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SUB-LAPLACIAN WITH DRIFT IN NILPOTENT LIE GROUPS

BY

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Abstract. We consider the heat kernel ϕ_t corresponding to the left invariant sub-Laplacian with drift term in the first commutator of the Lie algebra, on a nilpotent Lie group. We improve the results obtained by G. Alexopoulos in [1], [2] proving the "exact Gaussian factor" $\exp\big(-\frac{|g|^2}{4(1+\varepsilon)t}\big)$ in the large time upper Gaussian estimate for ϕ_t. We also obtain a large time lower Gaussian estimate for ϕ_t.

1. Introduction and statement of the results. Let G be a connected nilpotent Lie group, and let $X_1, \ldots, X_m$ be left invariant fields on G (i.e. $(Xf)_g = Xf_g$, $f_g(x) = f(gx)$) which satisfy the Hörmander condition, namely they generate, together with their successive Lie brackets $[X_{i_1}, [X_{i_2}, [\ldots, X_{i_l}] \ldots]]$, the Lie algebra $\mathfrak{g}$ of G.

A left invariant distance d on G, called the *control distance*, is associated to these vector fields (cf. [10]). We write $|g| = d(e, g)$, where e is the identity element of G, and denote by $V(t)$ the Haar measure of the ball $\{g \in G : |g| < t\}$.

Every connected nilpotent Lie group has polynomial volume growth (cf. [6]), i.e. there is an integer $D \geq 0$ such that

$$C^{-1} t^D \leq V(t) \leq C t^D, \quad t > 1.$$

We call D the *dimension at infinity* of G. Note that D does not depend on the choice of the Hörmander system.

The fields $X_1, \ldots, X_m$ induce on G the sub-Laplacian (with drift term)

$$L = -\sum_{i=1}^{m} X_i^2 + X_0, \quad X_0 \in \mathfrak{g}. \tag{1.1}$$

The operator L generates a diffusion semigroup e^{-tL} (cf. [3], [7]). We denote by ϕ_t the kernel of e^{-tL} with respect to the Haar measure on G, i.e.

$$T_t f(x) = \int_G \phi_t(y^{-1}x) f(y)\, dy, \quad t > 0,\ x \in G,\ f \in C_0^\infty(G).$$

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The behavior of ϕ_t for small time has been studied by N. Varopoulos in [14] and [15]. He proved that if the drift is of the special form

$$X_0 = \sum_{i=1}^{m} a_i X_i + \sum_{i,j=1}^{m} b_{i,j}[X_i, X_j] \tag{1.2}$$

then the kernel ϕ_t satisfies the following upper and lower estimate: for every $0 < \varepsilon < 1$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} C_\varepsilon^{-1} V(\sqrt{t})^{-1} \exp\left(-\frac{|g|^2}{4(1-\varepsilon)t}\right) &\le \phi_t(g) \\ &\le C_\varepsilon V(\sqrt{t})^{-1} \exp\left(-\frac{|g|^2}{4(1+\varepsilon)t}\right), \qquad g \in G,\ 0 < t < 1. \end{aligned} \tag{1.3}$$

For large time it is known by Alexopoulos's work [1], [2] that if $X_0 \in [\mathfrak{g}, \mathfrak{g}] + \mathfrak{k}$, where $[\mathfrak{g}, \mathfrak{g}]$ is the first commutator of the Lie algebra $\mathfrak{g}$ and $\mathfrak{k}$ is the Lie algebra of the maximal compact subgroup K, contained in the centre of G, so that G/K is a simply connected nilpotent Lie group (cf. [11, pp. 195–200]), then the kernel ϕ_t satisfies the upper Gaussian estimate

$$\phi_t(g) \le C V(\sqrt{t})^{-1} \exp\left(-\frac{|g|^2}{ct}\right), \qquad t > 1,\ g \in G. \tag{1.4}$$

This result has been proved by Alexopoulos in the more general context of Lie groups of polynomial volume growth. For an easier proof of (1.4) in the setting of nilpotent Lie groups, see [9].

When the drift has the special form (1.2) the lower estimate is an easy consequence of the upper estimate and Harnack inequality (cf. [16, pp. 47–50]). This method does not work, as far as we can see, if $X_0 \in [\mathfrak{g}, \mathfrak{g}]$ because the Harnack inequality holds just for large time (cf. Sect. 3).

In this paper we improve the above upper Gaussian estimate, obtaining the "exact Gaussian factor" $c = 4(1+\varepsilon)$ in $\exp\left(-\frac{|g|^2}{ct}\right)$, and we prove the lower Gaussian estimate:

THEOREM 1.1. *If $X_0 \in [\mathfrak{g}, \mathfrak{g}] + \mathfrak{k}$, then there exist $C, c > 0$ and for every $0 < \varepsilon < 1$ there exists $C_\varepsilon > 0$ such that*

$$\begin{aligned} C^{-1} V(\sqrt{t})^{-1} \exp\left(-\frac{|g|^2}{ct}\right) &\le \phi_t(g) \\ &\le C_\varepsilon V(\sqrt{t})^{-1} \exp\left(-\frac{|g|^2}{4(1+\varepsilon)t}\right), \qquad g \in G,\ t > 1. \end{aligned} \tag{1.5}$$

For every sub-Laplacian (1.1) there exists a multiplicative character $\chi : G \to \mathbb{R}^+$ and a constant $a \ge 0$ such that

$$\chi^{-1} L(\chi\cdot) = -\sum_{i=1}^{m} X_i^2 + Y + a,$$

where $Y \in [\mathfrak{g}, \mathfrak{g}] + \mathfrak{k}$ (for the construction of χ see e.g. [2]). Therefore the study of the behaviour of the kernel corresponding to (1.1) can be reduced to the case of drift in $[\mathfrak{g}, \mathfrak{g}] + \mathfrak{k}$. By Theorem 1.1, we have

THEOREM 1.2. *There exist* $C, c > 0$ *and for every* $0 < \varepsilon < 1$ *there exists* $C_\varepsilon > 0$ *such that*

$$C^{-1} V(\sqrt{t})^{-1} e^{-at} \chi(g) \exp\left(-\frac{|g|^2}{ct}\right) \leq \phi_t(g)$$
$$\leq C_\varepsilon V(\sqrt{t})^{-1} e^{-at} \chi(g) \exp\left(-\frac{|g|^2}{4(1+\varepsilon)t}\right), \qquad g \in G,\ t > 1.$$

Throughout this paper positive constants are denoted by c or C. These may differ from one line to another.

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2. Nilpotent Lie groups and stratified groups. The key observation in this paper is that any connected nilpotent Lie group is "covered" by a nilpotent Lie group that admits a dilation structure. The strategy of the proofs is to obtain all the results first in this special setting, and then to prove that we can "transfer" them to a general nilpotent group.

In this section we recall the basic properties of stratified groups and the link between stratified and nilpotent groups.

2.1. *Stratified groups and dilation structure.* A *stratified group* is a simply connected nilpotent group $\widetilde{G}$ whose Lie algebra $\widetilde{\mathfrak{g}}$ admits a direct sum decomposition

$$\widetilde{\mathfrak{g}} = V_1 \oplus \ldots \oplus V_r,$$

where V_i are vector subspaces of $\widetilde{\mathfrak{g}}$ such that $[V_1, V_{i-1}] = V_i$, $i \geq 2$. We say that V_i is the ith *slice* of the stratification of $\widetilde{\mathfrak{g}}$.

A stratified group $\widetilde{G}$ admits a one-parameter semigroup (see e.g. [5]) of homomorphisms (dilations) $\delta_t : \widetilde{G} \to \widetilde{G}$ such that

$$d\delta_t(\widetilde{X}) = t^i \widetilde{X}, \qquad \widetilde{X} \in V_i,\ t > 0,\ i = 1, \ldots, r.$$

We can fix on $\widetilde{G}$ a family $\widetilde{\mathbf{X}} = \{\widetilde{X}_1, \ldots, \widetilde{X}_m\}$ of left invariant vector fields satisfying the Hörmander condition and such that $\widetilde{\mathbf{X}} \subseteq V_1$. Then

$$|\delta_t(\widetilde{g})|_{\widetilde{G}} = t|\widetilde{g}|_{\widetilde{G}}, \qquad t > 0,\ \widetilde{g} \in \widetilde{G},$$

where $|\cdot|_{\widetilde{G}}$ is the distance on $\widetilde{G}$ induced by $\widetilde{\mathbf{X}}$.

Moreover since the Haar measure m on $\widetilde{G}$ is the image of the Lebesgue measure $\widetilde{\mathfrak{g}}$ under the exponential map $\exp : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ (cf. [11, pp. 195–200]) we know that for every measurable subset Ω of G,

$$m(\delta_t(\Omega)) = t^D m(\Omega), \qquad t > 0,$$

where $D = \sum_{i=1}^r i \dim V_i$. In particular $V(t) = Ct^D,\ t > 0$.

2.2. *Stratified groups and nilpotent groups.* Let G be a connected nilpotent Lie group and let $\mathbf{X} = \{X_1, \ldots, X_m\}$ be a family of left invariant vector fields on G satisfying the Hörmander condition. Then (cf. [13], and also [16, Chapter IV]) there is a stratified group $\widetilde{G}$, a family $\widetilde{\mathbf{X}} = \{\widetilde{X}_1, \ldots, \widetilde{X}_m\} \subseteq V_1$ (the first slice of the stratification) of fields on $\widetilde{G}$ satisfying the Hörmander condition and a surjective homomorphism

$$\pi : \widetilde{G} \to G \tag{2.1}$$

such that

$$d\pi(\widetilde{X}_i) = X_i, \qquad i = 1, \ldots, m.$$

Denote by $|\cdot|_{\widetilde{G}}$, $|\cdot|_G$ the distances on $\widetilde{G}$ and G induced by the Hörmander systems $\widetilde{\mathbf{X}}$ and $\mathbf{X}$, respectively. We have (see e.g. [5])

$$|x|_G = \inf_{\widetilde{x} \in \pi^{-1}(x)} |\widetilde{x}|_{\widetilde{G}}, \qquad x \in G. \tag{2.2}$$

3. The Harnack inequality. Let us recall the Harnack inequality for a sub-Laplacian (1.1) with drift in $[\mathfrak{g}, \mathfrak{g}]$, which will play a central role in this article:

THEOREM 3.1. *Let $0 < a < b < 1$ and $0 < \delta < 1$. Then there exists a constant $C > 0$ such that for all $g \in G$ and $R > 1$ and for every positive solution u of*

$$\left(\frac{\partial}{\partial t} + L\right) u = 0$$

in $(0, R) \times B(g, \sqrt{R})$, we have

$$\sup_{x \in B(g, \delta\sqrt{R})} u(aR, x) \le C \inf_{x \in B(g, \delta\sqrt{R})} u(bR, x). \tag{3.1}$$

This is a special case of Alexopoulos's Harnack inequality [2, Theorem 1.2.1], which holds in the more general context of Lie groups of polynomial volume growth. In the setting of nilpotent Lie groups the machinery of Alexopoulos is not needed and the proof is very easy (cf. [9]).

4. Probability estimates. In this section we recall an upper and a lower probability estimate obtained by Varopoulos in [15, Sect. 5] adapting the Varadhan–Vencel–Friedlin theory (cf. [12, Sect. 6]) to a family of subelliptic operators.

These estimates have been used by Varopoulos to prove the "exact Gaussian factor" $\exp\bigl(-\frac{|g|^2}{4(1-\varepsilon)t}\bigr)$ in the small time lower estimate (1.3) exploiting the stratified group dilation structure and the fact that drifts of special form (1.2) are "good" for local dilation. Note that if the drift is of the form $X_0 = \sum_{i,j=1}^m b_{i,j}[X_i, X_j]$ then the estimates (1.3) hold for $t > 0$ (in this way the Laplacian is "good" for global dilation).

Performing the local dilation procedure of [15, Sect. 6] (cf. also [16, Ch. V, Sect. 5]), one can show that the estimates (1.3) hold in more general contexts than that of nilpotent Lie groups.

In this paper we shall use these estimates to prove Theorem 1 using the fact that drifts in $[\mathfrak{g}, \mathfrak{g}]$ are "good" for large dilation.

Consider the family of subelliptic operators

$$L^s = -\sum_{j=1}^m X_j^2 + X_0^s, \tag{4.1}$$

where $(X_0^s)_{s\in S}$ is a family of vector fields which depend on a parameter $s \in S \subset \mathbb{R}$, and are bounded in the C^∞-topology. This is a special case of the families considered in [15, Sect. 5.6] (this is all we need). Here, only the drift depends on s.

We denote by $\mathcal{P}^s = \{\mathcal{D}(G), z_t, P_x^s\}$ the diffusions generated by L^s (cf. [3]). Recall that

$$P_x^s[z_t \in dy] = \phi_t^s(y^{-1}x)\,dy,$$

where ϕ_t^s are the kernels of the semigroups e^{-tL^s}. We have the following probability estimates:

(i) for every $0 < \varepsilon < 1$ there exist $C_\varepsilon, c_\varepsilon > 0$ such that

$$P_e^s[\sup_{0\le u\le t} |z_u| \ge 1] \le C_\varepsilon \exp\left(-\frac{1}{4(1+\varepsilon)t}\right), \qquad 0 < t < c_\varepsilon,\ s \in S, \tag{4.2}$$

(ii) for $0 < \varrho < 10^{-10}$ fixed there exist $C, c > 0$ such that

$$P_x^s[d(z_t, y) < \varrho] \ge C^{-1} \exp\left(-\frac{1}{ct}\right), \qquad 0 < t < 1,\ s \in S, \tag{4.3}$$

for all $x, y \in G$ with $10^{-10} \le d(x, y) \le 10^{10}$.

5. Proof of Theorem 1.1

5.1. *The upper estimate.* Since $X_0 \in [\mathfrak{g}, \mathfrak{g}] + \mathfrak{k}$ we can write

$$X_0 = X_{0,1} + X_{0,2},$$

where $X_{0,1} \in [\mathfrak{g}, \mathfrak{g}]$ and $X_{0,2} \in \mathfrak{k}$. Let ϕ_t^1 be the kernel of the semigroup e^{-tL^1} generated by $L^1 = -\sum_{i=1}^m X_i^2 + X_{0,1}$. Since $\mathfrak{k}$ is contained in the centre of the Lie algebra $\mathfrak{g}$, we have $e^{-tL} = e^{-tL^1} e^{-tX_{0,2}}$, and therefore

$$\phi_t(g) = \phi_t^1(ge^{tX_{0,2}}), \qquad g \in G,\ t > 0.$$

Moreover, since K is compact, $ge^{tX_{0,2}} \in B(g, c\sqrt{t})$ for $t > 1$ and c large enough. The Harnack inequality (Theorem 3.1 with $a = 1/4c^2$, $b = \frac{1+2\varepsilon}{1+\varepsilon} a$ and $\delta = 1/2$) applied to the kernel ϕ_t^1 on the ball $B(g, c\sqrt{t})$ yields

$$\phi_t^1(ge^{tX_{0,2}}) \le C_\varepsilon \phi^1_{\frac{1+2\varepsilon}{1+\varepsilon}t}(g), \qquad t > 1,\ g \in G,$$

whence

$$\phi_t(g) \le C_\varepsilon \phi^1_{\frac{1+2\varepsilon}{1+\varepsilon}t}(g), \qquad t > 1,\ g \in G.$$

Then it suffices to prove the right hand inequality in (1.5) for the kernel ϕ_t^1.

If $|g| < 6\sqrt{t}$, this reduces to the estimate

$$\phi_t^1(g) \le CV(\sqrt{t})^{-1}, \qquad |g| < 6\sqrt{t},\ t > 1,$$

which can be easily obtained by applying the Harnack inequality to the kernel ϕ_t^1 on the ball $B(e, 6\sqrt{t})$ in the following way:

$$\begin{aligned} V(6\sqrt{t})\phi_t^1(g) &= \int\limits_{B(e,6\sqrt{t})} \phi_t^1(g)\,dx \le C \int\limits_{B(e,6\sqrt{t})} \phi_{2t}^1(x)\,dx \\ &\le C \int\limits_G \phi_{2t}^1(x)\,dx = C, \qquad t > 1,\ g \in B(e, 6\sqrt{t}). \end{aligned}$$

The proof is therefore reduced to proving the estimate

$$\phi_t^1(g) \le C_\varepsilon V(\sqrt{t})^{-1} \exp\left(-\frac{|g|^2}{4(1+\varepsilon)t}\right), \qquad |g| > 6\sqrt{t},\ t > 1. \tag{5.1}$$

To prove (5.1) we first suppose that G is stratified and $\{X_1, \ldots, X_m\} \subset V_1$, the first slice of the stratification of G. The second step is to show that, for a general nilpotent Lie group, the estimate (5.1) "goes through" the homomorphism (2.1).

The case of stratified groups. Since $X_{0,1} \in [\mathfrak{g}, \mathfrak{g}]$, we can write

$$X_{0,1} = Y_2 + \ldots + Y_r$$

with $Y_i \in V_i$ (the ith slice of the stratification of G) for $i = 2, \ldots, r$ (the rank of G). Let $(X_0^s)_{s>1}$ be the family of vector fields

$$X_0^s = Y_2 + \frac{1}{\sqrt{s}} Y_3 + \ldots + \left(\frac{1}{\sqrt{s}}\right)^{r-2} Y_r$$

and let L^s be the family of sub-Laplacians

$$L^s = -\sum_{i=1}^m X_i^2 + X_0^s, \qquad s > 1. \tag{5.2}$$

Note that $d\delta_{\sqrt{s}}(L^s) = sL^1$.

We denote by ϕ_t^s the kernels of the semigroups e^{-tL^s}. We have the dilation formula

$$\phi^1_{st}(\delta_{\sqrt{s}}(x)) = s^{-D/2}\phi^s_t(x), \quad x \in G,\ t > 0,\ s > 1, \tag{5.3}$$

where D is the dimension at infinity of G. Indeed, to see (5.3), it suffices to verify that the semigroups T^s_t defined by convolution with the kernels $s^{D/2}\phi^1_{st}(\delta_{\sqrt{s}}(g))$ satisfy $T^s_t f(x) = T_{st}(f \circ \delta_{1/\sqrt{s}})(\delta_{\sqrt{s}}(x))$. Then T^s_t are the semigroups generated by L^s, i.e. $T^s_t = e^{-tL^s}$.

Let $\mathcal{P}^1 = \{\mathcal{D}(G), z_t, P^1_x\}$ and $\mathcal{P}^s = \{\mathcal{D}(G), z_t, P^s_x\}$ be the diffusions generated by L^1 and L^s respectively (cf. [3]). Because of the dilation formula (5.3) we have

$$\begin{aligned} P^s_e[z_t \in dy] &= \phi^s_t(y^{-1})dy = s^{D/2}\phi^1_{st}(\delta_{\sqrt{s}}(y)^{-1})dy \\ &= \phi^1_{st}(\delta_{\sqrt{s}}(y)^{-1})d(\delta_{\sqrt{s}}(y)) = P^1_e[z_{st} \in d(\delta_{\sqrt{s}}(y))]. \end{aligned}$$

Therefore

$$P^1_e[\sup_{0\le u\le ts} |z_u| \ge \sqrt{s}] = P^s_e[\sup_{0\le u\le t} |z_u| \ge 1].$$

Let $0 < \varepsilon < 1$. By the estimate (4.2) there exist $C_\varepsilon, c_\varepsilon > 0$ such that

$$P^1_e[\sup_{0\le u\le ts} |z_u| \ge \sqrt{s}] \le C_\varepsilon \exp\left(-\frac{1}{4(1+\varepsilon)t}\right), \quad 0 < t < c_\varepsilon,\ s > 1,$$

and changing ts to t we have

$$P^1_e[\sup_{0\le u\le t} |z_u| \ge \sqrt{s}] \le C_\varepsilon \exp\left(-\frac{s}{4(1+\varepsilon)t}\right), \quad 0 < t < c_\varepsilon s,\ s > 1. \tag{5.4}$$

Observe that if $t > c_\varepsilon s$ the above estimate holds for C_ε large enough.

Fix $g \in G$ with $|g| > 6\sqrt{t}$, and choose $\sqrt{s} = |g| - \varepsilon\sqrt{t}$. By (5.4) we have

$$\begin{aligned} \int_{B(g,\varepsilon\sqrt{t})} \phi^1_t(x)\,dx &\le P^1_e[\sup_{0\le u\le t} |z_u| \ge \sqrt{s}] \le C_\varepsilon \exp\left(-\frac{s}{4(1+\varepsilon)t}\right) \\ &\le C_\varepsilon \exp\left(-\frac{|g|^2}{4(1+\varepsilon)t} + \frac{\varepsilon}{2(1+\varepsilon)}\,\frac{|g|}{\sqrt{t}}\right) \\ &\le C_\varepsilon \exp\left(-\frac{|g|^2}{4(1+2\varepsilon)t}\right), \quad t > 1. \end{aligned}$$

Applying the Harnack inequality to the kernel ϕ_t on the ball $B(g, \varepsilon\sqrt{t})$ we deduce that

$$C_\varepsilon^{-1} V(\sqrt{t})\phi^1_{\frac{1+2\varepsilon}{1+4\varepsilon}t}(g) \le \int_{B(g,\sqrt{t})} \phi^1_t(x)\,dx \le C_\varepsilon \exp\left(-\frac{|g|^2}{4(1+2\varepsilon)t}\right), \quad t > 1.$$

General nilpotent Lie groups. Let $\widetilde{G}$ be the stratified group, $\widetilde{\mathbf{X}} = \{\widetilde{X}_1, \dots \widetilde{X}_m\} \subseteq V_1$ the family of fields satisfying the Hörmander condition, and $\pi : \widetilde{G} \to G$ the homomorphism defined in Section 2.2. By the surjectivity of π there exists a vector field $\widetilde{X}_0 \in [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] \subset \widetilde{\mathfrak{g}} = \mathrm{Lie}(\widetilde{G})$ such that $d\pi(\widetilde{X}_0) = X_{0,1}$. Let $\widetilde{L}$ be the sub-Laplacian on $\widetilde{G}$ given by

$$\widetilde{L} = \sum_{i=1}^{m} \widetilde{X}_i^2 + \widetilde{X}_0.$$

Note that $d\pi(\widetilde{L}) = L^1$.

Denote by $\widetilde{\phi}_t$ the kernel of the semigroup $e^{-t\widetilde{L}}$ and by $\widetilde{\mathcal{P}} = \{\mathcal{D}(\widetilde{G}), \widetilde{z}_t, \widetilde{P}_x\}$ the corresponding diffusion. The kernels ϕ_t^1 and $\widetilde{\phi}_t$ are related by the formula

$$\phi_t^1(g) = \int_H \widetilde{\phi}_t(\widetilde{g}h)\, dh, \qquad g \in G,\ \widetilde{g} \in \widetilde{G},\ \pi(\widetilde{g}) = g, \tag{5.5}$$

where $H = \ker \pi$. Indeed, to see (5.5), it suffices to verify that the semigroup $T_t f$, defined by convolution with the kernel $\int_H \widetilde{\phi}_t(\widetilde{g}h) dh$, satisfies $T_t f(x) = e^{-t\widetilde{L}}(f \circ \pi)(\widetilde{x})$, $\pi(\widetilde{x}) = x$. Then T_t is the semigroup generated by L^1, i.e. $T_t = e^{-tL^1}$.

Estimate (5.4) gives

$$\begin{aligned} \int_{\{\widetilde{g}\,:\,|\widetilde{g}| \geq \sqrt{s}\}} \widetilde{\phi}_t(\widetilde{g})\, d\widetilde{g} &\leq \widetilde{P}_e[\sup_{0 \leq u \leq t} |\widetilde{z}_u| \geq \sqrt{s}] \\ &\leq C_\varepsilon \exp\left(-\frac{s}{4(1+\varepsilon)t}\right), \qquad s > 1,\ t > 0. \end{aligned} \tag{5.6}$$

Recalling that $|\widetilde{g}|_{\widetilde{G}} \geq |\pi(\widetilde{g})|$ for all $\widetilde{g} \in \widetilde{G}$ (cf. Section 2.2), using the Haar measure disintegration formula (cf. [4])

$$\int_{\widetilde{G}} f(\widetilde{x})\, d\widetilde{x} = \int_G \int_H f(\widetilde{x}h)\, dh\, dx, \qquad \pi(\widetilde{x}) = x, \tag{5.7}$$

and the formula (5.5), we obtain

$$\begin{aligned} \int_{\{\widetilde{g}\,:\,|\widetilde{g}|_{\widetilde{G}} \geq \sqrt{s}\}} \widetilde{\phi}_t(\widetilde{g})\, d\widetilde{g} \geq \int_{\{\widetilde{g}\,:\,|\pi(\widetilde{g})| \geq \sqrt{s}\}} \widetilde{\phi}_t(\widetilde{g})\, d\widetilde{g} &\geq \int_{\{|g| \geq \sqrt{s}\}} \int_H \widetilde{\phi}_t(\widetilde{g}h)\, dh\, dg \\ &\geq \int_{\{g\,:\,|g| \geq \sqrt{s}\}} \phi_t^1(g)\, dg. \end{aligned}$$

Then by (5.6) we have the estimate

$$\int_{\{g\,:\,|g| \geq \sqrt{s}\}} \phi_t^1(g)\, dg \leq C_\varepsilon \exp\left(-\frac{s}{4(1+\varepsilon)t}\right), \qquad s > 1,\ t > 0,$$

and applying the Harnack inequality to the kernel ϕ_t^1 on the ball $B(g, \varepsilon\sqrt{t})$ as at the end of the proof in the stratified case yields the upper estimate of (1.5) for $|g| > 6\sqrt{t}$.

5.2. *The lower estimate.* Let us first remark that, arguing as in the proof of the upper estimate, using the Harnack inequality we can reduce the proof to the case of drift in $[\mathfrak{g}, \mathfrak{g}]$.

We prove the lower estimate when the drift is in $[\mathfrak{g}, \mathfrak{g}]$ in two steps. We first suppose that G is stratified and $\mathbf{X} \subset V_1$. Then, in the case of a general nilpotent Lie group, we prove that the lower estimate of (1.5) "goes through" the homomorphism (2.1).

The case of stratified groups. We begin by proving the following lemma that is an easy consequence of the upper Gaussian estimate (1.4):

LEMMA 5.1. *There exists $c > 0$ such that*

$$\int\limits_{B(e,c\sqrt{t})} \phi_t(g)\, dg \geq \frac{1}{2}, \qquad t > 1.$$

Proof. First observe that, because of the left invariance of the semigroup e^{-tL}, the total mass of ϕ_t is 1, i.e.

$$\int\limits_G \phi_t(g)\, dg = 1.$$

Indeed, by the left invariance of e^{-tL} we have $T_t\mathbf{1}(e) = T_t\mathbf{1}(g)$ for all $g \in G$, and therefore

$$\frac{\partial}{\partial t} T_t \mathbf{1} = -L\mathbf{1} = 0.$$

Since $(T_t\mathbf{1})_{t=0} = \mathbf{1}$ we have $T_t\mathbf{1} = \mathbf{1}$ for all $t > 0$, and this yields

$$T_t\mathbf{1}(e) = \int\limits_G \phi_t(g^{-1})\, dg = \int\limits_G \phi_t(g)\, dg = 1.$$

Then

$$\int\limits_{B(e,c\sqrt{t})} \phi_t(g)\, dg = 1 - \int\limits_{\{|g| \geq c\sqrt{t}\}} \phi_t(g)\, dg.$$

and by the upper Gaussian estimate (1.4) and the dilation structure

$$\begin{aligned} \int\limits_{\{|g| \geq c\sqrt{t}\}} \phi_t(g)\, dg &\leq Ct^{-D/2} \int\limits_{\{|g| \geq c\sqrt{t}\}} \exp\left(-\frac{|g|^2}{at}\right) dg \\ &\leq C \int\limits_{\{|x| \geq c\}} \exp\left(-\frac{|x|^2}{a}\right) dx \leq \frac{1}{2}, \qquad t > 1, \end{aligned}$$

for c large enough. Therefore

$$\int\limits_{B(e,c\sqrt{t})} \phi_t(g) dg \geq \frac{1}{2}, \qquad t > 1. \ \blacksquare$$

If $|g| < c\sqrt{t}$ the lower estimate in (1.5) reduces to the estimate

$$\phi_t(g) \geq C^{-1} V(\sqrt{t})^{-1}, \quad |g| < c\sqrt{t},\ t > 1, \tag{5.8}$$

which we can easily obtain by the previous lemma and the Harnack inequality (3.1) applied to ϕ_t on the ball $B(e, c\sqrt{t})$:

$$\frac{1}{2} \leq \int_{B(e,c\sqrt{t})} \phi_t(x)\, dx \leq CV(c\sqrt{t})\phi_{2t}(g), \quad |g| < c\sqrt{t}.$$

It suffices to show that

$$\phi_t(g) \geq C^{-1} V(\sqrt{t})^{-1} \exp\left(-\frac{|g|^2}{ct}\right), \quad |g| > c\sqrt{t},\ t > 1. \tag{5.9}$$

To this end, we first apply the dilation formula (5.3) with $t = 1$ to get

$$\phi_s(g) = \phi_s(\delta_{\sqrt{s}}(x)) = s^{-D/2}\phi_1^s(x),$$

where ϕ_t^s is the kernel of the semigroup e^{-tL^s} generated by the sub-Laplacian (5.2). Then the lower Gaussian estimate (5.9) is equivalent to the uniform (with respect to $s > 1$) lower Gaussian estimates for the kernels ϕ_t^s, at time $t = 1$:

$$\phi_1^s(g) \geq C^{-1} \exp\left(-\frac{|g|^2}{c}\right), \quad |g| > 1,\ s > 1. \tag{5.10}$$

To prove (5.10), fix $g \in G$ with $|g| > 1$, and let N be the integer such that $N \leq |g| \leq N + 1$. We can find a sequence $y_0 = e, y_1, \ldots, y_N = g$ such that $1 \leq d(y_i, y_{i+1}) \leq 3$. Fix $0 < \varrho < 10^{-10}$, and denote by B_i the ball of radius ϱ centred at y_i, and by $\mathcal{P}^s = \{\mathcal{D}(G), z_t, P_x^s\}$ the diffusions generated by the sub-Laplacians L^s, $s > 1$.

By the Markovian property of $\mathcal{P}^s$ and by the estimates (4.3) we have

$$\begin{aligned} P_e^s[z_1 \in B(g,\varrho)] &\geq P_e^s[z_{1/N} \in B_1] P_{x_1}^s[z_{1/N} \in B_2] \ldots P_{x_{N-1}}^s[z_{1/N} \in B(g,\varrho)] \\ &\geq C^{-N} \exp\left(-\frac{N^2}{c}\right) \geq C^{-1} \exp\left(-\frac{2N^2}{c}\right), \end{aligned}$$

where $x_i \in B_i$, $i = 1, \ldots, N-1$. And since $N \leq |g| \leq N+1$, we have

$$\int_{B(g,\varrho)} \phi_1^s(x)\, dx = P_e^s[z_1 \in B(g,\varrho)] \geq C^{-1} \exp\left(-\frac{|g|^2}{c}\right), \quad s > 1.$$

The uniform Harnack principle (cf. [16, Theorem III.2.4]) applied to ϕ_t^s on the ball $B(g, \varrho)$ gives

$$\phi_1^s(g) \geq C^{-1} \exp\left(-\frac{|g|^2}{c}\right), \quad s > 1$$

(the constant C does not depend on g because of the left invariance of L and of the distance $|\cdot|$).

General nilpotent Lie groups. Let $\widetilde{G}$ be the stratified group, $\widetilde{\mathbf{X}} = \{\widetilde{X}_1, \dots, \widetilde{X}_m\} \subseteq V_1$ the family of fields satisfying the Hörmander condition, and $\pi : \widetilde{G} \to G$ the homomorphism defined in Section 2.2.

By the surjectivity of π there exists a vector field $\widetilde{X}_0 \in [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] \subset \widetilde{\mathfrak{g}} = \mathrm{Lie}(\widetilde{G})$ such that $d\pi(\widetilde{X}_0) = X_0$. Let $\widetilde{L}$ be the sub-Laplacian on $\widetilde{G}$ given by

$$\widetilde{L} = \sum_{i=1}^{m} \widetilde{X}_i^2 + \widetilde{X}_0.$$

We denote by $\widetilde{\phi}_t$ the kernel of the semigroup $e^{-t\widetilde{L}}$. Recall that the kernels ϕ_t and $\widetilde{\phi}_t$ are related by the formula

$$\phi_t(g) = \int_H \widetilde{\phi}_t(\widetilde{g}h)\, dh, \qquad g \in G,\ \widetilde{g} \in \widetilde{G},\ \pi(\widetilde{g}) = g,$$

where $H = \ker \pi$.

If $g \in G$ and $t > 1$, we have

$$\begin{aligned}\phi_t(g) &= \int_H \widetilde{\phi}_t(\widetilde{g}h)\, dh \\ &\geq \int_{\{h \in H\,:\, |h|_{\widetilde{G}} < \sqrt{t}\}} \widetilde{\phi}_t(\widetilde{g}h)\, dh, \qquad g \in G,\ \widetilde{g} \in \widetilde{G},\ \pi(\widetilde{g}) = g.\end{aligned}$$

The Harnack inequality (3.1) applied to the kernel $\widetilde{\phi}_t$ on the ball $B_{\widetilde{G}}(\widetilde{g}, \sqrt{t})$ gives

$$\phi_t(g) \geq C^{-1} m_H\{h \in H : |h|_{\widetilde{G}} < \sqrt{t}\}\widetilde{\phi}_{t/2}(\widetilde{g}), \qquad g \in G,\ \widetilde{g} \in \widetilde{G},\ \pi(\widetilde{g}) = g,\ t > 1, \tag{5.11}$$

where m_H is the Haar measure of H.

By the results obtained in the stratified case we have

$$\widetilde{\phi}_t(\widetilde{g}) \geq C^{-1} \widetilde{V}(\sqrt{t})^{-1} \exp\left(-\frac{|\widetilde{g}|_{\widetilde{G}}^2}{ct}\right), \qquad \widetilde{g} \in \widetilde{G},\ t > 1,$$

where $\widetilde{V}(R)$ denotes the Haar measure of the ball $\{\widetilde{g} \in \widetilde{G} : |\widetilde{g}|_{\widetilde{G}} < R\}$. Therefore by (5.11),

$$\phi_t(g) \geq C^{-1} m_H\{h \in H : |h|_{\widetilde{G}} < \sqrt{t}\}\widetilde{V}(\sqrt{t})^{-1} \exp\left(-\frac{|\widetilde{g}|_{\widetilde{G}}^2}{ct}\right), \qquad g \in G,\ \widetilde{g} \in \widetilde{G},\ \pi(\widetilde{g}) = g,\ t > 1,$$

and by (2.2),

$$\phi_t(g) \geq C^{-1} m_H\{h \in H : |h|_{\widetilde{G}} < \sqrt{t}\}\widetilde{V}(\sqrt{t})^{-1} \exp\left(-\frac{|g|^2}{ct}\right), \qquad g \in G,\ t > 1.$$

To complete the proof we only need to verify the volume estimate

$$m_H\{h \in H : |h|_{\widetilde{G}} < \sqrt{t}\}\widetilde{V}(\sqrt{t})^{-1} \geq C^{-1}V(\sqrt{t})^{-1}, \qquad t > 1.$$

Let $t > 1$ be given. For any $\widetilde{g} \in \widetilde{G}$ we can choose $\widetilde{g}_0 \in \pi^{-1}(\pi(\widetilde{g}))$ such that $|\widetilde{g}_0|_{\widetilde{G}} < |\pi(\widetilde{g})| + \sqrt{t}$. We set $h_0 = \widetilde{g}_0^{-1}\widetilde{g} \in H$. If $\widetilde{g} \in B_{\widetilde{G}}(\widetilde{e}, \sqrt{t})$ then $|\widetilde{g}_0|_{\widetilde{G}} < |\widetilde{g}|_{\widetilde{G}} + \sqrt{t} < 2\sqrt{t}$ and $|h_0|_N < 3\sqrt{t}$. Therefore

$$B_{\widetilde{G}}(\widetilde{e}, \sqrt{t}) \subseteq \bigcup_{\{\widetilde{g}_0 \,:\, |\widetilde{g}_0|_{\widetilde{G}} < 2\sqrt{t}\}} \{\widetilde{g}_0 h : h \in H,\ |h|_{\widetilde{G}} < 3\sqrt{t}\}.$$

The Haar disintegration formula (5.7) gives

$$\begin{aligned} V_{\widetilde{G}}(\sqrt{t}) &= \int_{B_{\widetilde{G}}(\widetilde{e},\sqrt{t})} d\widetilde{g} \leq \int_{\{g \,:\, |g| < 2\sqrt{t}\}} \int_{\{h \,:\, |h|_{\widetilde{G}} < 3\sqrt{t}\}} dg\, dh \\ &\leq CV(\sqrt{t}) m_H\{h \in H : |h|_{\widetilde{G}} < \sqrt{t}\}. \end{aligned}$$

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