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STRUCTURE OF FLAT COVERS OF INJECTIVE MODULES

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Abstract. The aim of this paper is to discuss the flat covers of injective modules over a Noetherian ring. Let R be a commutative Noetherian ring and let E be an injective R-module. We prove that the flat cover of E is isomorphic to $\prod_{p\in \mathrm{Att}_R(E)} T_p$. As a consequence, we give an answer to Xu's question [10, 4.4.9]: for a prime ideal p, when does T_p appear in the flat cover of $E(R/\underline{m})$?

1. Introduction. The notion of flat covers of modules was introduced by Enochs in [6], but existence of flat covers was an open question. This question has been studied by several authors; see for example [1, 2, 12]. Recently, Bican, El Bashir and Enochs have proved that all modules have flat covers (see [3]).

The purpose of the present paper is to obtain information about the flat covers and minimal flat resolutions of injective modules over a Noetherian ring. Let R be a commutative Noetherian ring and let E be an injective R-module. Using [5] we see that the flat cover of E is of the form $\prod_{q\in \mathrm{Spec}(R)} T_q$. Here q is a prime ideal of R and T_q is the completion of a free R_q-module with respect to the qR_q-adic topology. We show, in 3.2, that if T_p appears in the flat cover of E, then p is an attached prime ideal of E. Now the answer to the question mentioned in the abstract is a consequence of 3.2. More precisely, we will prove that T_p appears in the flat cover of $E(R/\underline{m})$ exactly when $p \in \mathrm{Ass}_R(R)$. In the remainder of the paper, we focus on the minimal flat resolution of the injective R-module E. Firstly, we construct a minimal flat resolution for $0 :_E x$ from a given minimal flat resolution of E, when x is a non-unit and non-zero divisor of R. Secondly, we give a characterization of Cohen–Macaulay rings in terms of the vanishing property of the dual Bass numbers of E.

2. Preliminaries. In this section we recall some definitions and facts about the flat covers and minimal flat resolutions of modules. Throughout

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this paper R is a commutative ring with non-zero identity and M is an R-module.

DEFINITION 2.1. Let F be a flat R-module. A homomorphism $\phi : F \to M$ is called a *flat cover* of M if (1) for any homomorphism $\phi' : G \to M$ with G flat, there is a homomorphism $f : G \to F$ such that $\phi' = \phi f$ and (2) if $\phi = \phi f$ for some endomorphism of F, then f is an automorphism of F.

As mentioned in the introduction, it was proved in [3, 6] that M has a flat cover and it is unique up to isomorphism.

DEFINITION 2.2. An R-module C is said to be *cotorsion* if $\mathrm{Ext}^1_R(F, C) = 0$ for all flat modules F.

Note that if R is Noetherian and if F is a flat and cotorsion R-module, then it was proved in [5, p. 183] that F is uniquely a product $F = \prod T_p$, where T_p is the completion of a free R_p-module with respect to the pR_p-adic topology. Also note that a flat cover of a cotorsion R-module is flat and cotorsion, and the kernel of a flat cover $F \to M$ is cotorsion [5, Lemma 2.2]. Therefore, we have the following definitions.

DEFINITION 2.3. A *minimal flat resolution* of M is an exact sequence

$$\ldots \to F_i \stackrel{d_i}{\to} F_{i-1} \to \ldots \to F_1 \stackrel{d_1}{\to} F_0 \stackrel{d_0}{\to} M \to 0 \tag{1}$$

such that for each $i \geq 0$, F_i is a flat cover of $\mathrm{Im}(d_i)$.

By using the above remark, for each $i \geq 1$, F_i is flat and cotorsion, and thus it is a product of such T_p. For $i = 0$, F_0 is not cotorsion in general. But its pure injective envelope (or equivalently cotorsion envelope) $\mathrm{PE}(F_0)$ is flat and cotorsion [4, p. 352]. Hence $\mathrm{PE}(F_0)$ is a product of T_p.

DEFINITION 2.4. Let R be a commutative Noetherian ring, and let M admit a minimal flat resolution (1). For $i \geq 1$ and for a prime ideal p, $\pi_i(p, M)$ is defined to be the cardinality of the base of a free R_p-module whose completion is T_p in the product $F_i = \prod T_q$. For $i = 0$, $\pi_0(p, M)$ is defined similarly by using the pure injective envelope $\mathrm{PE}(F_0)$ instead of F_0 itself.

We note that the $\pi_i(p, M)$ are homologically independent and well defined. We call the $\pi_i(p, M)$ the *dual Bass numbers*.

3. The main results. Throughout this section, R will denote a commutative Noetherian ring. Let us recall the definition of the coassociated prime ideals of M. We say that an R-module L is *cocyclic* if L is a submodule of $E(R/\underline{m})$, where $E(R/\underline{m})$ is the injective envelope of $R/\underline{m}$ and $\underline{m}$ is a maximal ideal of R. A prime ideal p of R is called a *coassociated prime* of M

if there exists a cocyclic homomorphic image L of M such that $p = \mathrm{Ann}(L)$. The set of coassociated prime ideals of M is denoted by $\mathrm{Coass}_R(M)$.

Let A be a representable R-module. The set of attached prime ideals of A is denoted by $\mathrm{Att}_R(A)$. The reader is referred to [8] for details.

In order to prove our main result we need the following useful lemma.

LEMMA 3.1. *Let $\underline{a}$ be an ideal of R and let M have only finitely many coassociated prime ideals. Then $M = \underline{a}M$ if and only if there exists $x \in \underline{a}$ such that $M = xM$.*

Proof. The "if" part is clear. Hence we shall prove the "only if" half. Assume $M \neq xM$ for all $x \in \underline{a}$. Then, in view of [14, Theorem 1.13], $\underline{a} \subseteq \bigcup_{p\in\mathrm{Coass}_R(M)} p$. Thus there is a prime ideal p in $\mathrm{Coass}_R(M)$ such that $\underline{a} \subseteq p$, since $\mathrm{Coass}_R(M)$ is a finite set. Hence, by using the definition, M has a proper submodule N such that $p = \mathrm{Ann}_R(M/N)$. Thus $\underline{a}M \subseteq pM \subseteq N \subsetneq M$ contrary to assumption. ∎

We now come to the main theorem of this paper.

THEOREM 3.2. *If E is an injective R-module, then $\prod_{p\in\mathrm{Att}_R(E)} T_p$ is a flat cover of E.*

Proof. Note that E is cotorsion and so the flat cover of E, say F, is flat and cotorsion. Hence, as mentioned in the introduction, $F = \prod T_q$. Here T_q is the completion of a free R_q-module with respect to the qR_q-adic topology. First we show that $\mathrm{Coass}_{R_p}(\mathrm{Hom}_R(R_p, E))$ is a finite set for all $p \in \mathrm{Spec}(R)$. Note that since the zero submodule of R_p has a primary decomposition as an R_p-submodule, it has a primary decomposition as an R-submodule. Therefore, by using [13, Theorem 3.6], we see that

$$\mathrm{Coass}_R(\mathrm{Hom}_R(R_p, E)) = \{q \in \mathrm{Ass}_R(R_p) : q \subseteq q' \text{ for some } q' \in \mathrm{Ass}_R(E)\}.$$

Thus $\mathrm{Coass}_R(\mathrm{Hom}_R(R_p, E))$ is a finite set. Let $f : R \to R_p$ be the natural homomorphism and let $f^* : \mathrm{Spec}(R_p) \to \mathrm{Spec}(R)$ be the induced map. It is straightforward to see that

$$f^*\mathrm{Coass}_{R_p}(\mathrm{Hom}_R(R_p, E)) \subseteq \mathrm{Coass}_R(\mathrm{Hom}_R(R_p, E)).$$

Hence $\mathrm{Coass}_{R_p}(\mathrm{Hom}_R(R_p, E)$ is finite. Now assume that for a prime ideal p of R, T_p appears in the product of F. It follows from [7, Theorem 2.2] that

$$\mathrm{Hom}_R(R_p, E) \neq pR_p\mathrm{Hom}_R(R_p, E).$$

Thus, in view of 3.1 and [14, Theorem 1.13], we have

$$pR_p \subseteq \bigcup_{Q\in\mathrm{Coass}_{R_p}(\mathrm{Hom}_R(R_p,E))} Q, \quad \text{so} \quad pR_p \in \mathrm{Coass}_{R_p}(\mathrm{Hom}_R(R_p, E)).$$

Hence $p \in \mathrm{Coass}_R(\mathrm{Hom}_R(R_p, E))$. Therefore, we can deduce that $p \in \mathrm{Ass}_R(R_p)$ and $p \subseteq q$ for some $q \in \mathrm{Ass}_R(E)$. The claim now follows from [14, Lemma 1.17 and Theorem 1.14], that is, $p \in \mathrm{Att}_R(E)$. ∎

Let $(R, \underline{m})$ be a local ring. In [10, Remark 4.4.9], it was proved that if p is a minimal prime ideal of R, then T_p appears in the product of the flat cover of $E(R/\underline{m})$ (which is of the form $\prod T_q$). So a natural problem is to determine the set of prime ideals q for which T_q appears in the flat cover of $E(R/\underline{m})$. In the following consequence of 3.2 we answer this question.

THEOREM 3.3. *Let $(R, \underline{m})$ be a local ring and let $F = \prod T_q$ be a flat cover of $E(R/\underline{m})$. Then, for a prime ideal p of R, T_p appears in the product of F if and only if $p \in \operatorname{Att}_R(E(R/\underline{m}))$.*

Proof. By the previous theorem it is enough to show that if $p \in \operatorname{Att}_R(E(R/\underline{m}))$, then T_p appears in the product of F. Let $p \in \operatorname{Att}_R(E(R/\underline{m}))$ so that $p \in \operatorname{Ass}_R(R)$. In view of [9, Theorem 9.51] and using the fact that $E(R/\underline{m})$ is an injective cogenerator we have

$$0 \neq \operatorname{Hom}_R(\operatorname{Ext}^0_{R_p}(k(p), R_p), E(R/\underline{m})) \cong \operatorname{Tor}_0^{R_p}(k(p), \operatorname{Hom}_R(R_p, E(R/\underline{m})))$$

where $k(p)$ denotes the residue field of R_p. Hence by using [7, Theorem 2.2] it follows that $\pi_0(p, E(R/\underline{m})) \neq 0$. Thus T_p appears in the product of F. ∎

The following theorem is essential in the rest of the paper and we quote it for the convenience of the reader.

THEOREM 3.4 ([9, Theorem 9.37]). *If $(R, \underline{m})$ is a local ring and x is a non-unit and non-zero divisor of R, then for all $i \geq 0$,*

$$\operatorname{Ext}^i_{R/xR}(R/\underline{m}, R/xR) \cong \operatorname{Ext}^{i+1}_R(R/\underline{m}, R).$$

Proof. The exact sequence $0 \to R \xrightarrow{x} R \to R/xR \to 0$ induces the exact sequence

$$0 \to \operatorname{Hom}_R(R/\underline{m}, R) \xrightarrow{x} \operatorname{Hom}_R(R/\underline{m}, R) \to \operatorname{Hom}_R(R/\underline{m}, R/xR)$$
$$\to \operatorname{Ext}^1_R(R/\underline{m}, R) \xrightarrow{x} \operatorname{Ext}^1_R(R/\underline{m}, R) \to \dots$$

But $\operatorname{Hom}_R(1_{R/\underline{m}}, 1_R)$ is the identity mapping of $\operatorname{Hom}_R(R/\underline{m}, R)$ onto itself, and $x\operatorname{Hom}_R(1_{R/\underline{m}}, 1_R) = \operatorname{Hom}_R(x1_{R/\underline{m}}, 1_R)$; since $x \in \underline{m}$, it follows that $x1_{R/\underline{m}} = 0$ and $x\operatorname{Hom}_R(1_{R/\underline{m}}, 1_R)$ is zero. Thus the induced homomorphisms

$$\operatorname{Ext}^i_R(R/\underline{m}, R) \xrightarrow{x} \operatorname{Ext}^i_R(R/\underline{m}, R)$$

are zero for all $i \geq 0$. It follows that $\operatorname{Hom}_R(R/\underline{m}, R) = 0$ and

$$\operatorname{Hom}_R(R/\underline{m}, R/xR) \cong \operatorname{Ext}^1_R(R/\underline{m}, R).$$

But $R/\underline{m}$ and R/xR both have natural structures as R/xR-modules, and a mapping $\gamma : R/\underline{m} \to R/xR$ is an R-homomorphism if and only if it is an R/xR-homomorphism, thus

$$\operatorname{Hom}_R(R/\underline{m}, R/xR) = \operatorname{Hom}_{R/xR}(R/\underline{m}, R/xR).$$

Therefore,

$$\operatorname{Hom}_{R/xR}(R/\underline{m}, R/xR) \cong \operatorname{Ext}^1_R(R/\underline{m}, R).$$

Let

$$0 \to R \xrightarrow{\alpha} E^0 \xrightarrow{d_0} E^1 \to \ldots \to E^i \xrightarrow{d_i} E^{i+1} \to \ldots$$

be a minimal injective resolution for R. For each $i \geq 0$, define $0 :_{E^i} x = \{y \in E^i : xy = 0\}$. Then

$$0 \to R/xR \xrightarrow{\subseteq} 0 :_{E^1} x \xrightarrow{e_1} 0 :_{E^2} x \to \ldots \to 0 :_{E^i} x \xrightarrow{e_i} \ldots$$

is a minimal injective resolution for R/xR as an R/xR-module, where e_i is the restriction of d_i. There is a homomorphism of complexes of R-modules and R-homomorphisms:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & 0 :_{E^i} x & \longrightarrow & 0 :_{E^{i+1}} x & \longrightarrow & 0 :_{E^{i+2}} x & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_{i+2}} & & \\ \ldots & \longrightarrow & E^i & \longrightarrow & E^{i+1} & \longrightarrow & E^{i+2} & \longrightarrow & \ldots \end{array}$$

in which f_i is the inclusion map for all $i \geq 1$. Using the functor $\mathrm{Hom}_R(R/\underline{m}, -)$ we obtain the following homomorphism of complexes of R-modules and R-homomorphisms:

$$\begin{array}{ccccc} \mathrm{Hom}_{R/xR}(R/\underline{m}, 0 :_{E^i} x) & \longrightarrow & \mathrm{Hom}_{R/xR}(R/\underline{m}, 0 :_{E^{i+1}} x) & \longrightarrow & \mathrm{Hom}_{R/xR}(R/\underline{m}, 0 :_{E^{i+2}} x) \\ \downarrow{\scriptstyle \overline{f}_i} & & \downarrow{\scriptstyle \overline{f}_{i+1}} & & \downarrow{\scriptstyle \overline{f}_{i+2}} \\ \mathrm{Hom}_R(R/\underline{m}, E^i) & \longrightarrow & \mathrm{Hom}_R(R/\underline{m}, E^{i+1}) & \longrightarrow & \mathrm{Hom}_R(R/\underline{m}, E^{i+2}) \end{array}$$

Now it is straightforward to see that $\overline{f}_i = \mathrm{Hom}_R(1_{R/\underline{m}}, f_i)$ is an R- and R/xR-isomorphism for all $i \geq 1$. Hence

$$\mathrm{Ext}^i_{R/xR}(R/\underline{m}, R/xR) \cong \mathrm{Ext}^{i+1}_R(R/\underline{m}, R)$$

for all $i \geq 0$. This completes the proof of the theorem. ∎

THEOREM 3.5. *Let E be an injective R-module and let x be a non-unit and non-zero divisor of R. If $p \in \mathrm{Spec}(R)$ and $x \in p$, then for all $i \geq 0$,*

$$\pi_i(p/(x), 0 :_E x) = \pi_{i+1}(p, E).$$

Proof. Assume p is a prime ideal of R and $x \in p$. We let $\overline{R} = R/xR$, $\overline{p} = p/(x)$ and $k(\overline{p}) = \overline{R}_{\overline{p}}/\overline{p}\overline{R}_{\overline{p}}$ ($\cong k(p)$). Now

$$\begin{aligned} \mathrm{Hom}_{\overline{R}}(\overline{R}_{\overline{p}}, 0 :_E x) &\cong \mathrm{Hom}_{\overline{R}}(\overline{R}_{\overline{p}}, \mathrm{Hom}_R(\overline{R}, E)) \cong \mathrm{Hom}_R(\overline{R}_{\overline{p}} \otimes_{\overline{R}} \overline{R}, E) \\ &\cong \mathrm{Hom}_R(\overline{R}_{\overline{p}}, E). \end{aligned}$$

Moreover, for all $i \geq 0$,

$$\begin{aligned} \mathrm{Tor}_i^{\overline{R}_{\overline{p}}}(k(\overline{p}), \mathrm{Hom}_{\overline{R}}(\overline{R}_{\overline{p}}, 0 :_E x)) &\cong \mathrm{Tor}_i^{\overline{R}_{\overline{p}}}(k(\overline{p}), \mathrm{Hom}_R(\overline{R}_{\overline{p}}, E)) \\ &\cong \mathrm{Hom}_R(\mathrm{Ext}^i_{\overline{R}_{\overline{p}}}(k(\overline{p}), \overline{R}_{\overline{p}}), E) \end{aligned}$$

(see [9, Theorem 9.51]). On the other hand, in view of 3.4, we have

$$\begin{aligned}\mathrm{Hom}_R(\mathrm{Ext}^i_{\overline{R}_{\overline{p}}}(k(\overline{p}), \overline{R}_{\overline{p}}), E) &\cong \mathrm{Hom}_R(\mathrm{Ext}^{i+1}_{R_p}(k(p), R_p), E)\\ &\cong \mathrm{Tor}^{R_p}_{i+1}(k(p), \mathrm{Hom}_R(R_p, E)).\end{aligned}$$

Thus by using [7, Theorem 2.2] the result follows:

$$\begin{aligned}\pi_i(\overline{p}, 0 :_E x) &= \dim_{k(\overline{p})} \mathrm{Tor}^{\overline{R}_{\overline{p}}}_i(k(\overline{p}), \mathrm{Hom}_{\overline{R}}(\overline{R}_{\overline{p}}, 0 :_E x))\\ &= \dim_{k(p)} \mathrm{Tor}^{R_p}_{i+1}(k(p), \mathrm{Hom}_R(R_p, E)) = \pi_{i+1}(p, E). \blacksquare\end{aligned}$$

THEOREM 3.6. *Let E be an injective R-module and let x be a non-unit and non-zero divisor of R. Let*

$$\ldots \to F_i \xrightarrow{d_i} F_{i-1} \to \ldots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} E \to 0$$

be a minimal flat resolution for E. Let $K = \ker d_0$. Then $R/xR \otimes_R K \cong 0 :_E x$ as R- and R/xR-modules, and the induced complex of R/xR-modules and R/xR-homomorphisms

$$(2) \qquad \ldots \to F_i \otimes_R R/xR \to \ldots \to F_1 \otimes_R R/xR \to K \otimes_R R/xR \to 0$$

is a flat resolution for the R/xR-module $K \otimes_R R/xR$. Also, if

$$\ldots \to G_i \to G_{i-1} \to \ldots \to G_1 \to G_0 \to 0 :_E x \to 0$$

is a minimal flat resolution of $0 :_E x$ as an R/xR-module, then $G_i \cong F_{i+1} \otimes_R R/xR$ for all $i \geq 0$.

Proof. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & F_0 & \longrightarrow & E & \longrightarrow & 0\\ & & \downarrow x & & \downarrow x & & \downarrow x & & \\ 0 & \longrightarrow & K & \longrightarrow & F_0 & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

with exact rows induces an exact sequence

$$0 :_K x \to 0 :_{F_0} x \to 0 :_E x \to K/xK \to F_0/xF_0 \to E/xE.$$

Note that x is a non-zero divisor of R and F_0 is a flat R-module, hence $0 :_{F_0} x = 0$. We show $F_0 = xF_0$. In view of 3.2, $F_0 = \prod_{p \in \mathrm{Att}_R(E)} T_p$, so that

$$\begin{aligned}F_0 \otimes_R R/xR = \Big(\prod_{p \in \mathrm{Att}_R(E)} T_p\Big) \otimes_R R/xR &\cong \prod_{p \in \mathrm{Att}_R(E)} (T_p \otimes_R R/xR)\\ &\cong \prod_{x \notin p} T_p/xT_p = 0.\end{aligned}$$

Thus $F_0/xF_0 = 0$. Hence $0 :_E x \cong K/xK$ as R- and R/xR-modules. The exact sequence $F_2 \to F_1 \to K \to 0$ shows that (2) is exact at $K \otimes_R R/xR$ and at $F_1 \otimes_R R/xR$. If $n > 1$, the homology module of the complex

$$F_{i+1} \otimes_R R/xR \to F_i \otimes_R R/xR \to F_{i-1} \otimes_R R/xR$$

is isomorphic to $\mathrm{Tor}_i^R(E, R/xR)$, which is zero since the R-module R/xR has projective dimension ≤ 1. Thus (2) is exact. Also, $F_i \otimes_R R/xR$ is a flat R/xR-module for all $i \geq 1$. Hence, (2) is a flat resolution for $K \otimes_R R/xR$. The only thing left to do is to show that $G_i \cong F_{i+1} \otimes_R R/xR$. For this let $i \geq 0$ and let $F_{i+1} = \prod T_p$. By 3.5, $G_i = \prod_{x \in p} U_{p/(x)}$, where $U_{p/(x)}$ is the completion of a free $(R/xR)_{p/(x)}$-module with a base having the same cardinality of the base of the free R_p-module whose completion is T_p. On the other hand, $F_{i+1} \otimes_R R/xR = \prod_{x \in p} T_p/xT_p$ and it is easy to see that T_p/xT_p and $U_{p/(x)}$ have the same properties. Now we can deduce that G_i and $F_{i+1} \otimes_R R/xR$ are isomorphic. ∎

The next easy corollary is in fact an important "change of rings" result on flat dimension (which we write as f.dim).

Corollary 3.7. *If E is an injective R-module and x is a non-unit and non-zero divisor of R, then* $\mathrm{f.dim}_R E \geq \mathrm{f.dim}_{R/xR}(0 :_E x) + 1$.

For $n \in \mathbb{N}$, we say that R satisfies (S_n) if $\operatorname{depth} R_p \geq \min\{\operatorname{ht} p, n\}$ for every prime ideal p of R.

Theorem 3.8. *If R is a Noetherian ring*, *then the following statements are equivalent*:

(1) *R satisfies (S_n)*;

(2) *if E is an injective R-module*, *then $\pi_i(p, E) \neq 0$ implies that $\min\{\operatorname{ht} p, n\} \leq i$ for all prime ideals p and all $i \geq 0$*;

(3) *if $\pi_i(p, E(R/p)) \neq 0$, then $\min\{\operatorname{ht} p, n\} \leq i$ for all prime ideals p and all $i \geq 0$.*

Proof. (1)⇒(2). Let E be an injective R-module. Consider the minimal flat resolution of E:

$$\ldots \to F_i \to F_{i-1} \to \ldots \to F_1 \to F_0 \to E \to 0.$$

As mentioned before, for each $i \geq 0$, F_i is flat and cotorsion, so it is uniquely a product $\prod T_q$. We have to show that for a prime ideal p, if $\pi_i(p, E) \neq 0$ then $\min\{\operatorname{ht} p, n\} \leq i$. We use induction on i. If $\pi_0(p, E) \neq 0$ then T_p is a direct summand of F_0. Hence, in view of 3.2, $p \in \mathrm{Att}_R(E)$. So there is $q \in \mathrm{Ass}_R(R)$ such that $p \subseteq q$. Now we have $qRq \in \mathrm{Ass}_{R_q}(R_q)$ and

$$\min\{\operatorname{ht} p, n\} \leq \min\{\operatorname{ht} q, n\} \leq \operatorname{depth} R_q = 0.$$

Assume inductively that $k \geq 0$ and the result has been proved (for all choices of R and E satisfying the hypothesis) when $i = k$; let $\pi_{k+1}(p, E) \neq 0$. We may assume that $p \not\subseteq Z(R)$. Suppose that $x \in p - Z(R)$. It is easy to see that R/xR satisfies (S_{n-1}), and $0 :_E x$ is an injective R/xR-module. By using 3.5, we have $\pi_k(p/(x), 0 :_E x) \neq 0$. Hence, by the inductive hypothesis, $\min\{\operatorname{ht} p/(x), n-1\} \leq k$. Thus $\min\{\operatorname{ht} p, n\} \leq k+1$. The result follows by induction.

(2)⇒(3). This is trivial.

(3)⇒(1). Assume that $p \in \operatorname{Spec}(R)$ and $\operatorname{depth} R_p = i$. In view of [9, Theorem 9.51], and using the fact that $E(R/p)$ is an injective cogenerator R_p-module, we have

$$0 \neq \operatorname{Hom}_R(\operatorname{Ext}^i_{R_p}(k(p), R_p), E(R/p)) \cong \operatorname{Tor}_i^{R_p}(k(p), \operatorname{Hom}_R(R_p, E(R/p))).$$

Hence, by using [7, Theorem 2.2], it follows that $\pi_i(p, E(R/p)) \neq 0$ so that $\min\{\operatorname{ht} p, n\} \leq i = \operatorname{depth} R_p$. ∎

The next corollary is analogous to [11, Theorem 3.2] and provides an explicit description of the minimal flat resolution of an injective module over a Cohen–Macaulay ring.

Corollary 3.9. *If R is a Noetherian ring, then the following statements are equivalent*:

(1) *R is Cohen–Macaulay*;

(2) *if E is an injective R-module, then $\pi_i(p, E) \neq 0$ implies that $\operatorname{ht} p \leq i$ for all prime ideals p and all $i \geq 0$*;

(3) *if $\pi_i(p, E(R/p)) \neq 0$, then $\operatorname{ht} p \leq i$ for all prime ideals p and all $i \geq 0$.*

Proof. The proof is similar to that of 3.8. ∎

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