]) \leq \operatorname{diam}(\bigvee_{j=2}^{3} X_j) \leq 1/2$ for each $z \in I^q$.

Continuing this process, we have sequences $\{h_m : (I^q, \partial I^q) \to (\widetilde{\bigvee}_i X_i, x_\infty)\}_{m\geq 1}$ of maps and $\{H_m\}_{m\geq 1}$ of homotopies rel. ∂I^q from h_m to h_{m+1} $(m \geq 1)$ such that

$(m.1)$ $\operatorname{diam} H_m(\{z\} \times [0,1]) \leq \operatorname{diam}(\widetilde{\bigvee}_{i\geq m} X_i) \leq 1/2^{m-1}$ for each $z \in I^q$,

$(m.2)$ $H_m(z,t) = h_m(z)$ for all $z \in h_m^{-1}(\bigvee_{j=1}^{m} X_j)$ and $t \in [0,1]$. In particular, $h_{m+1}|h_m^{-1}(\bigvee_{j=1}^{m} X_j) = h_m|h_m^{-1}(\bigvee_{j=1}^{m} X_j)$.

The above condition $(m.1)$ implies that (h_m) forms a Cauchy sequence and $h_\infty = \lim_{m\to\infty} h_m$ exists and is continuous. By $(m.2)$, $h_\infty|h_m^{-1}(\bigvee_{j=1}^{m} X_j) = h_m|h_m^{-1}(\bigvee_{j=1}^{m} X_j)$. Also the limit $H_\infty = \lim_{m\to\infty} H_m * H_{m-1} * \cdots * H_1 * H_0$ exists and is a homotopy rel. ∂I^q between h_0 and h_∞. The map h_∞ is illustrated in Fig. 3 and is clearly homotopic to k rel. ∂I^q. Thus we have the desired equality.

Claim 3. $\varphi \circ i_\sharp \circ \lambda = \mathrm{id} : \prod_i \pi_q(X_i) \to \prod_i \pi_q(X_i)$.

Proof. For each $(\alpha_i = [f_i])_i \in \prod_i \pi_q(X_i)$, the element $(p_i)_\sharp \circ i_\sharp \circ \lambda((\alpha_i))$ is easily seen to be represented by the map $\bar{f}_i : (I^q, \partial I^q) \to (\widetilde{\bigvee}_i X_i, x_\infty)$ defined by $\bar{f}_i|I^{q-1} \times [a_i, a_{i+1}] = f_i$ and $\bar{f}_i|I^q \setminus (I^{q-1} \times [a_i, a_{i+1}]) \equiv x_\infty$. Obviously the map is homotopic to f_i rel. ∂I^q. This shows that $\varphi \circ i_\sharp \circ \lambda((\alpha_i)) = ((\alpha_i))$, completing the proof.

As $\varphi : \pi_q(\prod_i X_i) \to \prod_i \pi_q(X_i)$ is an isomorphism, Claim 3 implies that $i_\sharp$ is an epimorphism in each dimension and the conclusion of the theorem follows from the long exact sequence of homotopy groups of the pair $(\prod_i X_i, \widetilde{\bigvee}_i X_i)$.

COROLLARY 2.2.
$$\pi_{n+1}(\mathbb{H}_n) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^\omega \oplus \pi_{n+2}(S^n_\infty, \mathbb{H}_n) & \text{if } n \geq 3, \\ \mathbb{Z}^\omega \oplus \pi_4(S^2_\infty, \mathbb{H}_2) & \text{if } n = 2. \end{cases}$$

PROPOSITION 2.3. *The homomorphism $i_\sharp : \pi_n(\mathbb{H}_n) \to \pi_n(S^n_\infty)$ is an isomorphism.*

Proof. It suffices to prove that $i_\sharp$ is a monomorphism. Suppose that $i_\sharp(\alpha) = 0$ for $\alpha \in \pi_n(\mathbb{H}_n)$. By [9], α is represented by a map $f : (I^n, \partial I^n) \to (\mathbb{H}_n, \mathbf{o}_n)$ such that $f(I^{n-1} \times [a_i, a_{i+1}]) \subset S^n_i$ and $f(\partial(I^{n-1} \times [a_i, a_{i+1}])) = \mathbf{o}_n$ for each $i \geq 1$. The condition $i_\sharp(\alpha) = 0$ then implies that $f|I^{n-1} \times [a_i, a_{i+1}] \simeq 0$ rel. $\partial(I^{n-1} \times [a_i, a_{i+1}])$. Let $H_i : (I^{n-1} \times [a_i, a_{i+1}], \partial(I^{n-1} \times [a_i, a_{i+1}])) \times [0,1] \to (\mathbb{H}_n, \mathbf{o}_n)$ be a homotopy rel. $\partial(I^{n-1} \times [a_i, a_{i+1}])$ from $f|I^{n-1} \times [a_i, a_{i+1}]$ to the constant map $\mathbf{o}_n$. Again the fact $\operatorname{diam} S^n_i \to 0$ as $i \to \infty$ guarantees the continuity of the homotopy $H_\infty : I^n \times [0,1] \to \mathbb{H}_n$ defined by $H_\infty|I^{n-1} \times [a_i, a_{i+1}] = H_i$ $(i \geq 1)$ and $H_\infty|I^{n-1} \times \{1\} \equiv \mathbf{o}_n$. Hence f is null-homotopic rel. ∂I^n and $\alpha = 0$.

This completes the proof.

The above proposition and Theorem 2.1 imply that
$$\pi_{n+1}(S^n_\infty, \mathbb{H}_n) = \mathrm{H}_{n+1}(S^n_\infty, \mathbb{H}_n) = 0, \qquad \pi_{n+2}(S^n_\infty, \mathbb{H}_n) \cong \mathrm{H}_{n+2}(S^n_\infty, \mathbb{H}_n).$$
By [11], $\mathrm{H}_{n+1}(S^n_\infty) = 0$ for each $n \geq 2$ and so the connecting homomorphism $\partial : \mathrm{H}_{n+2}(S^n_\infty, \mathbb{H}_n) \to \mathrm{H}_{n+1}(\mathbb{H}_n)$ is an epimorphism for each $n \geq 2$. As $\mathrm{H}_3(\mathbb{H}_2)$ is nonzero ([1]), it follows that $\mathrm{H}_4(S^2_\infty, \mathbb{H}_2)$ is nonzero and $i_\sharp : \pi_3(\mathbb{H}_2) \to \pi_3(S^2_\infty)$ is not an isomorphism.

REMARK. The element $\gamma_t = \sum[\alpha_i, \beta_i] \in \pi_{t(n-1)+1}(\mathbb{H}_n)$ constructed in [1] belongs to $\operatorname{Ker} i_\sharp$.

Next we compute the $(n+1)$-st shape group $\check{\pi}_{n+1}(\mathbb{H}_n)$ via the Hilton–Milnor Theorem in the following form.

THEOREM 2.4 ([12, pp. 511–534]). *Let $S^n_j = \widetilde{\Sigma} S^{n-1}_j$ $(j = 1, \ldots, k)$ be the n-spheres $(n \geq 2)$. There exists an isomorphism*
$$\varphi_k : \bigoplus_{j=1}^{k} \pi_{n+1}(S^n_j) \oplus \prod_{r(w)\geq 2} \pi_{n+1}(\widetilde{\Sigma} w(S^{n-1}_1, \ldots, S^{n-1}_k)) \to \pi_{n+1}\Big(\bigvee_{j=1}^{k} S^n_j\Big)$$
given by the formula
$$\varphi_k((\beta_j)_{1\leq j\leq k}, (\gamma_w)_{r(w)\geq 2}) = \sum_{j=1}^{k} i^j \cdot \beta_j + \sum_{r(w)\geq 2} w(i^1, \ldots, i^k) \cdot \gamma_w,$$
where $r(w)$ denotes the weight of the basic product w of k generators, $w(S^{n-1}_1, \ldots, S^{n-1}_k)$ is the reduced join of $S^{n-1}_1, \ldots, S^{n-1}_k$ and $w(i^1, \ldots, i^k)$

is the iterated Whitehead product of the inclusions $i^j : S_j^n \to \bigvee_{j=1}^k S_j^n$ *associated with* w.

REMARK. Let w be a basic product with generators $x_1, \dots, x_k$. The space $w(S_1^{n-1}, \dots, S_k^{n-1})$ is homeomorphic to $S_1^{(n-1)a_w(1)} \wedge \dots \wedge S_k^{(n-1)a_w(k)}$ where $a_w(j)$ is the number of occurrences of x_j in the basic product w. Thus $\widetilde{\Sigma} w(S_1^{n-1}, \dots, S_k^{n-1})$ is homeomorphic to $S^{1+(n-1)\sum_{j=1}^k a_w(j)} \approx S^{1+(n-1)r(w)}$.

THEOREM 2.5.

$$\check{\pi}_{n+1}(\mathbb{H}_n) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^\omega & \text{if } n \geq 3, \\ \mathbb{Z}^\omega & \text{if } n = 2, \\ 0 & \text{if } n = 1. \end{cases}$$

Proof. Notice that $\mathbb{H}_n = \varprojlim(\bigvee_{i=1}^k S_i^n, p_k : \bigvee_{i=1}^{k+1} S_i^n \to \bigvee_{i=1}^k S_i^n)$, where p_k is the canonical retraction such that $p_k(S_{k+1}^n) = *$. So the $n = 1$ case follows directly from the definition. We divide our considerations into two cases. Let W_k be the set of all basic products of k generators $x_1, \dots, x_k$.

CASE 1: $n \geq 3$. As $S^{r(n-1)+1}$ is $r(n-1)$-connected and $r(n-1) \geq n+1$ for each $r \geq 2$,

$$\prod_{w \in W_k,\, r(w) \geq 2} \pi_{n+1}(\widetilde{\Sigma} w(S_1^{n-1}, \dots, S_k^{n-1})) = 0,$$

and the isomorphism φ_k of Theorem 2.4 is written as $\varphi_k : \bigoplus_{j=1}^k \pi_{n+1}(S_j^n) \to \pi_{n+1}(\bigvee_{j=1}^k S_j^n)$ such that $\varphi_k((\beta_j)) = \sum_{j=1}^k i^j \cdot \beta_j$ (for $k \geq 2$). Clearly, the diagram

$$\begin{tikzcd} \bigoplus_{j=1}^{k+1} \pi_{n+1}(S_j^n) \arrow[r, "\varphi_{k+1}"] \arrow[d, "\mathrm{proj}_k"'] & \pi_{n+1}(\bigvee_{j=1}^{k+1} S_j^n) \arrow[d, "(p_k)_\sharp"'] \\ \bigoplus_{j=1}^{k} \pi_{n+1}(S_j^n) \arrow[r, "\varphi_k"] & \pi_{n+1}(\bigvee_{j=1}^{k} S_j^n) \end{tikzcd}$$

is commutative where proj_k is the canonical projection of $\bigoplus_{j=1}^{k+1} \pi_{n+1}(S_j^n)$ onto $\bigoplus_{j=1}^k \pi_{n+1}(S_j^n)$. Hence $\check{\pi}_{n+1}(\mathbb{H}_n) \cong \pi_{n+1}(S^n)^\omega \cong (\mathbb{Z}/2\mathbb{Z})^\omega$.

CASE 2: $n = 2$. As $\pi_3(S^{r+1}) = 0$ for each $r \geq 3$,

$$\prod_{w \in W_k,\, r(w) \geq 3} \pi_3(\widetilde{\Sigma} w(S_1^1, \dots, S_k^1)) = \pi_3(S^{1+r(w)}) = 0,$$

and the isomorphism φ_k of Theorem 2.4 is of the form

$$\varphi_k : \bigoplus_{j=1}^k \pi_3(S_j^2) \oplus \prod_{w \in W_k,\, r(w)=2} \pi_3(\widetilde{\Sigma} w(S_1^1, \dots, S_k^1)) \to \pi_3\Big(\bigvee_{j=1}^k S_j^2\Big)$$

with $\widetilde{\Sigma} w(S^1_1, \ldots, S^1_k) \approx S^3$ and

$$\varphi_k((\beta_j)_{1\le j\le k}, (\gamma_w)_{w\in W_k, r(w)=2}) = \sum_{j=1}^{k} i^j \cdot \beta_j + \sum_{w\in W_k, r(w)=2} w(i^1, \ldots, i^k) \cdot \gamma_w.$$

By the bilinearity of the Whitehead product, we see that

$$\begin{aligned}
(p_k)_\sharp \circ \varphi_{k+1}&((\beta_j)_{1\le j\le k+1},\, (\gamma_w)_{w\in W_{k+1},\, r(w)=2}) \\
&= (p_k)_\sharp\Big(\sum_{j=1}^{k+1} i^j \cdot \beta_j + \sum_{w\in W_{k+1},\, r(w)=2} w(i^1, \ldots, i^{k+1}) \cdot \gamma_w\Big) \\
&= \sum_{j=1}^{k} i^j \cdot \beta_j + \sum_{x_{k+1} \text{ does not appear in } w} w(i^1, \ldots, i^{k+1}) \cdot \gamma_j \\
&= \sum_{j=1}^{k} i^j \cdot \beta_j + \sum_{w\in W_k,\, r(w)=2} w(i^1, \ldots, i^k) \cdot \gamma_w \\
&= \varphi_k((\beta_j)_{1\le j\le k}, (\gamma_w)_{w\in W_k,\, r(w)=2}).
\end{aligned}$$

That means that the diagram

$$\begin{tikzcd}
\bigoplus_{j=1}^{k+1} \pi_3(S^2_j) \oplus \bigoplus_{w\in W_{k+1},\, r(w)=2} \pi_3(S^3) \arrow[r, "\varphi_{k+1}"] \arrow[d, "\mathrm{proj}_k"'] & \pi_3(\bigvee_{j=1}^{k+1} S^2_j) \arrow[d, "(p_k)_\sharp"] \\
\bigoplus_{j=1}^{k} \pi_3(S^2_j) \oplus \bigoplus_{w\in W_k,\, r(w)=2} \pi_3(S^3) \arrow[r, "\varphi_k"] & \pi_3(\bigvee_{j=1}^{k} S^2_j)
\end{tikzcd}$$

is commutative and hence $\check{\pi}_3(\mathbb{H}_2) \cong (\pi_3(S^2))^\omega \oplus (\pi_3(S^3))^\omega \cong \mathbb{Z}^\omega$.

This completes the proof.

3. $\pi_{n+1}(\widehat{\mathbb{H}}_n), n \geq 3$. As stated in the introduction, $\mathrm{H}_*(\widehat{\mathbb{H}}_n)$ is easily computed, and the computation of $\pi_{n+1}(\widehat{\mathbb{H}}_n)$ depends on the result and the following theorem. For an abelian group A, let $\Gamma_{n+1}(A)$ be $A \otimes \mathbb{Z}/2\mathbb{Z}$ if $n \geq 3$, and $\Gamma(A)$, the Whitehead quadratic group, if $n = 2$.

Theorem 3.1 ([13], cf. [2, p. 36]). *Suppose that X is an $(n-1)$-connected space with $n \geq 2$. There exists a natural exact sequence*

$$\mathrm{H}_{n+2}(X) \to \Gamma_{n+1}(\mathrm{H}_n(X)) \xrightarrow{i} \pi_{n+1}(X) \xrightarrow{\theta} \mathrm{H}_{n+1}(X) \to 0$$

where θ is the Hurewicz homomorphism.

Thus, for each $n \geq 3$, $\pi_{n+1}(\widehat{\mathbb{H}}_n) \cong \mathrm{H}_1(\mathbb{H}_1) \otimes (\mathbb{Z}/2\mathbb{Z})$.

Theorem 3.2. *For each $n \geq 3$, $\pi_{n+1}(\widehat{\mathbb{H}}_n) \cong (\mathbb{Z}/2\mathbb{Z})^\omega \oplus \bigoplus_c (\mathbb{Z}/2\mathbb{Z})$.*

In the next lemma, we follow the notation of [10].

LEMMA 3.3. *Let J_p be the p-adic integers (p a prime) and A_p be the p-adic completion of the direct sum $\bigoplus_\tau J_p$ ($\tau \geq \omega$). Then*

$$A_p/qA_p \cong \begin{cases} \bigoplus_\tau (\mathbb{Z}/p\mathbb{Z}) & \text{if } q = p, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall that the p-adic completion of an abelian group B is given by the projective limit

$$\varprojlim(B/pB \leftarrow B/p^2B \leftarrow \ldots \leftarrow B/p^nB \leftarrow B/p^{n+1}B \leftarrow \ldots),$$

where the bonding maps are the canonical projections.

In what follows, we fix $\tau\,(\geq \omega)$, and the τ-fold direct sum $\bigoplus_\tau$ is abbreviated to $\bigoplus$ for simplicity. Multiplication by p, $\times p : \bigoplus J_p \to \bigoplus J_p$, induces a homomorphism $f_n : \bigoplus J_p / \bigoplus p^n J_p \to \bigoplus J_p / \bigoplus p^n J_p$ and it is easy to see that

$$\begin{aligned} \operatorname{Ker} f_n &= \bigoplus p^{n-1} J_p / \bigoplus p^n J_p, \\ \operatorname{Coker} f_n &= \Big(\bigoplus J_p / \bigoplus p^n J_p\Big) / \Big(\bigoplus pJ_p / \bigoplus p^n J_p\Big) \cong \bigoplus (J_p/pJ_p). \end{aligned}$$

Let $\varrho_n : \bigoplus J_p / \bigoplus p^{n+1} J_p \to \bigoplus J_p / \bigoplus p^n J_p$ be the canonical projection. As $\operatorname{Ker} f_{n+1} \subset \operatorname{Ker} \varrho_n$, the projection ϱ_n induces an epimorphism $\overline{\varrho}_n : \operatorname{Im} f_{n+1} \to \operatorname{Im} f_n$. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f_{n+1} \arrow[r] \arrow[d, "0"'] & \bigoplus J_p / \bigoplus p^{n+1} J_p \arrow[r, "f_{n+1}"] \arrow[d, "\varrho_n"'] & \operatorname{Im} f_{n+1} \arrow[r] \arrow[d, "\overline{\varrho}_n"'] & 0 \\ 0 \arrow[r] & \operatorname{Ker} f_n \arrow[r] & \bigoplus J_p / \bigoplus p^n J_p \arrow[r, "f_n"] & \operatorname{Im} f_n \arrow[r] & 0 \end{tikzcd}$$

Each row above is obviously exact. Taking the projective limits of the vertical sequences, we see that $f_\infty = \varprojlim f_n : A_p \to \varprojlim(\operatorname{Im} f_n, \overline{\varrho}_n)$ is an isomorphism.

Next we consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Im} f_{n+1} \arrow[r, "j_{n+1}"] \arrow[d, "\overline{\varrho}_n"'] & \bigoplus J_p / \bigoplus p^{n+1} J_p \arrow[r] \arrow[d, "\varrho_n"'] & \operatorname{Coker} f_{n+1} \arrow[r, "\cong"] \arrow[d, "\widetilde{\varrho}_n"'] & \bigoplus J_p/pJ_p \arrow[r] \arrow[d, "="'] & 0 \\ 0 \arrow[r] & \operatorname{Im} f_n \arrow[r, "j_n"] & \bigoplus J_p/p^n J_p \arrow[r] & \operatorname{Coker} f_n \arrow[r, "\cong"] & \bigoplus J_p/pJ_p \arrow[r] & 0 \end{tikzcd}$$

where j_n and j_{n+1} are inclusions. Again each row is obviously exact and we take the projective limits of the vertical sequences to obtain the following exact sequence (notice that each $\overline{\varrho}_n$ is an epimorphism, so the first derived limit $\varprojlim^1(\operatorname{Im} f_n)$ is zero):

$$0 \to \varprojlim(\operatorname{Im} f_n, \overline{\varrho}_n) \xrightarrow{\varprojlim j_n} A_p \to \varprojlim \operatorname{Coker} f_n \overset{\cong}{\to} \bigoplus J_p/pJ_p \to 0.$$

It is easy to see that the composition $(\varprojlim j_n) \circ f_\infty : A_p \to \varprojlim(\operatorname{Im} f_n) \to A_p$ coincides with multiplication by p and so

$$0 \to A_p \xrightarrow{\times p} A_p \to \bigoplus (J_p/pJ_p) \to 0$$

is exact. Thus $A_p/pA_p \cong \bigoplus(J_p/pJ_p) \cong \bigoplus(\mathbb{Z}/p\mathbb{Z})$. This completes the proof of the first conclusion.

If q is a prime distinct from p, then it is easy to see that $\times q : J_p/p^n J_p \to J_p/p^n J_p$ is an isomorphism for each n and hence $\times q : A_p \to A_p$ is an isomorphism. The second conclusion follows. This completes the proof of the lemma.

Proof of Theorem 3.2. By Theorem 3.1 and the fact that $\mathrm{H}_{n+2}(\mathbb{H}_n) = 0$, we obtain, for each $n \geq 3$, an isomorphism $\pi_{n+1}(\mathbb{H}_n) \cong \mathrm{H}_1(\mathbb{H}_1) \otimes (\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{H}_1(\mathbb{H}_1)/2\mathrm{H}_1(\mathbb{H}_1)$. By [8], $\mathrm{H}_1(\mathbb{H}_1) \cong \mathbb{Z}^\omega \oplus \bigoplus_c \mathbb{Q} \oplus \prod_{p:\,\mathrm{prime}} A_p$, where A_p is the p-adic completion of $\bigoplus_c J_p$. Therefore,

$$\begin{aligned}
\pi_{n+1}(\mathbb{H}_n) &\cong (\mathbb{Z}/2\mathbb{Z})^\omega \oplus \bigoplus\nolimits_c \mathbf{Q}/2\mathbf{Q} \oplus \prod_{p:\,\mathrm{prime}} A_p/2A_p \\
&\cong (\mathbb{Z}/2\mathbb{Z})^\omega \oplus (A_2/2A_2) \\
&\cong (\mathbb{Z}/2\mathbb{Z})^\omega \oplus \bigoplus\nolimits_c (\mathbb{Z}/2\mathbb{Z}).
\end{aligned}$$

The last two isomorphisms follow from Lemma 3.3.

4. The Whitehead quadratic group of the first singular homology groups of one-dimensional spaces. Theorem 3.1 and $\mathrm{H}_3(\widehat{\mathbb{H}}_2) = \mathrm{H}_4(\widehat{\mathbb{H}}_2) = 0$ imply that $\pi_3(\widehat{\mathbb{H}}_2) \cong \mathit{\Gamma}(\mathrm{H}_2(\widehat{\mathbb{H}}_2)) \cong \mathit{\Gamma}(\mathrm{H}_1(\mathbb{H}_1))$. The results of [4] and [7] show that

$$\pi_3(\widehat{\mathbb{H}}_2) \cong \operatorname{Ker}\Big(\bigtimes\nolimits_\sigma \mathbb{Z} \otimes \bigtimes\nolimits_\sigma \mathbb{Z} \to \bigtimes\nolimits_\sigma \mathbb{Z};\ g \otimes h \mapsto ghg^{-1}h^{-1} \Big),$$

where $\otimes$ denotes the noncommutative tensor product introduced in [4]. However, the author has not succeeded in determining the explicit structure of this group. Here we provide an exact sequence including $\mathit{\Gamma}(G^{\mathrm{ab}})$ of the abelianization G^{ab} of a locally free group G when G^{ab} is torsion free. A group is said to be *locally free* if every finitely generated subgroup is free. By [5], the fundamental group of every one-dimensional separable metric space is locally free, so the first homology groups of such spaces, being torsion free by [9], are examples of G^{ab} as above.

In order to state the result, we need some notation and facts. Let G be a group, and let $r : G \to A = G^{\mathrm{ab}}$ and $\varrho : [G,G] \to [G,G]^{\mathrm{ab}}$ be the projections to the abelianizations of G and of the commutator subgroup $[G,G]$ respectively. The abelianization $A = G^{\mathrm{ab}}$ acts on $[G,G]^{\mathrm{ab}}$ by

$$r(g) \cdot \varrho(x) = \varrho(g^{-1}xg) \quad (g \in G,\ x \in [G,G]).$$

It is easy to see that this is a well defined action which makes $[G,G]^{\mathrm{ab}}$ a $\mathbb{Z}A$-module. For a $\mathbb{Z}A$-module M, let $M_A = M/\langle \gamma\cdot x - x \mid \gamma\in\mathbb{Z}A,\ x\in M\rangle$.

Theorem 4.1. *Let G be a locally free group and let $A = G^{\mathrm{ab}}$. If A is torsion free, then there exists an exact sequence*

$$A\otimes([G,G]^{\mathrm{ab}})_A \to \mathrm{H}_1(A;[G,G]^{\mathrm{ab}}) \to \Gamma(A) \to A\otimes A \to ([G,G]^{\mathrm{ab}})_A \to 0.$$

It follows from [5] that for each one-dimensional separable metric space X, $\mathrm{H}_q(X)=0$ for each $q\ge 2$, and it is clear that the unreduced suspension ΣX satisfies $\mathrm{H}_{q+1}(\Sigma X)\cong \mathrm{H}_q(X)$ $(q\ge 1)$. These together with Theorem 3.1 and the above remark imply that $\pi_3(\Sigma X)\cong \Gamma(\mathrm{H}_1(X))$. Thus we have the following corollary.

Corollary 4.2. *For each one-dimensional separable metric space X, we have an exact sequence*

$$\begin{aligned}\mathrm{H}_1(X)\otimes([\Pi,\Pi]^{\mathrm{ab}})_{\mathrm{H}_1(X)} &\to \mathrm{H}_1(\mathrm{H}_1(X);[\Pi,\Pi]^{\mathrm{ab}}) \to \pi_3(\Sigma X)\\ &\to \mathrm{H}_1(X)\otimes \mathrm{H}_1(X) \to ([\Pi,\Pi]^{\mathrm{ab}})_{\mathrm{H}_1(X)} \to 0,\end{aligned}$$

where $\Pi = \pi_1(X)$.

Example. Let X be the figure-eight. Then Π is the free group of rank 2 and $A=\mathrm{H}_1(X)$ is the free abelian group of rank 2. Let α and β be the generators of $\mathrm{H}_1(X)$, represented by the two cycles of X. Let $L=\{(x,y)\in\mathbb{R}^2 \mid x \text{ or } y \text{ in } \mathbb{Z}\}$. Then $[\Pi,\Pi]\cong\pi_1(L)$ and hence $[\Pi,\Pi]^{\mathrm{ab}}\cong \mathrm{H}_1(L)$. The group $\mathrm{H}_1(L)$ is generated by $\{[\gamma_{m,n}] \mid m,n\in\mathbb{Z}\}$, where $\gamma_{m,n}$ denotes the loop which passes through the four vertices $(m,n),(m+1,n),(m+1,n+1)$, $(m,n+1)$ in this order. The action of $\mathrm{H}_1(X)$ on $\mathrm{H}_1(L)$ is given (upon exchanging α and β if necessary) by the formulas

$$\alpha\cdot[\gamma_{m,n}] = [\gamma_{m+1,n}],\qquad \beta\cdot[\gamma_{m,n}]=[\gamma_{m,n+1}].$$

Hence, as a $\mathbb{Z}A$-module, $\mathrm{H}_1(L)$ is isomorphic to $\mathbb{Z}A$ and generated by $[\gamma_{1,1}]$. Thus $[\Pi,\Pi]^{\mathrm{ab}}_A$ is isomorphic to $\mathbb{Z}$. Therefore $\mathrm{H}_q(A;[\Pi,\Pi]^{\mathrm{ab}})\cong \mathrm{H}_q(A;\mathbb{Z}A)$ $=0$ for each $q\ge 1$. Hence the exact sequence of Theorem 4.1 reduces to

$$(\mathbb{Z}\oplus\mathbb{Z})\otimes\mathbb{Z} \to \mathrm{H}_1(A;\mathbb{Z}A)=0 \to \Gamma(\mathbb{Z}\oplus\mathbb{Z}) \to (\mathbb{Z}\oplus\mathbb{Z})\otimes(\mathbb{Z}\oplus\mathbb{Z}) \to \mathbb{Z}\to 0.$$

Thus $\Gamma(\mathbb{Z}\oplus\mathbb{Z})$ is a free abelian group of rank 3. Actually a formula for the direct sum ([13]) shows that $\Gamma(\mathbb{Z}\oplus\mathbb{Z})\cong\Gamma(\mathbb{Z})\oplus\Gamma(\mathbb{Z})\oplus\mathbb{Z}\otimes\mathbb{Z}\cong\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$.

Proof of Theorem 4.1. Let $M(A,2)$ be the Moore space of type $(A,2)$. By [2, p. 36], $\Gamma(A)\cong\pi_3(M(A,2))\cong \mathrm{H}_4(K(A,2))$. We compute $\mathrm{H}_4(K(A,2))$ via the path fibration

$$K(A,1)\to E\simeq *\to K(A,2).$$

We examine the differentials of the Leray–Serre spectral sequence

$$E_2^{pq} = \mathrm{H}_p(K(A,2);H_q(A;\mathbb{Z}))\Rightarrow H_{p+q}(E).$$

Since $\widetilde{\mathrm{H}}_*(E) = 0$, it is easy to see that $d^4_{4,0} : E^4_{4,0} \to E^4_{0,3}$ is an isomorphism. In order to describe $E^4_{0,3}$, we start with $E^2_{0,3} = \mathrm{H}_3(A;\mathbb{Z})$. As A is torsion free, we have $\mathrm{H}_3(K(A,2);A) \cong \mathrm{H}_3(K(A,2)) \otimes A \oplus \mathrm{Tor}(\mathrm{H}_2(K(A,2)),A) = 0$ and hence $E^2_{3,1} = E^3_{3,1} = 0$. It follows that $E^3_{0,3} = \operatorname{Coker} d^2_{2,2} = E^4_{0,3} = 0$. Thus the sequence

$$A \otimes \mathrm{H}_2(A,\mathbb{Z}) = E^2_{2,2} \xrightarrow{d^2_{2,2}} E^2_{0,3} = \mathrm{H}_3(A,\mathbb{Z}) \xrightarrow{l} E^4_{0,3} \to 0$$

is exact. Now consider the sequence

$$\begin{aligned}\mathrm{H}_4(K(A,2)) = E^2_{4,0} \xrightarrow{d^2_{4,0}} E^2_{2,1} = A \otimes A \xrightarrow{d^2_{2,1}} E^2_{0,2} = \mathrm{H}_2(A;\mathbb{Z})\\ \to \operatorname{Coker} d^2_{2,1} = E^3_{0,2} = E^\infty_{0,2} = 0.\end{aligned}$$

As $\operatorname{Ker} d^2_{2,1}/\operatorname{Im} d^2_{4,0} = E^3_{2,1} = E^\infty_{2,1} = 0$, the above sequence is exact. Combining these two sequences, we have the commutative diagram (with $\mathbb{Z}$ as the coefficients):

$$\begin{tikzcd}
A \otimes \mathrm{H}_2(A) \arrow[r] & \mathrm{H}_3(A) \arrow[r] \arrow[d, "l"'] & \mathrm{H}_4(K(A,2)) \arrow[r, "d^2_{4,0}"] & A \otimes A \arrow[r] & \mathrm{H}_2(A) \arrow[r] & 0 \\
& E^4_{0,3} & E^4_{4,0} = \operatorname{Ker} d^2_{4,0} \arrow[l, "d^4_{4,0}\cong"'] \arrow[u, "i"]
\end{tikzcd}$$

(i is the inclusion) with the top row being exact. So the theorem follows from the above and the following lemma.

Lemma 4.3. *Let G be a locally free group and $A = G^{\mathrm{ab}}$. For each $n \geq 2$, we have an isomorphism $\mathrm{H}_n(A;\mathbb{Z}) \cong \mathrm{H}_{n-2}(A;[G,G]^{\mathrm{ab}})$, $\mathbb{Z}$ being regarded as a trivial A-module.*

Proof. Case 1: $n = 2$. Consider the five-term exact sequence of [3, p. 171]:

$$\mathrm{H}_2(G) \to \mathrm{H}_2(A) \to \mathrm{H}_1([G,G])_A \to \mathrm{H}_1(G) \xrightarrow{\alpha} \mathrm{H}_1(A) \to 0.$$

All coefficients are $\mathbb{Z}$ being regarded as trivial modules. It follows from a footnote in [6] that $\mathrm{H}_2(G) = 0$ and, by the definition of A, α is an isomorphism. So $\mathrm{H}_2(A) \cong \mathrm{H}_1([G,G])_A = [G,G]^{\mathrm{ab}}_A \cong \mathrm{H}_0(A;[G,G]^{\mathrm{ab}})$. This finishes the proof for the case $n = 2$.

Case 2: $n \geq 3$. We apply the Lyndon–Hochschild–Serre spectral sequence to

$$1 \to [G,G] \to G \to A \to 1.$$

Since $[G,G]$ is locally free, $\mathrm{H}_q([G,G];\mathbb{Z}) = 0$ for each $q \geq 2$ (the footnote of [6]) and hence $E^2_{p,q} = 0$ for each $q \geq 2$. The differential $d^2_{n,0} : H_n(A;\mathbb{Z}) \to E^2_{n-2,1} = \mathrm{H}_{n-2}(A;[G,G]^{\mathrm{ab}})$ has the property $\operatorname{Ker} d^2_{n,0} = E^2_{n,0} = E^2_{n,0} = 0$

and so $d^2_{n,0}$ is a monomorphism. Also $E^2_{n-2,1}/\operatorname{Im} d^2_{n,0} = E^3_{n-2,1} = E^\infty_{n-2,1} = 0$ since $n \geq 3$. Hence $d^2_{n,0}$ is an epimorphism.

This completes the proof.

REFERENCES

[1] M. G. Barratt and J. Milnor, *An example of anomalous singular theory*, Proc. Amer. Math. Soc. 13 (1962), 293–297.

[2] H.-J. Baues, *Homotopy Type and Homology*, Oxford Sci. Publ., 1996.

[3] K. S. Brown, *Cohomology of Groups*, Grad. Texts in Math. 87, Springer, 1982.

[4] R. Brown and J.-L. Loday, *Van Kampen theorems for diagrams of spaces*, Topology 26 (1987), 311–335.

[5] M. L. Curtis and M. K. Fort, *Homotopy groups of one-dimensional spaces*, Proc. Amer. Math. Soc. 8 (1957), 577–579.

[6] —, —, *Singular homology of one-dimensional spaces*, Ann. of Math. 69 (1959), 309–313.

[7] K. Eda, *Free σ-products and noncommutative slender groups*, J. Algebra 148 (1992), 243–263.

[8] K. Eda and K. Kawamura, *The singular homology of the Hawaiian earring*, J. London Math. Soc. 62 (2000), 305–310.

[9] —, —, *Homotopy and homology groups of the n-dimensional Hawaiian earring*, Fund. Math. 165 (2000), 17–28.

[10] L. Fuchs, *Infinite Abelian Groups*, Vol. 1, Academic Press, 1970.

[11] K. Kawamura, *A note on singular homology groups of infinite products of compacta*, preprint.

[12] G. W. Whitehead, *Elements of Homotopy Theory*, Grad. Texts in Math. 61, Springer, 1978.

[13] J. H. C. Whitehead, *A certain exact sequence*, Ann. of Math. 52 (1950), 51–110.

[14] J. Wu, *Combinatorial descriptions of homotopy groups of certain spaces*, Math. Proc. Cambridge Philos. Soc. 130 (2001), 489–413.

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