STRUCTURE OF FLAT COVERS OF INJECTIVE MODULES

by

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Abstract. The aim of this paper is to discuss the flat covers of injective modules over a Noetherian ring. Let $R$ be a commutative Noetherian ring and let $E$ be an injective $R$-module. We prove that the flat cover of $E$ is isomorphic to $\prod_{p \in \text{Att}_R(E)} T_p$. As a consequence, we give an answer to Xu’s question [10, 4.4.9]: for a prime ideal $p$, when does $T_p$ appear in the flat cover of $E(R/m)$?

1. Introduction. The notion of flat covers of modules was introduced by Enochs in [6], but existence of flat covers was an open question. This question has been studied by several authors; see for example [1, 2, 12]. Recently, Bican, El Bashir and Enochs have proved that all modules have flat covers (see [3]).

The purpose of the present paper is to obtain information about the flat covers and minimal flat resolutions of injective modules over a Noetherian ring. Let $R$ be a commutative Noetherian ring and let $E$ be an injective $R$-module. Using [5] we see that the flat cover of $E$ is of the form $\prod_{q \in \text{Spec}(R)} T_q$. Here $q$ is a prime ideal of $R$ and $T_q$ is the completion of a free $R_q$-module with respect to the $qR_q$-adic topology. We show, in 3.2, that if $T_p$ appears in the flat cover of $E$, then $p$ is an attached prime ideal of $E$. Now the answer to the question mentioned in the abstract is a consequence of 3.2. More precisely, we will prove that $T_p$ appears in the flat cover of $E(R/m)$ exactly when $p \in \text{Ass}_R(R)$. In the remainder of the paper, we focus on the minimal flat resolution of the injective $R$-module $E$. Firstly, we construct a minimal flat resolution for $0 :_E x$ from a given minimal flat resolution of $E$, when $x$ is a non-unit and non-zero divisor of $R$. Secondly, we give a characterization of Cohen–Macaulay rings in terms of the vanishing property of the dual Bass numbers of $E$.

2. Preliminaries. In this section we recall some definitions and facts about the flat covers and minimal flat resolutions of modules. Throughout
this paper $R$ is a commutative ring with non-zero identity and $M$ is an $R$-module.

**Definition 2.1.** Let $F$ be a flat $R$-module. A homomorphism $\phi : F \to M$ is called a flat cover of $M$ if (1) for any homomorphism $\phi' : G \to M$ with $G$ flat, there is a homomorphism $f : G \to F$ such that $\phi' = \phi f$ and (2) if $\phi = \phi f$ for some endomorphism of $F$, then $f$ is an automorphism of $F$.

As mentioned in the introduction, it was proved in [3, 6] that $M$ has a flat cover and it is unique up to isomorphism.

**Definition 2.2.** An $R$-module $C$ is said to be cotorsion if $\text{Ext}^1_R(F, C) = 0$ for all flat modules $F$.

Note that if $R$ is Noetherian and if $F$ is a flat and cotorsion $R$-module, then it was proved in [5, p. 183] that $F$ is uniquely a product $F = \prod T_p$, where $T_p$ is the completion of a free $R_p$-module with respect to the $pR_p$-adic topology. Also note that a flat cover of a cotorsion $R$-module is flat and cotorsion, and the kernel of a flat cover $F \to M$ is cotorsion [5, Lemma 2.2]. Therefore, we have the following definitions.

**Definition 2.3.** A minimal flat resolution of $M$ is an exact sequence

\[
\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0
\]

such that for each $i \geq 0$, $F_i$ is a flat cover of $\text{Im}(d_i)$.

By using the above remark, for each $i \geq 1$, $F_i$ is flat and cotorsion, and thus it is a product of such $T_p$. For $i = 0$, $F_0$ is not cotorsion in general. But its pure injective envelope (or equivalently cotorsion envelope) $\text{PE}(F_0)$ is flat and cotorsion [4, p. 352]. Hence $\text{PE}(F_0)$ is a product of $T_p$.

**Definition 2.4.** Let $R$ be a commutative Noetherian ring, and let $M$ admit a minimal flat resolution (1). For $i \geq 1$ and for a prime ideal $p$, $\pi_i(p, M)$ is defined to be the cardinality of the base of a free $R_p$-module whose completion is $T_p$ in the product $F_i = \prod T_q$. For $i = 0$, $\pi_0(p, M)$ is defined similarly by using the pure injective envelope $\text{PE}(F_0)$ instead of $F_0$ itself.

We note that the $\pi_i(p, M)$ are homologically independent and well defined. We call the $\pi_i(p, M)$ the dual Bass numbers.

**3. The main results.** Throughout this section, $R$ will denote a commutative Noetherian ring. Let us recall the definition of the coassociated prime ideals of $M$. We say that an $R$-module $L$ is cocyclic if $L$ is a submodule of $E(R/m)$, where $E(R/m)$ is the injective envelope of $R/m$ and $m$ is a maximal ideal of $R$. A prime ideal $p$ of $R$ is called a coassociated prime of $M$.
if there exists a cocyclic homomorphic image $L$ of $M$ such that $p = \text{Ann}(L)$. The set of coassociated prime ideals of $M$ is denoted by $\text{Coass}_R(M)$.

Let $A$ be a representable $R$-module. The set of attached prime ideals of $A$ is denoted by $\text{Att}_R(A)$. The reader is referred to [8] for details.

In order to prove our main result we need the following useful lemma.

**Lemma 3.1.** Let $a$ be an ideal of $R$ and let $M$ have only finitely many coassociated prime ideals. Then $M = aM$ if and only if there exists $x \in a$ such that $M = xM$.

**Proof.** The “if” part is clear. Hence we shall prove the “only if” half. Assume $M \neq xM$ for all $x \in a$. Then, in view of [14, Theorem 1.13], $a \subseteq \bigcup_{p \in \text{Coass}_R(M)} p$. Thus there is a prime ideal $p$ in $\text{Coass}_R(M)$ such that $a \subseteq p$, since $\text{Coass}_R(M)$ is a finite set. Hence, by using the definition, $M$ has a proper submodule $N$ such that $p = \text{Ann}_R(M/N)$. Thus $aM \subseteq pM \subseteq N \subseteq M$ contrary to assumption. 

We now come to the main theorem of this paper.

**Theorem 3.2.** If $E$ is an injective $R$-module, then $\prod_{p \in \text{Att}_R(E)} T_p$ is a flat cover of $E$.

**Proof.** Note that $E$ is cotorsion and so the flat cover of $E$, say $F$, is flat and cotorsion. Hence, as mentioned in the introduction, $F = \prod T_q$. Here $T_q$ is the completion of a free $R_q$-module with respect to the $qR_q$-adic topology. First we show that $\text{Coass}_{R_p}(\text{Hom}_R(R_p, E))$ is a finite set for all $p \in \text{Spec}(R)$. Note that since the zero submodule of $R_p$ has a primary decomposition as an $R_p$-submodule, it has a primary decomposition as an $R$-submodule. Therefore, by using [13, Theorem 3.6], we see that

$\text{Coass}_R(\text{Hom}_R(R_p, E)) = \{ q \in \text{Ass}_R(R_p) : q \subseteq q' \text{ for some } q' \in \text{Ass}_R(E) \}.$

Thus $\text{Coass}_R(\text{Hom}_R(R_p, E))$ is a finite set. Let $f : R \to R_p$ be the natural homomorphism and let $f^* : \text{Spec}(R_p) \to \text{Spec}(R)$ be the induced map. It is straightforward to see that

$f^* \text{Coass}_{R_p}(\text{Hom}_R(R_p, E)) \subseteq \text{Coass}_R(\text{Hom}_R(R_p, E)).$

Hence $\text{Coass}_{R_p}(\text{Hom}_R(R_p, E))$ is finite. Now assume that for a prime ideal $p$ of $R$, $T_p$ appears in the product of $F$. It follows from [7, Theorem 2.2] that

$\text{Hom}_R(R_p, E) \neq pR_p \text{Hom}_R(R_p, E).$

Thus, in view of 3.1 and [14, Theorem 1.13], we have

$pR_p \subseteq \bigcup_{Q \in \text{Coass}_{R_p}(\text{Hom}_R(R_p, E))} Q,$

so $pR_p \in \text{Coass}_{R_p}(\text{Hom}_R(R_p, E))$.

Hence $p \in \text{Coass}_R(\text{Hom}_R(R_p, E))$. Therefore, we can deduce that $p \in \text{Ass}_R(R_p)$ and $p \subseteq q$ for some $q \in \text{Ass}_R(E)$. The claim now follows from [14, Lemma 1.17 and Theorem 1.14], that is, $p \in \text{Att}_R(E)$. 

Let \((R, m)\) be a local ring. In [10, Remark 4.4.9], it was proved that if \(p\) is a minimal prime ideal of \(R\), then \(T_p\) appears in the product of the flat cover of \(E(R/m)\) (which is of the form \(\prod T_q\)). So a natural problem is to determine the set of prime ideals \(q\) for which \(T_q\) appears in the flat cover of \(E(R/m)\). In the following consequence of 3.2 we answer this question.

**Theorem 3.3.** Let \((R, m)\) be a local ring and let \(F = \prod T_q\) be a flat cover of \(E(R/m)\). Then, for a prime ideal \(p\) of \(R\), \(T_p\) appears in the product of \(F\) if and only if \(p \in \text{Att}_R(E(R/m))\).

**Proof.** By the previous theorem it is enough to show that if \(p \in \text{Att}_R(E(R/m))\), then \(T_p\) appears in the product of \(F\). Let \(p \in \text{Att}_R(E(R/m))\) so that \(p \in \text{Ass}_R(R)\). In view of [9, Theorem 9.51] and using the fact that \(E(R/m)\) is an injective cogenerator we have

\[
0 \neq \text{Hom}_R(\text{Ext}^0_{R_p}(k(p), R_p), E(R/m)) \cong \text{Tor}^R_0(k(p), \text{Hom}_R(R_p, E(R/m)))
\]

where \(k(p)\) denotes the residue field of \(R_p\). Hence by using [7, Theorem 2.2] it follows that \(\pi_0(p, E(R/m)) \neq 0\). Thus \(T_p\) appears in the product of \(F\). \(\blacksquare\)

The following theorem is essential in the rest of the paper and we quote it for the convenience of the reader.

**Theorem 3.4 ([9, Theorem 9.37]).** If \((R, m)\) is a local ring and \(x\) is a non-unit and non-zero divisor of \(R\), then for all \(i \geq 0\),

\[
\text{Ext}^i_{R/xR}(R/m, R/xR) \cong \text{Ext}^{i+1}_R(R/m, R).
\]

**Proof.** The exact sequence \(0 \to R \xrightarrow{x} R \to R/xR \to 0\) induces the exact sequence

\[
0 \to \text{Hom}_R(R/m, R) \xrightarrow{x} \text{Hom}_R(R/m, R) \to \text{Hom}_R(R/m, R/xR)
\]

\[
\to \text{Ext}^1_R(R/m, R) \xrightarrow{x} \text{Ext}^1_R(R/m, R) \to \ldots
\]

But \(\text{Hom}_R(1_{R/m}, 1_R)\) is the identity mapping of \(\text{Hom}_R(R/m, R)\) onto itself, and \(x\text{Hom}_R(1_{R/m}, 1_R) = \text{Hom}_R(x1_{R/m}, 1_R)\); since \(x \in m\), it follows that \(x1_{R/m} = 0\) and \(x\text{Hom}_R(1_{R/m}, 1_R)\) is zero. Thus the induced homomorphisms

\[
\text{Ext}^i_R(R/m, R) \xrightarrow{x} \text{Ext}^i_R(R/m, R)
\]

are zero for all \(i \geq 0\). It follows that \(\text{Hom}_R(R/m, R) = 0\) and

\[
\text{Hom}_R(R/m, R/xR) \cong \text{Ext}^1_R(R/m, R).
\]

But \(R/m\) and \(R/xR\) both have natural structures as \(R/xR\)-modules, and a mapping \(\gamma : R/m \to R/xR\) is an \(R\)-homomorphism if and only if it is an \(R/xR\)-homomorphism, thus

\[
\text{Hom}_R(R/m, R/xR) = \text{Hom}_{R/xR}(R/m, R/xR).
\]

Therefore,

\[
\text{Hom}_{R/xR}(R/m, R/xR) \cong \text{Ext}^1_R(R/m, R).
\]
Let
\[ 0 \to R \xrightarrow{\alpha} E^0 \xrightarrow{d_0} E^1 \to \ldots \to E^i \xrightarrow{d_i} E^{i+1} \to \ldots \]
be a minimal injective resolution for \( R \). For each \( i \geq 0 \), define \( 0 : E^i x = \{ y \in E^i : xy = 0 \} \). Then
\[ 0 \to R / xR \xhookleftarrow{} 0 : E^i x \xrightarrow{e_i} 0 : E^{i+1} x \to \ldots \to 0 : E^i x \xrightarrow{e_i} \]
is a minimal injective resolution for \( R / xR \) as an \( R / xR \)-module, where \( e_i \) is the restriction of \( d_i \). There is a homomorphism of complexes of \( R \)-modules and \( R \)-homomorphisms:
\[
\begin{array}{c}
\ldots \longrightarrow 0 : E^i x \longrightarrow 0 : E^{i+1} x \longrightarrow 0 : E^{i+2} x \longrightarrow \ldots \\
\downarrow f_i \quad \downarrow f_{i+1} \quad \downarrow f_{i+2} \\
\ldots \longrightarrow E^i \longrightarrow E^{i+1} \longrightarrow E^{i+2} \longrightarrow \ldots 
\end{array}
\]
in which \( f_i \) is the inclusion map for all \( i \geq 1 \). Using the functor \( \text{Hom}_R(R / m, -) \) we obtain the following homomorphism of complexes of \( R \)-modules and \( R \)-homomorphisms:
\[
\text{Hom}_{R / xR}(R / m, 0 : E^i x) \longrightarrow \text{Hom}_{R / xR}(R / m, 0 : E^{i+1} x) \longrightarrow \text{Hom}_{R / xR}(R / m, 0 : E^{i+2} x)
\]
\[
\begin{array}{c}
\downarrow \overline{f}_i \quad \downarrow \overline{f}_{i+1} \quad \downarrow \overline{f}_{i+2} \\
\text{Hom}_R(R / m, E^i) \longrightarrow \text{Hom}_R(R / m, E^{i+1}) \longrightarrow \text{Hom}_R(R / m, E^{i+2}) 
\end{array}
\]
Now it is straightforward to see that \( \overline{f}_i = \text{Hom}_R(1_R / m, f_i) \) is an \( R \)- and \( R / xR \)-isomorphism for all \( i \geq 1 \). Hence
\[
\text{Ext}^i_{R / xR}(R / m, R / xR) \cong \text{Ext}^{i+1}_R(R / m, R)
\]
for all \( i \geq 0 \). This completes the proof of the theorem. \( \blacksquare \)

**Theorem 3.5.** Let \( E \) be an injective \( R \)-module and let \( x \) be a non-unit and non-zero divisor of \( R \). If \( p \in \text{Spec}(R) \) and \( x \in p \), then for all \( i \geq 0 \),
\[
\pi_i(p / (x), 0 : E x) = \pi_{i+1}(p, E).
\]

**Proof.** Assume \( p \) is a prime ideal of \( R \) and \( x \in p \). We let \( \overline{R} = R / xR \), \( \overline{p} = p / (x) \) and \( k(\overline{p}) = \overline{R}_p / \overline{p} \overline{R}_p (\cong k(p)) \). Now
\[
\text{Hom}_R(\overline{R}_p, 0 : E x) \cong \text{Hom}_R(\overline{R}_p, \text{Hom}_R(\overline{R}, E)) \cong \text{Hom}_R(\overline{R}_p \otimes_R \overline{R}, E)
\]
\[
\cong \text{Hom}_R(\overline{R}_p, E).
\]
Moreover, for all \( i \geq 0 \),
\[
\text{Tor}^i_{\overline{R}_p}(k(\overline{p}), \text{Hom}_R(\overline{R}_p, 0 : E x)) \cong \text{Tor}^i_{\overline{R}_p}(k(\overline{p}), \text{Hom}_R(\overline{R}_p, E))
\]
\[
\cong \text{Hom}_R(\text{Ext}^i_{\overline{R}_p}(k(\overline{p}), \overline{R}_p), E)
\]
(see [9, Theorem 9.51]). On the other hand, in view of 3.4, we have
\[\text{Hom}_R(\text{Ext}^i_{R_p}(k(p), \overline{R_p}), E) \cong \text{Hom}_R(\text{Ext}^{i+1}_{R_p}(k(p), R_p), E)\]
\[\cong \text{Tor}^{R_p}_{i+1}(k(p), \text{Hom}_R(R_p, E)).\]
Thus by using [7, Theorem 2.2] the result follows:
\[\pi_i(\overline{p}, 0 : E x) = \dim_{k(\overline{p})} \text{Tor}^R_{i+1}(k(p), \text{Hom}_R(R_p, E)) = \pi_{i+1}(p, E).\]

**Theorem 3.6.** Let \( E \) be an injective \( R \)-module and let \( x \) be a non-unit and non-zero divisor of \( R \). Let
\[\ldots \rightarrow F_i \xrightarrow{d_i} F_{i-1} \rightarrow \ldots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} E \rightarrow 0\]
be a minimal flat resolution for \( E \). Let \( K = \ker d_0 \). Then \( R/xR \otimes_R K \cong 0 : E x \) as \( R \)- and \( R/xR \)-modules, and the induced complex of \( R/xR \)-modules and \( R/xR \)-homomorphisms
\[(2) \quad \ldots \rightarrow F_i \otimes_R R/xR \rightarrow \ldots \rightarrow F_1 \otimes_R R/xR \rightarrow K \otimes_R R/xR \rightarrow 0\]
is a flat resolution for the \( R/xR \)-module \( K \otimes_R R/xR \). Also, if
\[\ldots \rightarrow G_i \rightarrow G_{i-1} \rightarrow \ldots \rightarrow G_1 \rightarrow G_0 \rightarrow 0 : E x \rightarrow 0\]
is a minimal flat resolution of \( 0 : E x \) as an \( R/xR \)-module, then \( G_i \cong F_{i+1} \otimes_R R/xR \) for all \( i \geq 0 \).

**Proof.** The commutative diagram
\[
\begin{array}{ccc}
0 & \rightarrow & K \\
\downarrow x & & \downarrow x \\
0 & \rightarrow & K \\
\downarrow x & & \downarrow x \\
0 & \rightarrow & F_0 \rightarrow E \rightarrow 0
\end{array}
\]
with exact rows induces an exact sequence
\[0 : K x \rightarrow 0 : F_0 x \rightarrow 0 : E x \rightarrow K/xK \rightarrow F_0/xF_0 \rightarrow E/xE.\]
Note that \( x \) is a non-zero divisor of \( R \) and \( F_0 \) is a flat \( R \)-module, hence \( 0 : F_0 x = 0 \). We show \( F_0 = xF_0 \). In view of 3.2, \( F_0 = \prod_{p \in \text{Att}_R(E)} T_p \), so that
\[F_0 \otimes_R R/xR = \left( \prod_{p \in \text{Att}_R(E)} T_p \right) \otimes_R R/xR \cong \prod_{p \in \text{Att}_R(E)} (T_p \otimes_R R/xR)\]
\[\cong \prod_{x \notin \mathfrak{p}} T_p/xT_p = 0.\]
Thus \( F_0/xF_0 = 0 \). Hence \( 0 : E x \cong K/xK \) as \( R \)- and \( R/xR \)-modules. The exact sequence \( F_2 \rightarrow F_1 \rightarrow K \rightarrow 0 \) shows that (2) is exact at \( K \otimes_R R/xR \) and at \( F_1 \otimes_R R/xR \). If \( n > 1 \), the homology module of the complex
\[F_{i+1} \otimes_R R/xR \rightarrow F_i \otimes_R R/xR \rightarrow F_{i-1} \otimes_R R/xR\]
is isomorphic to \( \text{Tor}_i^R(E, R/xR) \), which is zero since the \( R \)-module \( R/xR \) has projective dimension \( \leq 1 \). Thus (2) is exact. Also, \( F_i \otimes_R R/xR \) is a flat \( R/xR \)-module for all \( i \geq 1 \). Hence, (2) is a flat resolution for \( K \otimes_R R/xR \).

The only thing left to do is to show that \( G_i \cong F_{i+1} \otimes_R R/xR \). For this let \( i \geq 0 \) and let \( F_{i+1} = \prod T_p \). By 3.5, \( G_i = \prod_{x \in p} U_{p/(x)} \), where \( U_{p/(x)} \) is the completion of a free \((R/xR)_{p/(x)}\)-module with a base having the same cardinality of the base of the free \( R_p \)-module whose completion is \( T_p \). On the other hand, \( F_{i+1} \otimes_R R/xR = \prod_{x \in p} T_p / xT_p \) and it is easy to see that \( T_p / xT_p \) and \( U_{p/(x)} \) have the same properties. Now we can deduce that \( G_i \) and \( F_{i+1} \otimes_R R/xR \) are isomorphic. \( \blacksquare \)

The next easy corollary is in fact an important “change of rings” result on flat dimension (which we write as \( f.\text{dim} \)).

**Corollary 3.7.** If \( E \) is an injective \( R \)-module and \( x \) is a non-unit and non-zero divisor of \( R \), then \( f.\text{dim}_R E \geq f.\text{dim}_{R/xR} (0 : E x) + 1 \).

For \( n \in \mathbb{N} \), we say that \( R \) satisfies \( (S_n) \) if \( \text{depth} R_p \geq \min \{ \text{ht} p, n \} \) for every prime ideal \( p \) of \( R \).

**Theorem 3.8.** If \( R \) is a Noetherian ring, then the following statements are equivalent:

1. \( R \) satisfies \( (S_n) \);
2. if \( E \) is an injective \( R \)-module, then \( \pi_i(p, E) \neq 0 \) implies that \( \min \{ \text{ht} p, n \} \leq i \) for all prime ideals \( p \) and all \( i \geq 0 \);
3. if \( \pi_i(p, E(R/p)) \neq 0 \), then \( \min \{ \text{ht} p, n \} \leq i \) for all prime ideals \( p \) and all \( i \geq 0 \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( E \) be an injective \( R \)-module. Consider the minimal flat resolution of \( E \):

\[
\ldots \rightarrow F_i \rightarrow F_{i-1} \rightarrow \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow E \rightarrow 0.
\]

As mentioned before, for each \( i \geq 0 \), \( F_i \) is flat and cotorsion, so it is uniquely a product \( \prod T_q \). We have to show that for a prime ideal \( p \), if \( \pi_i(p, E) \neq 0 \) then \( \min \{ \text{ht} p, n \} \leq i \). We use induction on \( i \). If \( \pi_0(p, E) \neq 0 \) then \( T_p \) is a direct summand of \( F_0 \). Hence, in view of 3.2, \( p \in \text{Att}_R(E) \). So there is \( q \in \text{Ass}_R(R) \) such that \( p \subseteq q \). Now we have \( qRq \in \text{Ass}_{R_q}(R_q) \) and

\[
\min \{ \text{ht} p, n \} \leq \min \{ \text{ht} q, n \} \leq \text{depth} R_q = 0.
\]

Assume inductively that \( k \geq 0 \) and the result has been proved (for all choices of \( R \) and \( E \) satisfying the hypothesis) when \( i = k \); let \( \pi_{k+1}(p, E) \neq 0 \). We may assume that \( p \not\subseteq Z(R) \). Suppose that \( x \in p - Z(R) \). It is easy to see that \( R/xR \) satisfies \( (S_{n-1}) \), and \( 0 : E x \) is an injective \( R/xR \)-module. By using 3.5, we have \( \pi_k(p/(x), 0 : E x) \neq 0 \). Hence, by the inductive hypothesis, \( \min \{ \text{ht} p/(x), n - 1 \} \leq k \). Thus \( \min \{ \text{ht} p, n \} \leq k + 1 \). The result follows by induction.
This is trivial.

(3)⇒(1). Assume that $p \in \text{Spec}(R)$ and $\text{depth } R_p = i$. In view of [9, Theorem 9.51], and using the fact that $E(R/p)$ is an injective cogenerator $R_p$-module, we have

$$0 \neq \text{Hom}_R(\text{Ext}_R^i(k(p), R_p), E(R/p)) \cong \text{Tor}_R^1(k(p), \text{Hom}_R(R_p, E(R/p))).$$

Hence, by using [7, Theorem 2.2], it follows that $\pi_i(p, E(R/p)) \neq 0$ so that $\min\{\text{ht } p, n\} \leq i = \text{depth } R_p$.

The next corollary is analogous to [11, Theorem 3.2] and provides an explicit description of the minimal flat resolution of an injective module over a Cohen–Macaulay ring.

**Corollary 3.9.** If $R$ is a Noetherian ring, then the following statements are equivalent:

1. $R$ is Cohen–Macaulay;
2. if $E$ is an injective $R$-module, then $\pi_i(p, E) \neq 0$ implies that $\text{ht } p \leq i$ for all prime ideals $p$ and all $i \geq 0$;
3. if $\pi_i(p, E(R/p)) \neq 0$, then $\text{ht } p \leq i$ for all prime ideals $p$ and all $i \geq 0$.

**Proof.** The proof is similar to that of 3.8.

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