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SYMMETRIC SPECIAL BISERIAL ALGEBRAS OF EUCLIDEAN TYPE

BY

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Abstract. We classify (up to Morita equivalence) all symmetric special biserial algebras of Euclidean type, by algebras arising from Brauer graphs.

Introduction and the main result. Throughout the paper K will denote a fixed algebraically closed field. By an *algebra* we mean a finite-dimensional K-algebra with identity, which we shall assume (without loss of generality) to be basic and connected. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules and by D the standard duality $\operatorname{Hom}_K(-,K)$ on $\operatorname{mod} A$. The *Cartan matrix* C_A of A is the matrix $(\dim_K \operatorname{Hom}_A(P_i,P_j))_{1\le i,j\le n}$ for a complete family $P_1,\dots,P_n$ of pairwise nonisomorphic indecomposable projective A-modules.

An algebra A is called *selfinjective* if $A \cong D(A)$ in $\operatorname{mod} A$, that is, the projective A-modules are injective. Further, A is called *symmetric* if A and $D(A)$ are isomorphic as A-bimodules. For a selfinjective algebra A, we denote by Γ_A^{s} the *stable Auslander–Reiten quiver* of A, obtained from the Auslander–Reiten quiver Γ_A of A by removing all projective modules and the arrows attached to them. We also note that if A is symmetric then the Auslander–Reiten translation $\tau_A = D\operatorname{Tr}$ in $\operatorname{mod} A$ is the square Ω_A^2 of the Heller syzygy operator Ω_A. An important class of selfinjective algebras is formed by the algebras of the form $\widehat{B}/G$, where $\widehat{B}$ is the *repetitive algebra* [8] (locally finite-dimensional, without identity)

$$\widehat{B} = \bigoplus_{m\in\mathbb{Z}} (B_m \oplus Q_m)$$

of an algebra B, where $B_m = B$ and $Q_m = D(B)$ for all $m\in\mathbb{Z}$, the multiplication in $\widehat{B}$ is defined by

$$(a_m,f_m)_m\cdot(b_m,g_m)_m = (a_mb_m, a_mg_m + f_mb_{m+1})_{m\in\mathbb{Z}}$$

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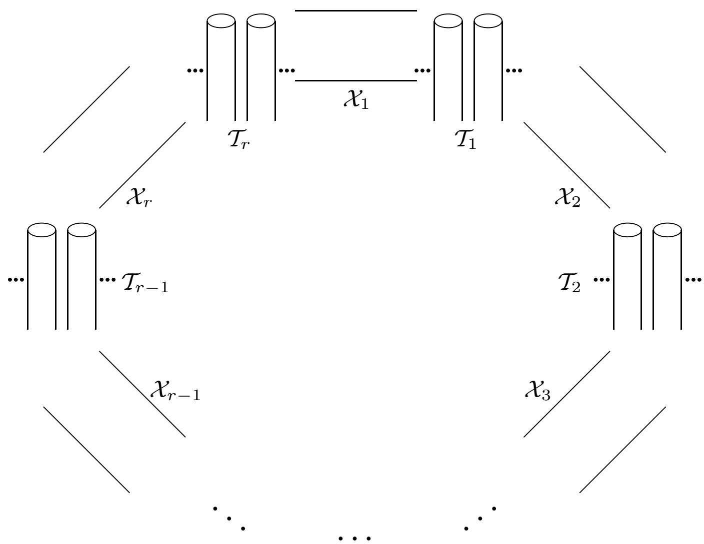
for $a_m, b_m \in B_m$, $f_m, g_m \in Q_m$, and G is an admissible group of K-automorphisms of $\widehat{B}$. In particular, if $\nu_{\widehat{B}} : \widehat{B} \to \widehat{B}$ is the *Nakayama automorphism* of $\widehat{B}$ given by the identity shifts $B_m \to B_{m+1}$ and $Q_m \to Q_{m+1}$, then the infinite cyclic group $(\nu_{\widehat{B}})$ generated by $\nu_{\widehat{B}}$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the *trivial extension* $T(B) = B \ltimes D(B)$ of B by $D(B)$, and is a symmetric algebra.

We are concerned with the problem of classifying all *selfinjective algebras of Euclidean type*, that is, of the form $\widehat{B}/G$, where B is a tilted algebra of Euclidean type $\Delta \in \{\widetilde{\mathbb{A}}_m, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ and G is an admissible group of K-automorphisms of $\widehat{B}$. It is known (see [2], [16]) that if $A = \widehat{B}/G$ and B is tilted of Euclidean type Δ then the stable Auslander–Reiten quiver Γ_A^{s} has the following "clock structure":



where $r \geq 1$ and for each $p \in \{1, \ldots, r\}$, $\mathcal{X}_p$ is of the form $\mathbb{Z}\Delta$ and $\mathcal{T}_p$ is a $\mathbb{P}_1(K)$-family of stable tubes. In fact, if A is symmetric then $r \leq 2$, and $r = 2$ if $A = T(B) = \widehat{B}/(\nu_{\widehat{B}})$. It has been proved in [12] that every symmetric algebra of Euclidean type $\Delta \in \{\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ is isomorphic to the trivial extension $T(B)$ of a (representation-infinite) tilted algebra B of type Δ. But this is not the case for the Euclidean types $\widetilde{\mathbb{A}}_m$ and $\widetilde{\mathbb{D}}_n$ (see [16, 2.6, 2.7]).

The aim of this paper is to describe all symmetric algebras of Euclidean types $\widetilde{\mathbb{A}}_m$, $m \geq 1$. It is known (see [3], [16]) that the class of selfinjective algebras of Euclidean type $\widetilde{\mathbb{A}}_m$ coincides with the class of representation-infinite special biserial algebras of polynomial growth. Recall that following [17] an algebra A is called *special biserial* if it is isomorphic to a bound quiver algebra KQ/I where the bound quiver (Q, I) satisfies the following conditions:

(SP1) The number of arrows in Q with a given source or sink is at most two.

(SP2) For any arrow α of Q, there is at most one arrow β and at most one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ are not in I.

We refer to [6] and [13] for the structure and representation theory of special biserial selfinjective algebras.

If K is of characteristic $p > 0$ and G is a finite group, we know by Dade [4], Janusz [9] and Kupisch [11] (see also [1]) that the representation-finite blocks of the group algebra KG are Morita equivalent to special biserial algebras arising from Brauer trees with one distinguished vertex. In fact, it was shown later in [10] and [15] that every symmetric special biserial algebra is Morita equivalent to a special biserial algebra arising from a Brauer graph which is locally embedded in the plane. Following this idea we associate (see Section 1 for details) to any Brauer tree T with two distinguished vertices v_1 and v_2 a symmetric special biserial algebra $\Lambda(T, v_1, v_2)$, and to any Brauer graph T with exactly one cycle a symmetric special biserial algebra $\Lambda'(T)$ (resp. $\Lambda''(T)$) according as the unique cycle in T has an odd (resp. even) number of edges. The following main results of the paper give a complete description of all symmetric algebras of Euclidean types $\widetilde{\mathbb{A}}_m$ (equivalently, symmetric special biserial algebras of Euclidean type).

THEOREM 1. *Let A be a basic connected algebra. Then the following conditions are equivalent*:

(i) *A is a symmetric algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ and the Cartan matrix of A is nonsingular.*

(ii) *A is isomorphic to an algebra of the form $\widehat{B}/(\varphi)$, where B is a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ and φ is a square root of the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$, but A is not isomorphic to the four-dimensional local algebra $K\langle x, y\rangle/(x^2, y^2, xy + yx)$ if $\operatorname{char} K \neq 2$.*

(iii) *A is isomorphic to an algebra of the form $\Lambda(T, v_1, v_2)$ for a Brauer tree T with two distinguished vertices v_1 and v_2, or to $\Lambda'(T)$ for a Brauer graph T having a unique cycle, and the cycle has an odd number of edges.*

THEOREM 2. *Let A be a basic connected algebra. Then the following conditions are equivalent*:

(i) *A is a symmetric algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ and the Cartan matrix of A is singular.*

(ii) *A is isomorphic to the trivial extension $T(B)$, where B is a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{A}}_m$.*

(iii) *A is isomorphic to an algebra of the form $\Lambda''(T)$, where T is a Brauer graph having a unique cycle, and the cycle has an even number of edges.*

As a consequence of our proofs we also obtain the following description of all weakly symmetric algebras of Euclidean types $\widetilde{\mathbb{A}}_m$ which are not symmetric.

COROLLARY 3. *Let A be a basic connected algebra. Then the following conditions are equivalent*:

(i) *A is a weakly symmetric but nonsymmetric algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ for some m.*

(ii) *A is isomorphic to the four-dimensional local algebra $K\langle x,y\rangle/(x^2,y^2, xy-\lambda yx)$ for some $\lambda \in K\setminus\{0,1\}$.*

Recall that an algebra A is called *weakly symmetric* if the socle $\operatorname{soc} P$ of any indecomposable projective A-module P is isomorphic to its top $P/\operatorname{rad} P$.

For general background concerning representation theory of algebras and selfinjective algebras applied here we refer to [1], [5], [6], [14] and [19].

1. Brauer quiver algebras. In this paper, by a *Brauer graph* we mean only (for a general definition see [10], [15]) a finite connected undirected graph T with at most one cycle, possibly with a loop or a double edge, together with a circular ordering of the edges issuing from each vertex, which we put in a concrete form by drawing T in the plane in such a way that the edges issuing from any vertex have the clockwise cyclic order. A Brauer graph T defines a *Brauer quiver* Q_T such that:

(a) Q_T is the union of (oriented) cycles.

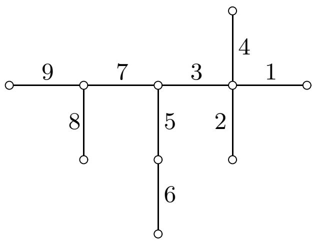
(b) Every vertex of Q_T belongs to exactly two cycles.

The vertices of Q_T are the edges of T, and there is an arrow $i\to j$ in Q_T if and only if the edges i and j have a common vertex v and j is the immediate successor of i in the circular ordering of the edges issuing from v. Therefore, the vertices of T correspond to the oriented cycles of Q_T.

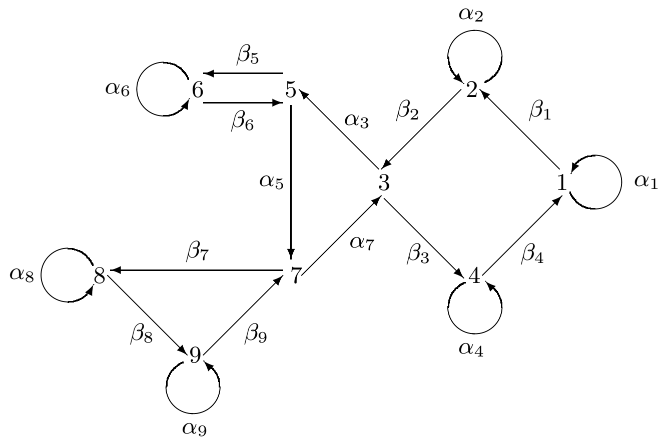
Let T be a Brauer tree. Then the simple cycles of the Brauer quiver Q_T may be divided into two camps, the *α-camp* and *β-camp*, in such a way that any two cycles which intersect nontrivially belong to different camps. We denote by α_i (resp. β_i) the arrow of the α-camp (resp. β-camp) of Q_T starting at the vertex i, and by $\alpha(i)$ (resp. $\beta(i)$) the end vertex of α_i (resp. β_i). We also denote by A_i (resp. B_i) the cycle from i to i going once around the α-cycle (resp. β-cycle) through i, that is,

$$A_i=\alpha_i\alpha_{\alpha(i)}\ldots\alpha_{\alpha^{-1}(i)},\qquad B_i=\beta_i\beta_{\beta(i)}\ldots\beta_{\beta^{-1}(i)}.$$

EXAMPLE 1.1. Let T be a Brauer tree of the form



Then Q_T is (up to choice of α-camps and β-camps) of the form



Let T be a Brauer tree with a set $V = \{v_1, \ldots, v_t\}$ of distinguished (pairwise different) vertices, marked by $\bullet$. Then the associated Brauer quiver Q_T has *exceptional cycles* given by the edges of T issuing from the vertices $v_1, \ldots, v_t$. We define $\Lambda(T, V)$ as the bound quiver algebra $KQ_T/I(T, V)$, where KQ_T is the path algebra of the quiver Q_T and $I(T, V)$ is the ideal in KQ_T generated by:

(1) $\alpha_i\beta_{\alpha(i)}$, $\beta_i\alpha_{\beta(i)}$ for all vertices i of Q_T,

(2) $A_j - B_j$ if neither the α-cycle nor the β-cycle through the vertex j are exceptional,

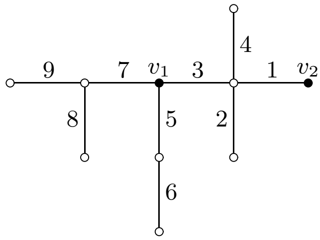
(3) $A_j^2 - B_j$ if the α-cycle through j is exceptional but the β-cycle through j is not,

(4) $A_j - B_j^2$ if the β-cycle through j is exceptional but the α-cycle through j is not,

(5) $A_j^2 - B_j^2$ if the α-cycle and β-cycle through j are exceptional.

We write frequently $\Lambda(T, v_1, \ldots, v_t)$ instead of $\Lambda(T, V)$, and $I(T, v_1, \ldots, v_t)$ instead of $I(T, V)$.

EXAMPLE 1.2. Let T be the following Brauer tree with two distinguished vertices v_1 and v_2:



Then the algebra $\Lambda(T, v_1, v_2)$ is given by the quiver Q_T (described in 1.1) and the ideal $I(T, v_1, v_2)$ in KQ_T generated by: $\alpha_1\beta_1$, $\beta_1\alpha_2$, $\alpha_2\beta_2$, $\beta_2\alpha_3$, $\alpha_7\beta_3$, $\beta_3\alpha_4$, $\alpha_4\beta_4$, $\beta_4\alpha_1$, $\alpha_3\beta_5$, $\beta_5\alpha_6$, $\alpha_6\beta_6$, $\beta_6\alpha_5$, $\alpha_5\beta_7$, $\beta_9\alpha_7$, $\alpha_8\beta_8$, $\beta_7\alpha_8$, $\alpha_9\beta_9$, $\beta_8\alpha_9$, $\alpha_1^2 - \beta_1\beta_2\beta_3\beta_4$, $\alpha_2 - \beta_2\beta_3\beta_4\beta_1$, $\alpha_4 - \beta_4\beta_1\beta_2\beta_3$, $(\alpha_3\alpha_5\alpha_7)^2 - \beta_3\beta_4\beta_1\beta_2$, $(\alpha_5\alpha_7\alpha_3)^2 - \beta_5\beta_6$, $(\alpha_7\alpha_3\alpha_5)^2 - \beta_7\beta_8\beta_9$, $\alpha_6 - \beta_6\beta_5$, $\alpha_8 - \beta_8\beta_9\beta_7$, $\alpha_9 - \beta_9\beta_7\beta_8$.

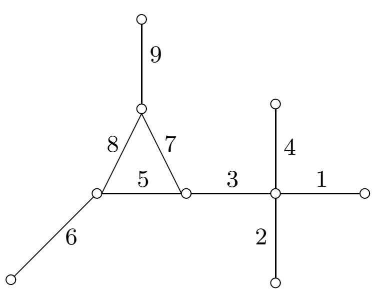
Let T be a Brauer graph with exactly one cycle and let the cycle have an odd number of edges. Assume first that the cycle is not a loop (so has at least two vertices). We fix a vertex on the cycle and denote by γ_i the arrow of the associated simple cycle of Q_T starting at a vertex i, and by $\gamma(i)$ the end vertex of γ_i. Then the remaining (simple) cycles of Q_T may be divided into two camps, the α-camp and β-camp, in such a way that any two cycles which intersect nontrivially belong to different camps. We define the cycles A_i and B_i as above. We also denote by C_i the simple cycle from i to i going once around the γ-cycle through i, that is,

$$C_i = \gamma_i\gamma_{\gamma(i)}\dots\gamma_{\gamma^{-1}(i)}.$$

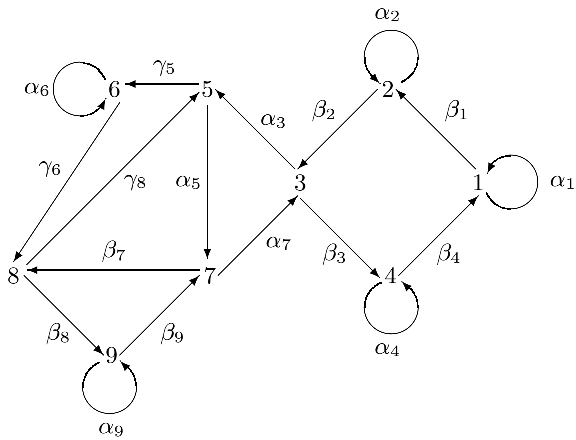
We define $\Lambda'(T)$ as the bound quiver algebra $KQ_T/I'(T)$, where $I'(T)$ is the ideal generated by:

(1) $\alpha_i\beta_{\alpha(i)}$, $\beta_i\alpha_{\beta(i)}$, $\alpha_i\gamma_{\alpha(i)}$, $\gamma_i\alpha_{\gamma(i)}$, $\gamma_i\beta_{\gamma(i)}$, $\beta_i\gamma_{\beta(i)}$ for all vertices i of Q_T,
(2) $A_j - B_j$ if j is the intersection of an α-cycle and a β-cycle,
(3) $A_j - C_j$ if j is the intersection of an α-cycle and a γ-cycle,
(4) $B_j - C_j$ if j is the intersection of a β-cycle and a γ-cycle.

Example 1.3. Let T be the following Brauer graph with one cycle:



Then Q_T is the quiver



and $\Lambda'(T)$ is given by the above quiver and the ideal $I'(T)$ generated by: $\alpha_1\beta_1$, $\beta_1\alpha_2$, $\alpha_2\beta_2$, $\beta_2\alpha_3$, $\alpha_7\beta_3$, $\beta_3\alpha_4$, $\alpha_4\beta_4$, $\beta_4\alpha_1$, $\alpha_3\gamma_5$, $\gamma_5\alpha_6$, $\alpha_5\beta_7$, $\beta_9\alpha_7$, $\alpha_9\beta_9$, $\beta_8\alpha_9$, $\beta_7\gamma_8$, $\gamma_8\alpha_5$, $\gamma_6\beta_8$, $\alpha_6\gamma_6$, $\alpha_1-\beta_1\beta_2\beta_3\beta_4$, $\alpha_2-\beta_2\beta_3\beta_4\beta_1$, $\alpha_3\alpha_5\alpha_7-\beta_3\beta_4\beta_1\beta_2$, $\alpha_4-\beta_4\beta_1\beta_2\beta_3$, $\alpha_5\alpha_7\alpha_3-\gamma_5\gamma_6\gamma_8$, $\alpha_6-\gamma_6\gamma_8\gamma_5$, $\alpha_7\alpha_3\alpha_5-\beta_7\beta_8\beta_9$, $\gamma_8\gamma_5\gamma_6-\beta_8\beta_9\beta_7$, $\alpha_9-\beta_9\beta_7\beta_8$.

Now assume that the cycle is a loop (so has only one vertex). We fix this vertex on the loop of T. The associated (nonsimple) cycle of Q_T is a composition of two simple shorter cycles with exactly one intersection vertex. We denote by γ_i (resp. δ_i) the arrows of the first (resp. second) of them starting at a vertex i, and by $\gamma(i)$ (resp. $\delta(i)$) the end vertex of γ_i (resp. δ_i). Then the remaining (simple) cycles of Q_T may be divided into two camps, the α-camp and β-camp, in such a way that any two cycles which intersect nontrivially belong to different camps. Again, we denote by A_i (resp. B_i) the cycle from i to i going once around the α-cycle (resp. β-cycle) through i. We denote by v the intersection vertex of the γ-cycle and δ-cycle and by C_v (resp. D_v) the simple cycle from v to v going once around the γ-cycle (resp. δ-cycle) through v, that is,

$$C_v=\gamma_v\gamma_{\gamma(v)}\dots\gamma_{\gamma^{-1}(v)},\qquad D_v=\delta_v\delta_{\delta(v)}\dots\delta_{\delta^{-1}(v)}.$$

We define $\Lambda'(T)$ as the bound quiver algebra $KQ_T/I'(T)$, where $I'(T)$ is the ideal generated by:

(1) $\alpha_i\beta_{\alpha(i)}$, $\beta_i\alpha_{\beta(i)}$, $\alpha_i\gamma_{\alpha(i)}$, $\gamma_i\alpha_{\gamma(i)}$, $\gamma_i\beta_{\gamma(i)}$, $\beta_i\gamma_{\beta(i)}$, $\alpha_i\delta_{\alpha(i)}$, $\delta_i\alpha_{\delta(i)}$, $\delta_i\beta_{\delta(i)}$, $\beta_i\delta_{\beta(i)}$ for all vertices i of Q_T,

(2) A_j-B_j if j is the intersection of an α-cycle and a β-cycle,

(3) $A_j-\gamma_j\gamma_{\gamma(j)}\dots\gamma_{\gamma^{-1}(v)}D_v\gamma_v\dots\gamma_{\gamma^{-1}(j)}$ if j is the intersection of an α-cycle and the γ-cycle,

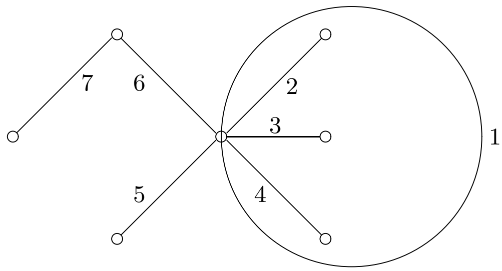
(4) $A_j-\delta_j\delta_{\delta(j)}\dots\delta_{\delta^{-1}(v)}C_v\delta_v\dots\delta_{\delta^{-1}(j)}$ if j is the intersection of an α-cycle and the δ-cycle,

(5) $B_j - \delta_j \delta_{\delta(j)} \ldots \delta_{\delta^{-1}(v)} C_v \delta_v \ldots \delta_{\delta^{-1}(j)}$ if j is the intersection of a β-cycle and the δ-cycle,

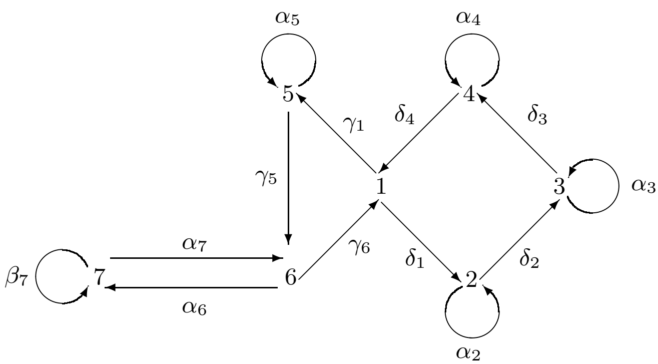
(6) $B_j - \gamma_j \gamma_{\gamma(j)} \ldots \gamma_{\gamma^{-1}(v)} D_v \gamma_v \ldots \gamma_{\gamma^{-1}(j)}$ if j is the intersection of a β-cycle and the γ-cycle,

(7) $\gamma_{\gamma^{-1}(v)}\gamma_v$, $\delta_{\delta^{-1}(v)}\delta_v$, $C_v D_v - D_v C_v$.

Example 1.4. Let T be the following Brauer graph with one loop:



Then Q_T is the quiver



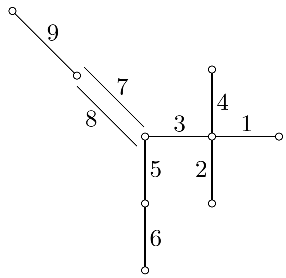
and $\Lambda'(T)$ is given by the above quiver and the ideal $I'(T)$ generated by: $\delta_1\alpha_2$, $\alpha_2\delta_2$, $\delta_2\alpha_3$, $\alpha_3\delta_3$, $\delta_3\alpha_4$, $\alpha_4\delta_4$, $\gamma_1\alpha_5$, $\alpha_5\gamma_5$, $\gamma_5\alpha_6$, $\alpha_6\beta_7$, $\beta_7\alpha_7$, $\alpha_7\gamma_6$, $\delta_4\delta_1$, $\gamma_6\gamma_1$, $\alpha_2 - \delta_2\delta_3\delta_4\gamma_1\gamma_5\gamma_6\delta_1$, $\alpha_3 - \delta_3\delta_4\gamma_1\gamma_5\gamma_6\delta_1\delta_2$, $\alpha_4 - \delta_4\gamma_1\gamma_5\gamma_6\delta_1\delta_2\delta_3$, $\alpha_5 - \gamma_5\gamma_6\delta_1\delta_2\delta_3\delta_4\gamma_1$, $\alpha_6\alpha_7 - \gamma_6\delta_1\delta_2\delta_3\delta_4\gamma_1\gamma_5$, $\beta_7 - \alpha_7\alpha_6$ $\delta_1\delta_2\delta_3\delta_4\gamma_1\gamma_5\gamma_6 - \gamma_1\gamma_5\gamma_6\delta_1\delta_2\delta_3\delta_4$.

Let T be a Brauer graph with exactly one cycle, and let the cycle have an even number of edges. Then the simple cycles of the Brauer quiver Q_T may be divided into two camps, the α-camp and β-camp, in such a way that any two cycles which intersect nontrivially belong to different camps. We define A_i and B_i as before. We let $\Lambda''(T)$ be the bound quiver algebra $KQ_T/I''(T)$, where $I''(T)$ is generated by:

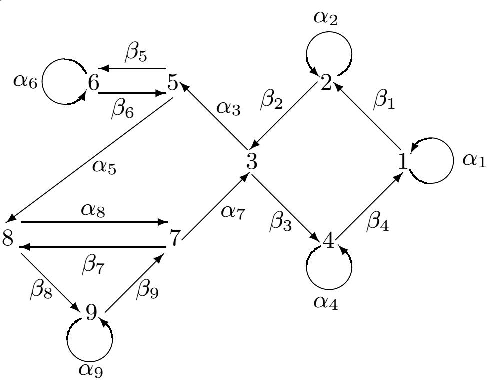
(1) $\alpha_i\beta_{\alpha(i)}$, $\beta_i\alpha_{\beta(i)}$ for all vertices i of Q_T,

(2) $A_j - B_j$ if j is the intersection of an α-cycle and a β-cycle.

EXAMPLE 1.5. Let T be the following Brauer graph with one cycle:



Then Q_T is the quiver



and $\Lambda''(T)$ is given by the above quiver and the ideal $I''(T)$ generated by: $\alpha_1\beta_1$, $\beta_1\alpha_2$, $\alpha_2\beta_2$, $\beta_2\alpha_3$, $\alpha_7\beta_3$, $\beta_3\alpha_4$, $\alpha_4\beta_4$, $\beta_4\alpha_1$, $\alpha_3\beta_5$, $\beta_5\alpha_6$, $\alpha_6\beta_6$, $\beta_6\alpha_5$, $\alpha_5\beta_8$, $\beta_8\alpha_9$, $\alpha_9\beta_9$, $\beta_9\alpha_7$, $\alpha_8\beta_7$, $\beta_7\alpha_8$, $\alpha_1 - \beta_1\beta_2\beta_3\beta_4$, $\alpha_2 - \beta_2\beta_3\beta_4\beta_1$, $\alpha_3\alpha_5\alpha_8\alpha_7 - \beta_3\beta_4\beta_1\beta_2$, $\alpha_4 - \beta_4\beta_1\beta_2\beta_3$, $\alpha_5\alpha_8\alpha_7\alpha_3 - \beta_5\beta_6$, $\alpha_6 - \beta_6\beta_5$, $\alpha_7\alpha_3\alpha_5\alpha_8 - \beta_7\beta_8\beta_9$, $\alpha_8\alpha_7\alpha_3\alpha_5 - \beta_8\beta_9\beta_7$, $\alpha_9 - \beta_9\beta_7\beta_8$.

2. Cartan matrix of the algebra $\Lambda(T, V)$. Let T be a Brauer tree with e edges, V a set of distinguished vertices of T, and $t = |V|$. The main aim of this section is to prove the following formula for the determinant of the Cartan matrix of $\Lambda(T, V)$.

PROPOSITION 2.1. *In the above notation, we have*

$$\det C_{\Lambda(T,V)} = 2^t(e - t + 1) + 2^{t-1}t.$$

We need a technical lemma. For integers $x, a_1, \ldots, a_n$ we denote by $[x, a_1, \ldots, a_n]$ the $n \times n$-matrix

$$\begin{bmatrix} a_1 + x & x & \ldots & x \\ x & a_2 + x & \ldots & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \ldots & a_n + x \end{bmatrix}.$$

LEMMA 2.2. *We have the equality*

$$\det[x, a_1, \ldots, a_n] = a_1 \ldots a_n + x \sum_{i=1}^{n} a_1 \ldots \widehat{a}_i \ldots a_n,$$

where $\widehat{a}_i = 1$.

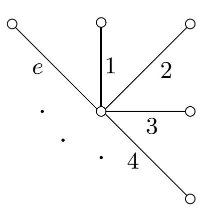
Proof. We proceed by induction on n. For $n = 1$, the claim is obvious. For $n \geq 2$, we have the equalities

$$\begin{aligned}
&\det[x, a_1, \ldots, a_n, a_{n+1}] \\
&= (a_1 + x) \det[x, a_2, \ldots, a_n, a_{n+1}] - x \sum_{i=2}^{n+1} \det[x, a_2, \ldots, \widehat{a}_i, \ldots, a_n, a_{n+1}] \\
&= a_1 \Big(a_2 \ldots a_{n+1} + x \sum_{i=2}^{n+1} a_2 \ldots \widehat{a}_i \ldots a_{n+1} \Big) \\
&\quad + x \Big(a_2 \ldots a_{n+1} + x \sum_{i=2}^{n+1} a_2 \ldots \widehat{a}_i \ldots a_{n+1} \Big) - x^2 \sum_{i=2}^{n+1} a_2 \ldots \widehat{a}_i \ldots a_{n+1} \\
&= a_1 a_2 \ldots a_{n+1} + x \sum_{i=2}^{n+1} a_1 a_2 \ldots \widehat{a}_i \ldots a_{n+1} + x a_2 \ldots a_{n+1} \\
&\quad + x^2 \sum_{i=2}^{n+1} a_2 \ldots \widehat{a}_i \ldots a_{n+1} - x^2 \sum_{i=2}^{n+1} a_2 \ldots \widehat{a}_i \ldots a_{n+1} \\
&= a_1 a_2 \ldots a_{n+1} + x \sum_{i=1}^{n+1} a_2 \ldots \widehat{a}_i \ldots a_{n+1}. \quad \blacksquare
\end{aligned}$$

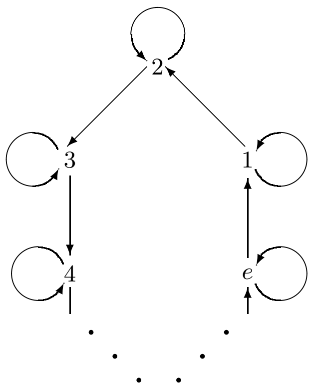
Proof of Proposition 2.1. We argue in several steps, by induction on the number k of vertices of T having at least two neighbours.

(1) Assume $k = 0$. Then the Brauer tree T consists of one edge, $0 \leq t \leq 2$, and hence $C_{\Lambda(T,V)}$ is of the form either $[2]$, $[3]$, or $[4]$. Since $2 = 2^0(1 - 0 + 1) + 0$, $3 = 2^1(1 - 1 + 1) + 1$ and $4 = 2^2(1 - 2 + 1) + 2^1 \cdot 2$, the required formula holds.

(2) Assume $k = 1$. Then T is a star of the form



and Q_T is of the form



We have two cases to consider.

(a) Assume that the middle of the star T is a distinguished vertex. Then $C_{\Lambda(T,V)}$ is of the form

$$\overbrace{\hphantom{4 \quad \ddots \quad 4}}^{t-1}$$
$$\begin{bmatrix} 4 & & & & & 2 \\ & \ddots & & & & \\ & & 4 & & & \\ & & & 3 & & \\ & & & & \ddots & \\ 2 & & & & & 3 \end{bmatrix}$$

with 2's everywhere off the main diagonal. From Lemma 2.2 we have the equalities

$$\begin{aligned} \det C_{\Lambda(T,V)} &= \det[2, \underbrace{2, \ldots, 2}_{t-1}, \underbrace{1, \ldots, 1}_{e-t+1}] \\ &= 2^{t-1} + 2(2^{t-1}(e-t+1) + 2^{t-2}(t-1)) \\ &= 2^{t-1} + 2^t(e-t+1) + 2^{t-1}(t-1) = 2^t(e-t+1) + 2^{t-1}t. \end{aligned}$$

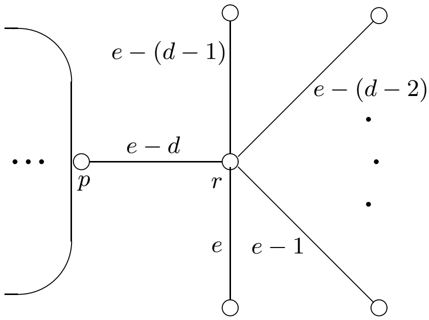
(b) Assume that the middle of the star T is an ordinary vertex. Then $C_{\Lambda(T,V)}$ is of the form

$$\overbrace{\hphantom{3 \quad \ddots \quad 3}}^{t}$$
$$\begin{bmatrix} 3 & & & & & 1 \\ & \ddots & & & & \\ & & 3 & & & \\ & & & 2 & & \\ & & & & \ddots & \\ 1 & & & & & 2 \end{bmatrix}$$

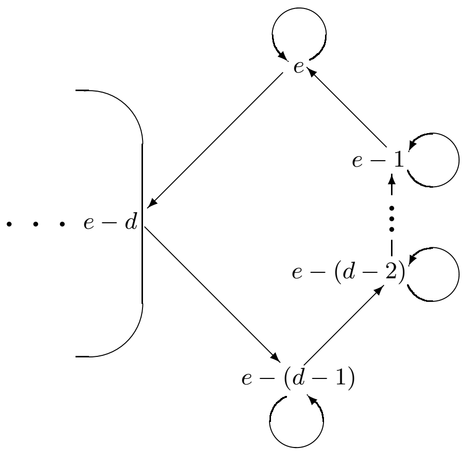
with 1's off the main diagonal. From Lemma 2.2 we have

$$\begin{aligned} \det C_{\Lambda(T,V)} &= \det[1, \underbrace{2, \ldots, 2}_{t}, \underbrace{1, \ldots, 1}_{e-t}] \\ &= 2^t + 2^t(e-t) + 2^{t-1}t = 2^t(e-t+1) + 2^{t-1}t. \end{aligned}$$

(3) Assume $k \geq 2$ and the required formula holds for the Brauer trees having at most $k-1$ vertices with at least two neighbours. Let T be a Brauer tree having k vertices with at least two neighbours and let r be the last vertex of T which is not an end, that is, T is of the form

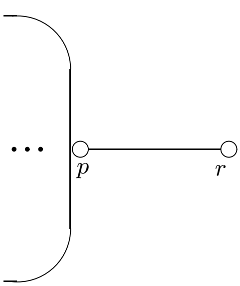


Denote by p the neighbour of the vertex r which connects r with the second part of T. Then Q_T is of the form

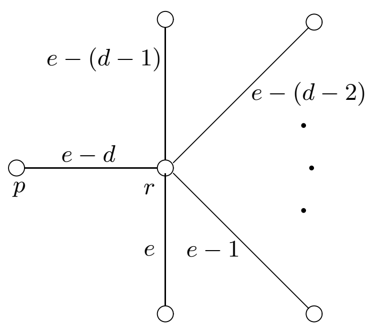


Consider the following two subtrees of T:

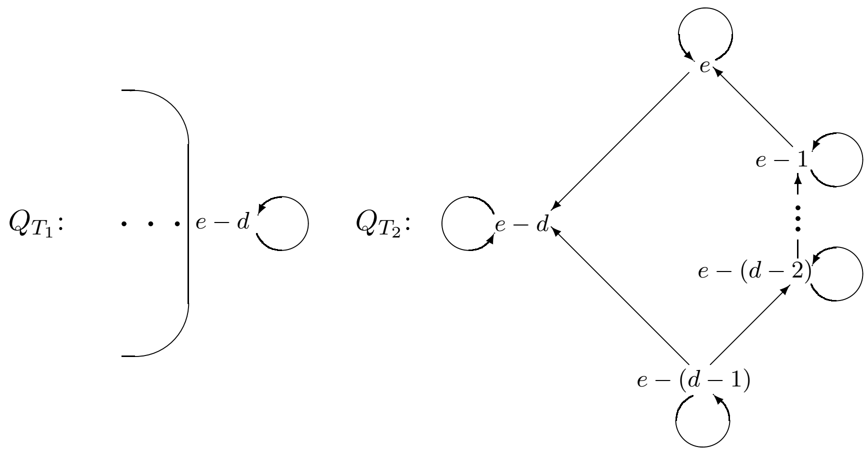
T_1:



T_2:



The Brauer quivers Q_{T_1} and Q_{T_2} are



Then the Cartan matrix $C_{\Lambda(T,V)}$ is of the block form

$$\begin{bmatrix} X & \vdots & 0 \\ \cdots & z & \cdots \\ 0 & \vdots & Y \end{bmatrix}$$

where X is the Cartan matrix of $\Lambda(T_1, V_1)$, Y is the Cartan matrix of $\Lambda(T_2, V_2)$ and z is the only common nonzero coefficient of the matrices X and Y. Then, applying [1, Lemma 24.4], we obtain

$$\det C_{\Lambda(T,V)} = \det(X)\det(Y_0) + \det(X_0)\det(Y) - z\det(X_0)\det(Y_0),$$

where X_0 is obtained from X by erasing the last row and last column, and Y_0 is obtained from Y by erasing the first row and first column.

Assume that T_2 has m distinguished vertices. We have four cases to consider.

(a) Assume that p and r are ordinary vertices. Then $z = 2$, and using our inductive assumption, we obtain

$$\begin{aligned} \det C_{\Lambda(T,V)} = {} & \det(X)\det(Y_0) + \det(X_0)\det(Y) - 2\det(X_0)\det(Y_0) \\ = {} & (2^{t-m}(e-d-(t-m)+1) + 2^{t-m-1}(t-m)) \\ & \times (2^m(d-m+1) + 2^{m-1}m) \\ & + (2^{t-m}(e-d-1-(t-m)+1) + 2^{t-m-1}(t-m)) \\ & \times (2^m(d+1-m+1) + 2^{m-1}m) \\ & - 2(2^{t-m}(e-d-1-(t-m)+1) + 2^{t-m-1}(t-m)) \\ & \times (2^m(d-m+1) + 2^{m-1}m) \\ = {} & 2^t(e-t+1) + 2^{t-1}t. \end{aligned}$$

(b) Assume that p is an ordinary vertex and r is a distinguished vertex. Then $z=3$, and using our inductive assumption, we obtain

$$\begin{aligned}\det C_{\Lambda(T,V)} &= (2^{t-m+1}(e-d-(t-m+1)+1)+2^{t-m}(t-m+1))\\ &\quad\times(2^m(d-m+1)+2^{m-1}m)\\ &\quad+(2^{t-m}(e-d-1-(t-m)+1)+2^{t-m-1}(t-m))\\ &\quad\times(2^m(d+1-m+1)+2^{m-1}m)\\ &\quad-3(2^{t-m}(e-d-1-(t-m)+1)+2^{t-m-1}(t-m))\\ &\quad\times(2^m(d-m+1)+2^{m-1}m)\\ &= 2^t(e-t+1)+2^{t-1}t.\end{aligned}$$

(c) Assume that p is a distinguished vertex and r is an ordinary vertex. Then $z=3$, and using our inductive assumption, we obtain

$$\begin{aligned}\det C_{\Lambda(T,V)} &= (2^{t-m}(e-d-(t-m)+1)+2^{t-m-1}(t-m))\\ &\quad\times(2^m(d-m+1)+2^{m-1}m)\\ &\quad+(2^{t-m}(e-d-1-(t-m)+1)+2^{t-m-1}(t-m))\\ &\quad\times(2^{m+1}(d+1-(m+1)+1)+2^m(m+1))\\ &\quad-3(2^{t-m}(e-d-1-(t-m)+1)+2^{t-m-1}(t-m))\\ &\quad\times(2^m(d-m+1)+2^{m-1}m)\\ &= 2^t(e-t+1)+2^{t-1}t.\end{aligned}$$

(d) Assume finally that p and r are distinguished vertices. Then $z=4$, and invoking our inductive assumption, we obtain

$$\begin{aligned}\det C_{\Lambda(T,V)} &= (2^{t-m+1}(e-d-(t-m+1)+1)+2^{t-m}(t-m+1))\\ &\quad\times(2^m(d-m+1)+2^{m-1}m)\\ &\quad+(2^{t-m}(e-d-1-(t-m)+1)+2^{t-m-1}(t-m))\\ &\quad\times(2^{m+1}(d+1-(m+1)+1)+2^m(m+1))\\ &\quad-4(2^{t-m}(e-d-1-(t-m)+1)+2^{t-m-1}(t-m))\\ &\quad\times(2^m(d-m+1)+2^{m-1}m)\\ &= 2^t(e-t+1)+2^{t-1}t. \ \blacksquare\end{aligned}$$

We record some immediate consequences of Proposition 2.1:

COROLLARY 2.3. *Let T be a Brauer tree with two distinguished vertices v_1 and v_2 and with e edges. Then*

$$\det C_{\Lambda(T,v_1,v_2)} = 4e.$$

COROLLARY 2.4. *Let T be a Brauer tree with one distinguished vertex v and with e edges. Then*

$$\det C_{\Lambda(T,v)} = 2e + 1.$$

COROLLARY 2.5. *Let T be a Brauer tree without distinguished vertices and with e edges. Then*

$$\det C_{\Lambda(T)} = e + 1.$$

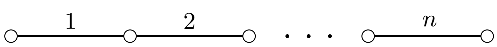
3. Cartan matrices of the algebras $\Lambda'(T)$ and $\Lambda''(T)$. The aim of this section is to prove the following formulas on the determinant of the Cartan matrices of the algebras $\Lambda'(T)$ and $\Lambda''(T)$.

PROPOSITION 3.1. *Let T be a Brauer graph with exactly one cycle. Then*:

(1) $\det C_{\Lambda'(T)} = 4$ *if the number of edges on the cycle is odd.*
(2) $\det C_{\Lambda''(T)} = 0$ *if the number of edges on the cycle is even.*

In order to prove the proposition we need several technical facts.

Let $n \geq 4$. Denote by $E_0(n)$ the Cartan matrix of the algebra $\Lambda(T) = \Lambda(T, \emptyset)$, where T is a tree, without distinguished vertices, of the shape



We define two square $n \times n$ matrices:

$$E_1(n) = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{bmatrix},$$

$$E_2(n) = \begin{bmatrix} 1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

LEMMA 3.2. *In the above notation*:

(1) $\det E_1(n) = (-1)^{n+1} + n$,
(2) $\det E_2(n) = 1 + (-1)^{n+1} n$.

Proof. (1) Applying the Laplace formula to the first row of $E_1(n)$, we obtain

$$\det E_1(n) = \det E_0(n-1) - \det D,$$

where

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{bmatrix}.$$

It is easy to check that $\det D = (-1)^n$. Then we conclude from Corollary 2.5 that $\det E_1(n) = n - (-1)^n = (-1)^{n+1} + n$.

(2) Applying the Laplace formula to the first column of $E_2(n)$, we obtain

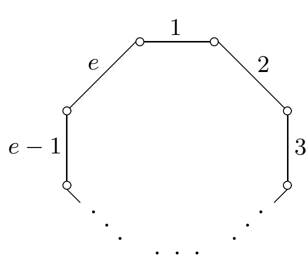
$$\det E_2(n) = \det D + (-1)^{n+1} \det E_0(n-1),$$

where

$$D = \begin{bmatrix} 1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

It is clear that $\det D = 1$. Then we conclude from Corollary 2.5 that $\det E_2(n) = 1 + (-1)^{n+1} n$. ■

LEMMA 3.3. *Let T be the Brauer graph with exactly one cycle of the form*



Then:

(1) $\det C_{\Lambda'(T)} = 4$ *if the cycle has an odd number of edges.*
(2) $\det C_{\Lambda''(T)} = 0$ *if the cycle has an even number of edges.*

Proof. We have four cases to consider.

(a) Assume $e = 1$. Then T consists of one loop and hence $C_{\Lambda'(T)}$ is of the form $[4]$. So $\det C_{\Lambda'(T)} = 4$.

(b) Assume $e = 2$. Then T is a cycle having two edges and $C_{\Lambda''(T)} = \left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]$. Hence $\det C_{\Lambda''(T)} = 0$.

(c) Assume $e = 3$. Then

$$C_{\Lambda'(T)} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

and hence $\det C_{\Lambda'(T)} = 4$.

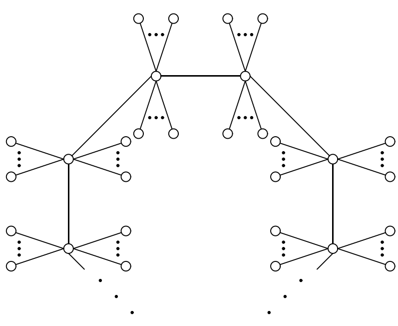
(d) Assume $e \geq 4$. Set $A(e) = \Lambda'(T)$ for e odd and $A(e) = \Lambda''(T)$ for e even. Then

$$C_{A(e)} = \begin{bmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{bmatrix}.$$

We apply the Laplace formula to the first row and Lemma 3.2 to obtain

$$\begin{aligned} \det C_{A(e)} &= 2\det E_0(e-1) - \det E_1(e-1) + (-1)^{e+1}\det E_2(e-1) \\ &= 2e - ((-1)^e + e - 1) + (-1)^{e+1}(1 + (-1)^e(e-1)) \\ &= 2 + 2(-1)^{e+1} = \begin{cases} 4 & \text{if } e \text{ is odd,} \\ 0 & \text{if } e \text{ is even.} \end{cases} \quad \blacksquare \end{aligned}$$

Let T be a Brauer graph (with exactly one cycle) of the form

($\star$) 

Denote by e the number of edges of T, by $\mathcal{R}$ the unique cycle in T, and by t the number of edges in $\mathcal{R}$. For each vertex v of T we denote by $l(v)$ the number of edges having v as one of its ends. Define

$$s = \max\{l(v) \mid v \text{ is a vertex of } \mathcal{R}\}.$$

Then the Cartan matrix of $A(e) = \Lambda'(T)$ (for t odd) or $A(e) = \Lambda''(T)$ (for t even) is of the form

$$C_{A(e)} = \left[\begin{array}{cccccccccc} 2 & 1 & 1 & \dots & 1 & \overset{s}{1} & a_1 & \dots & a_{e-s} \\ 1 & 2 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 2 & \dots & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 2 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 2 & b_1 & \dots & b_{e-s} \\ a_1 & 0 & 0 & \dots & 0 & b_1 & * & \dots & * \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{e-s} & 0 & 0 & \dots & 0 & b_{e-s} & * & \dots & * \end{array}\right],$$

where $a_i, b_j \in \{0,1\}$ for $i,j = 1,\dots,e-s$. Let $C_{A(e)} = (\alpha_{i,j})_{i,j=1}^{e}$. For $t \geq 3$ and $s \geq 2$, we define the matrix $E_3(e) = (\gamma_{i,j})_{i,j=1}^{e}$, where

$$\gamma_{i,j} = \begin{cases} \alpha_{i,j} & \text{if } i \neq 1 \text{ or } (i = 1 \text{ and } 2 \leq j \leq s), \\ 1 & \text{if } i = j = 1, \\ 0 & \text{if } i = 1 \text{ and } s+1 \leq j \leq e. \end{cases}$$

Thus

$$E_3(e) = \left[\begin{array}{cccccccccc} 1 & 1 & 1 & \dots & 1 & \overset{s}{1} & a_1 & \dots & a_{e-s} \\ 1 & 2 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 2 & \dots & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 2 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 2 & b_1 & \dots & b_{e-s} \\ 0 & 0 & 0 & \dots & 0 & b_1 & * & \dots & * \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & b_{e-s} & * & \dots * & * \end{array}\right].$$

LEMMA 3.4.

$$\det E_3(e) = \begin{cases} 2 & \textit{if } t \textit{ is odd}, \\ 0 & \textit{if } t \textit{ is even}. \end{cases}$$

Proof. We prove the lemma by induction on the number k of edges of T which are not in $\mathcal{R}$. If $k = 0$ then

$$E_3(e) = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 & 1 \\ 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{bmatrix}.$$

We apply the Laplace formula to the first row to obtain

$$\det E_3(e) = \det E_0(e-1) - \det E_0(e-2) + (-1)^{e+1} \det D,$$

where

$$D = \begin{bmatrix} 1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

Since $t = e$ and $\det D = 1$, from Corollary 2.5 we have

$$\begin{aligned} \det E_3(e) &= e - (e-1) + (-1)^{e+1} = 1 + (-1)^{e+1} \\ &= \begin{cases} 2 & \text{if } t \text{ is odd,} \\ 0 & \text{if } t \text{ is even.} \end{cases} \end{aligned}$$

Assume $k \geq 1$ and the lemma holds for all Brauer graphs of the form $(\star)$ having $k-1$ edges which are not in $\mathcal{R}$. Let T be a Brauer graph of the form $(\star)$ having k edges which are not in $\mathcal{R}$. Then

$$\det E_3(e) = \det \begin{bmatrix} 1 & 0 & 1 & \dots & 1 & \overset{s}{1} & a_1 & \dots & a_{e-s} \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 2 & \dots & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 0 & 1 & \dots & 2 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 1 & 2 & b_1 & \dots & b_{e-s} \\ 0 & 0 & 0 & \dots & 0 & b_1 & * & \dots & * \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & b_{e-s} & * & \dots & * \end{bmatrix}$$

$$= \det E_3(e-1) = \begin{cases} 2 & \text{if } t \text{ is odd,} \\ 0 & \text{if } t \text{ is even.} \end{cases} \blacksquare$$

For the Cartan matrix $C_{A(e)}$, $t \geq 3$ and $s \geq 2$, we define $E_4(e) = (\delta_{i,j})_{i,j=1}^{e}$, where

$$\delta_{i,j} = \begin{cases} \alpha_{i,j} & \text{if } i \neq s \text{ or } (i = s \text{ and } 2 \leq j \leq s), \\ 1 & \text{if } i = j = s, \\ 0 & \text{if } i = s \text{ and } s+1 \leq j \leq e. \end{cases}$$

Then $E_4(e)$ is of the form

$$\begin{bmatrix} 2 & 1 & 1 & \dots & 1 & \overset{s}{1} & a_1 & \dots & a_{e-s} \\ 1 & 2 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 2 & \dots & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 2 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 & b_1 & \dots & b_{e-s} \\ a_1 & 0 & 0 & \dots & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{e-s} & 0 & 0 & \dots & 0 & 0 & * & \dots & * \end{bmatrix}.$$

Lemma 3.5.

$$\det E_4(e) = \begin{cases} 2 & \textit{if } t \textit{ is odd}, \\ 0 & \textit{if } t \textit{ is even}. \end{cases}$$

Proof. We have

$$\det E_4(e) = \det \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & \overset{s}{1} & b_1 & \dots & b_{e-s} \\ 1 & 2 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 2 & \dots & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 2 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 2 & a_1 & \dots & a_{e-s} \\ 0 & 0 & 0 & \dots & 0 & a_1 & * & \dots & * \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{e-s} & * & \dots & * \end{bmatrix}$$

$$= (-1)^{s-2} \det E_3(e) = \begin{cases} 2 & \text{if } t \text{ is odd}, \\ 0 & \text{if } t \text{ is even}. \end{cases} \blacksquare$$

For the Cartan matrix $C_{A(e)}$, $t \geq 3$ and $s \geq 3$, we define $E_5(e) = (\varepsilon_{i,j})_{i,j=1}^{e}$, where

$$\varepsilon_{i,j} = \begin{cases} \alpha_{i,j} & \text{if } i \neq 2 \text{ or } j \neq 2, \\ 1 & \text{if } i = j = 2. \end{cases}$$

Thus

$$E_5(e) = \begin{bmatrix} 2 & 1 & 1 & \dots & 1 & \overset{s}{1} & a_1 & \dots & a_{e-s} \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 2 & \dots & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 2 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 2 & b_1 & \dots & b_{e-s} \\ a_1 & 0 & 0 & \dots & 0 & b_1 & * & \dots & * \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{e-s} & 0 & 0 & \dots & 0 & b_{e-s} & * & \dots & * \end{bmatrix}.$$

LEMMA 3.6.

(1) $\det C_{A(e)} = \begin{cases} 4 & \text{if } t \text{ is odd}, \\ 0 & \text{if } t \text{ is even}, \end{cases}$

(2) $\det E_5(e) = 0$.

Proof. The proof is divided into three parts.

(a) Assume $t = 1$. Then

$$C_{A(e)} = \begin{bmatrix} 4 & 2 & 2 & 2 & \dots & 2 & 2 \\ 2 & 2 & 1 & 1 & \dots & 1 & 1 \\ 2 & 1 & 2 & 1 & \dots & 1 & 1 \\ 2 & 1 & 1 & 2 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 1 & 1 & 1 & \dots & 2 & 1 \\ 2 & 1 & 1 & 1 & \dots & 1 & 2 \end{bmatrix},$$

and we have $\det C_{A(e)} = 4$.

(b) Assume $t = 2$. Then

$$C_{A(e)} = \begin{bmatrix} 2 & 1 & 1 & \dots & 1 & 2 \\ * & * & * & \dots & * & * \\ * & * & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & * \\ 2 & 1 & 1 & \dots & 1 & 2 \end{bmatrix},$$

and obviously $\det C_{A(e)} = 0$.

(c) Assume $t \geq 3$. We prove the lemma by induction on the number k of edges of T which are not in $\mathcal{R}$. For $k = 0$, the statement (1) follows from Lemma 3.3, and the matrix $E_5(e)$ is not defined. Assume $k \geq 1$ and the lemma holds for all Brauer graphs of the form $(\star)$ having $k - 1$ edges which are not in $\mathcal{R}$. Let T be a Brauer graph having k edges which are not in $\mathcal{R}$.

The Laplace formula applied to the second row of $C_{A(e)}$ and Lemmas 3.4 and 3.5 yield

$$\begin{aligned}\det C_{A(e)} = & -\det E_3(e-1) + 2\det C_{A(e-1)} \\ & + (s-3)\det E_5(e-1) + (-1)^{s+2}\det E_4(e-1) \\ = & \begin{cases} -2+8+0+2(-1)^{s+2} \\ 0 \end{cases} = \begin{cases} 6-2 \\ 0 \end{cases} = \begin{cases} 4 & \text{if } t \text{ is odd,} \\ 0 & \text{if } t \text{ is even.} \end{cases}\end{aligned}$$

Therefore, it remains to prove that $\det E_5(e) = 0$. If $s = 3$, then

$$\begin{aligned}\det E_5(e) &= -\det E_3(e-1) + \det C_{A(e-1)} - \det E_4(e-1) \\ &= \begin{cases} -2+4-2 \\ 0 \end{cases} = 0.\end{aligned}$$

If $s \geq 4$, then

$$\det E_5(e) = \det \begin{bmatrix} 2 & 1 & 1 & 1 & \dots & 1 & \overset{s}{1} & a_1 & \dots & a_{e-s} \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 2 & \dots & 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 2 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 & 2 & b_1 & \dots & b_{e-s} \\ a_1 & 0 & 0 & 0 & \dots & 0 & b_1 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{e-s} & 0 & 0 & 0 & \dots & 0 & b_{e-s} & * & \dots & * \end{bmatrix}$$

$$= \det E_5(e-1) = 0. \blacksquare$$

Proof of Proposition 3.1. The proofs of (1) and (2) are similar. Let $\mathcal{R}$ be the unique cycle in T. Denote by $d(T)$ the maximal distance of the vertices of T to the cycle $\mathcal{R}$, so $d(T) = 0$ if and only if $T = \mathcal{R}$. For Brauer graphs T such that $d(T) = 0$, the proposition follows from Lemma 3.3. Assume that $d(T) \geq 1$. Let T' be the maximal subgraph of T such that $d(T') = 1$. Then T' is of the form $(\star)$ and T is the union of T' and (connected) Brauer trees $T_1, \dots, T_{k(T)}$ having exactly one common vertex with $\mathcal{R}$. We prove the proposition by induction on $k(T)$. For $k(T) = 0$, the proposition follows from Lemma 3.6. Let T be a Brauer graph having $k(T) \geq 1$ and assume that the required formula holds for all Brauer graphs T'' such that $k(T'') < k(T)$. Let T' be the maximal subgraph of T with $d(T') = 1$ and $T_1, \dots, T_{k(T)}$ the Brauer trees having exactly one common vertex with $\mathcal{R}$, such that T is the union of $T', T_1, \dots, T_{k(T)}$. We denote by T_0 the Brauer graph which is the union of $T', T_2, \dots, T_{k(T)}$. The Cartan matrix $C_{\Lambda'(T)}$ (resp. $C_{\Lambda''(T)}$) is of the block form

$$\begin{bmatrix} X & \vdots & 0 \\ \dots & 2 & \dots \\ 0 & \vdots & Y \end{bmatrix},$$

where X is the Cartan matrix of $\Lambda'(T_0)$ (resp. $\Lambda''(T_0)$), Y is the Cartan matrix of $\Lambda(T_1)$ and 2 is the only common nonzero coefficient of the matrices X and Y. Then, applying [1, Lemma 24.4], we obtain for both $\det C_{\Lambda'(T)}$ and $\det C_{\Lambda''(T)}$ the formula

$$\det(X)\det(Y_0) + \det(X_0)\det(Y) - 2\det(X_0)\det(Y_0),$$

where X_0 is obtained from X by erasing the last row and last column, and Y_0 is obtained from Y by erasing the first row and first column. Let r be the number of edges in T_1. Then

$$\det C_{\Lambda'(T)} = 4\cdot r + 4\cdot(r+1) - 8\cdot r = 4,$$
$$\det C_{\Lambda''(T)} = 0\cdot r + 0\cdot(r+1) - 2\cdot 0\cdot 4 = 0.$$

4. Proofs of the main results. Recall from [14, (4.9)] that an algebra B is a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ if and only if B is a tubular extension or a tubular coextension of a hereditary algebra of type $\widetilde{\mathbb{A}}_p$ for some $p \le m$. Moreover, we know from [2] that the class of repetitive algebras $\widehat{B}$ of representation-infinite tilted algebras B of Euclidean types $\widetilde{\mathbb{A}}_m$, $m \ge 1$, coincides with the class of repetitive algebras $\widehat{B}$ of tubular extensions B (equivalently, tubular coextensions) of hereditary algebras of Euclidean types $\widetilde{\mathbb{A}}_p$, $p \ge 1$.

Let B be a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ and $e_1, \dots, e_n$ a complete set of primitive orthogonal idempotents of B such that $1_B = e_1 + \dots + e_n$. Then we have the canonical set $\mathcal{E} = \{e_{k,i} \mid 1 \le i \le n, k \in \mathbb{Z}\}$ of primitive orthogonal idempotents of the repetitive algebra $\widehat{B}$ such that $e_{k,1}+e_{k,2}+\dots+e_{k,n}$ is the identity of the diagonal algebra $B_k = B$ of $\widehat{B}$. By an *automorphism* of $\widehat{B}$ we mean a K-algebra automorphism of $\widehat{B}$ which fixes the set $\mathcal{E}$. A group G of automorphisms of $\widehat{B}$ is called *admissible* if G acts freely on the set $\mathcal{E}$ and has finitely many orbits. Then the orbit algebra $\widehat{B}/G$ is defined (see [7] for details) and is a (finite-dimensional) selfinjective algebra. The action of the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$ on the set $\mathcal{E}$ is given by $\nu_{\widehat{B}}(e_{k,i}) = e_{k+1,i}$ for $(k,i) \in \mathbb{Z} \times \{1,\dots,n\}$, the infinite cyclic group $(\nu_{\widehat{B}})$ is admissible, and $\widehat{B}/(\nu_{\widehat{B}})$ is isomorphic to the trivial extension $T(B) = B \ltimes D(B)$. An automorphism σ of $\widehat{B}$ is said to be *rigid* [16] if for any $(k,i) \in \mathbb{Z} \times \{1,\dots,n\}$ there exists $j \in \{1,\dots,n\}$ such that $\sigma(e_{k,i}) = e_{k,j}$. Following [16] the tilted algebra B is said to be *exceptional* if there exists an automorphism φ of $\widehat{B}$ such that $\varphi^2 = \varrho\nu_{\widehat{B}}$ for a rigid automorphism ϱ of $\widehat{B}$.

We need the following special case of the description of admissible groups of automorphisms of the repetitive algebras of tilted algebras of Euclidean types established in [16, 2.13].

PROPOSITION 4.1. *Let B be a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ and G an admissible group of automorphisms of $\widehat{B}$. Then G is an infinite cyclic group generated by an automorphism $\sigma\varphi^k$ for some $k \geq 1$, where σ is a rigid automorphism of $\widehat{B}$ and φ is an automorphism of $\widehat{B}$ such that $\varphi^d = \varrho\nu_{\widehat{B}}$ for some $d \in \{1, 2\}$ and a rigid automorphism ϱ of $\widehat{B}$. Moreover, if B is not exceptional, we may take $\varphi = \nu_{\widehat{B}}$.*

Let B be a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{A}}_m$, G an admissible group of automorphisms of $\widehat{B}$, and $A = \widehat{B}/G$ the associated selfinjective algebra of type $\widetilde{\mathbb{A}}_m$. Without loss of generality we may assume B is a tubular extension of a hereditary algebra H of type $\widetilde{\mathbb{A}}_p$ for some $p \leq m$.

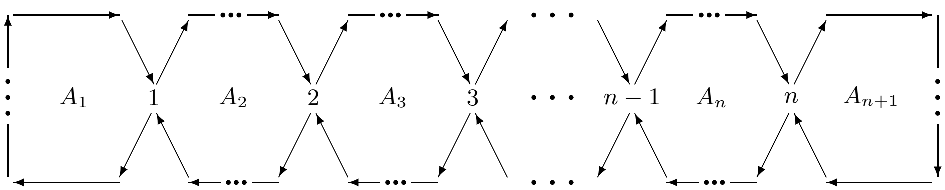
Assume that A is weakly symmetric. Since for any indecomposable projective A-module P the socle of P is isomorphic to the top of P, invoking Proposition 4.1 we conclude that one of the following two cases holds:

(1) B is exceptional, $G = (\sigma\psi)$ for a rigid automorphism σ of $\widehat{B}$ and an automorphism ψ of $\widehat{B}$ such that $\psi^2 = \varrho\nu_{\widehat{B}}$ for some rigid automorphism ϱ of $\widehat{B}$, and moreover $(\sigma\psi)^2$ acts trivially on the set $\mathcal{E}$.

(2) $G = (\sigma\nu_{\widehat{B}})$ for some rigid automorphism σ of $\widehat{B}$, and G acts trivially on $\mathcal{E}$.

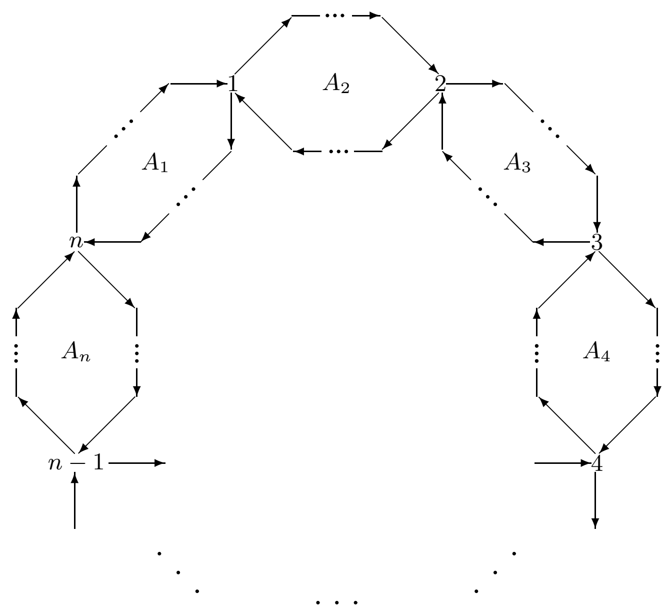
If (2) holds, then since B is a tubular extension of a hereditary algebra H of Euclidean type $\widetilde{\mathbb{A}}_p$ for some p, we easily deduce that $A = \widehat{B}/(\sigma\nu_{\widehat{B}}) \cong \widehat{B}/(\nu_{\widehat{B}}) \cong T(B)$. Similarly, if (1) holds and A is not local, then $A = \widehat{B}/(\sigma\psi) \cong \widehat{B}/(\varphi)$ for an automorphism φ of $\widehat{B}$ such that $\varphi^2 = \nu_{\widehat{B}}$. Assume now that A is local. Then (1) holds, $B = H$ is the hereditary algebra of type $\widetilde{\mathbb{A}}_1$, given by the Kronecker quiver $\bullet \rightrightarrows \bullet$, and consequently $A = \widehat{B}/(\sigma\varphi)$ is isomorphic to the four-dimensional algebra $A_\lambda = K\langle x, y\rangle/(x^2, y^2, xy - \lambda yx)$ for some $\lambda \in K \setminus \{0\}$. Moreover, $A \cong A_\lambda$ is symmetric if and only if $\lambda = 1$ (see [5, Chapter III]).

Assume now that $A = \widehat{B}/(\varphi)$ for an automorphism φ of $\widehat{B}$ such that $\varphi^2 = \nu_{\widehat{B}}$. It follows from [3] that $\widehat{B}$ is special biserial, and hence A is selfinjective and special biserial. Further, since $\varphi^2 = \nu_{\widehat{B}}$, it follows from [16] that the stable Auslander–Reiten quiver Γ_A^{s} consists of one component of the form $\mathbb{Z}\widetilde{\mathbb{A}}_m$ and a $\mathbb{P}_1(K)$-family of stable tubes. Moreover, the one-parameter families of indecomposable modules are given by the images of the one-parameter families of indecomposable modules over the hereditary algebra H of type $\widetilde{\mathbb{A}}_p$ under the push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$ associated to the Galois covering $F : \widehat{B} \to \widehat{B}/(\varphi) = A$. In fact, the bound quiver, say (Q, I), of A admits a unique primitive walk (in the sense of [18]) which is the image of the unique cycle (with underlying graph $\widetilde{\mathbb{A}}_p$) of the Gabriel quiver of B. This primitive walk in (Q, I) is formed by the corresponding paths of one of the bound quivers

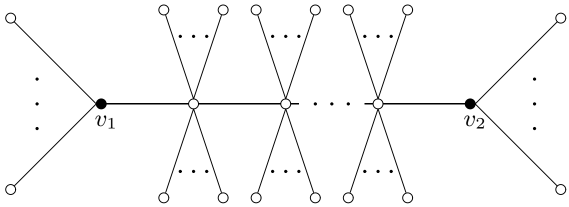


with the relations $A_1^2 - A_2$, $A_n - A_{n+1}^2$, $A_j - A_{j+1}$ for $j = 2, \ldots, n$ if $n \geq 2$, $A_1^2 - A_2^2$ if $n = 1$, and $\alpha_{n_j,j}\alpha_{1,j+1}$, $\alpha_{n_{j+1},j+1}\alpha_{1,j}$ for $j = 1, \ldots, n$, $\alpha_{i,j}\alpha_{i+1,j} \ldots \alpha_{n_j,j}\alpha_{1,j} \ldots \alpha_{i-1,j}\alpha_{i,j}$ for $i = 1, \ldots, n_j$, $j = 2, \ldots, n$, $\alpha_{i,j}\alpha_{i+1,j} \ldots \alpha_{n_j,j}A_j\alpha_{1,j} \ldots \alpha_{i-1,j}\alpha_{i,j}$ for $i = 1, \ldots, n_j$, $j = 1, n+1$, where n_j is the number of arrows on the cycle A_j and $\alpha_{i,j}$ is the arrow on the cycle A_j starting at the vertex i, or

(⋆⋆)

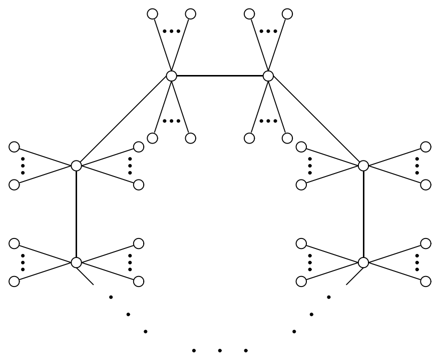


with the relations $A_j - A_{j+1}$ for $j = 1, \ldots, n$ (and $n + 1 = 1$), $\alpha_{n_j,j}\alpha_{1,j+1}$, $\alpha_{n_{j+1},j+1}\alpha_{1,j}$ for $j = 1, \ldots, n$, $\alpha_{i,j}\alpha_{i+1,j} \ldots \alpha_{n_j,j}\alpha_{1,j} \ldots \alpha_{i-1,j}\alpha_{i,j}$ for $i = 1, \ldots, n_j$, where n_j is the number of arrows on the cycle A_j, $\alpha_{i,j}$ is the arrow on the cycle A_j starting at the vertex i, and the number of simple cycles is odd. The first algebra is an algebra of the form $\Lambda(T_0, v_1, v_2)$ for the Brauer tree T_0 of the form



while the second one is of the form $\Lambda'(T_0')$ for the Brauer graph T_0'

(⋆)



with one cycle, and the cycle has an odd number of edges. Since $A = KQ/I$ is special biserial, and (Q, I) contains exactly one primitive walk (described above), we deduce that $Q = Q_T$ and $I = I(T, v_1, v_2)$ for a Brauer tree T with two distinguished vertices v_1 and v_2, containing the Brauer tree T_0 as a full convex subtree, or that $Q = Q_{T'}$ and $I = I'(T')$ for a Brauer graph T' with one cycle containing the Brauer graph T_0' as a full convex subgraph.

Assume now that $A = T(B) = \widehat{B}/(\nu_{\widehat{B}})$. Then again $T(B)$ is a selfinjective (even symmetric) special biserial algebra but the stable Auslander–Reiten quiver consists of two components of type $\mathbb{Z}\widetilde{\mathbb{A}}_m$ and two $\mathbb{P}_1(K)$-families of stable tubes (see [2]). Then the bound quiver (Q, I) of $A = T(B)$ contains exactly two primitive walks, and both contain all sources and sinks of the unique cycle of B (of type $\widetilde{\mathbb{A}}_p$) as vertices. Hence these primitive walks are formed by the corresponding paths of the quiver of the form (⋆⋆) and an even number of simple cycles. Clearly, this is an algebra of the form $\Lambda''(T_0'')$ for a Brauer graph (⋆) with one cycle, and the cycle has an even number of edges. Since A is a symmetric special biserial algebra with exactly two primitive walks (described above) we infer that $A = KQ_T/I''(T)$ for a Brauer graph T'' with one cycle containing the Brauer graph T_0'' as a full convex subgraph.

Finally, assume that A is an algebra of one of the forms $\Lambda(T, v_1, v_2)$, $\Lambda'(T)$, or $\Lambda''(T)$. Then clearly A is a special biserial algebra whose bound quiver contains at most two primitive walks, and consequently A is of domestic type (see [6], [16]). Applying now [16] we infer that A is a selfinjective algebra of Euclidean type $\widetilde{\mathbb{A}}_m$. Moreover, A is symmetric, because we have canonical symmetrizing linear forms $\varphi : \Lambda(T, v_1, v_2) \to K$, $\varphi' : \Lambda'(T) \to K$, $\varphi'' : \Lambda''(T) \to K$ assigning 1 to any maximal nonzero path and 0 to the remaining paths of the bound quiver $(Q_T, I(T, v_1, v_2))$, $(Q_T, I'(T))$, $(Q_T, I''(T))$, respectively (see [5] and [19] for characterizations of symmetric algebras). We also know from Propositions 2.1 and 3.1 that the Cartan matrices of the algebras $\Lambda(T, v_1, v_2)$ and $\Lambda'(T)$ are nonsingular while that of $\Lambda''(T)$ is singular.

Summing up our considerations above, we obtain the assertions of Theorems 1 and 2, and obviously also of Corollary 3.

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