ON FLUCTUATIONS IN THE MEAN OF A SUM-OF-DIVISORS FUNCTION, II

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Abstract. I give explicit values for the constant implied by an Omega-estimate due to Chen and Chen [CC] on the average of the sum of the divisors of $n$ which are relatively coprime to any given integer $a$.

Let $a$ be a positive integer, and consider the sum-of-divisors function

$$
\sigma_{(a)}(n) := \sum_{d|n \atop (d,a)=1} d
$$

(when $a = 1$ this is the classical $\sigma(n)$). This function is known to be on average of size $\frac{\pi^2 \phi(a)}{6a} n$, in the sense that the difference

$$
E_a(x) := \sum_{n \leq x} \sigma_{(a)}(n) - \frac{\pi^2 \phi(a)}{12a} x^2
$$

is small compared to $x^2$. Exactly how small is a difficult problem. In this note I confine myself to establishing explicit lower bounds for the oscillations of $F_a(x)$, where

$$
F_a(x) := \sum_{n \leq x} \frac{\sigma_{(a)}(n)}{n} - \frac{\pi^2 \phi(a)}{6a} x + \frac{1}{2} \log x \sum_{d|a} \mu(d).
$$

Equivalent bounds for $E_a(x)$ will then follow in view of the relation

$$
\frac{E_a(x)}{x} - F_a(x) = O(1),
$$

which is proven for $a > 1$ in Lemma 5 of [CC] (the case $a = 1$ is well known, and is for instance easy to obtain by adapting the previous one). In [P1] I proved the two-sided $\Omega$–estimate

(1) $F_a(x) = \Omega_{\pm}(\log \log x)$

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when \(a = 1\). In [ACM] Adhikari, Coppola and Mukhopadhyay established (1) in the case where \(a\) is a prime number, and finally in [CC] Chen and Chen proved that (1) holds for every positive integer \(a\).

I am now interested in providing explicit values for the constants implied by the estimates (1). Define the positive numbers \(R_0\) and \(R_1\) by

\[ R_i := \limsup_{x \to \infty} \frac{(-1)^i F_a(x)}{\log \log x} \quad (i = 0, 1). \]

In [P2] I proved that

\[ R_i \geq \frac{e^{\gamma}}{2} \quad (i = 0, 1) \]

when \(a = 1\), and in [P3] that

\[ R_i \geq \frac{e^{\gamma}}{2} \cdot \frac{P - 1}{P + 1} \quad (i = 0, 1), \]

when \(a = P\) is a prime number.

In this note I establish the following generalization. (In what follows, the symbols \(P, P_i, p\) will denote prime numbers.)

**Theorem.** For every positive integer \(a\) we have

\[ R_i \geq \frac{e^{\gamma}}{2} \cdot \prod_{P | a} P \frac{P - 1}{P + 1} \quad (i = 0, 1). \]

**Preliminary remark.** If \(a_0\) is the squarefree core of \(a\) (i.e. if \(a_0\) is squarefree and satisfies \(p | a_0 \iff p | a\)), then it is easy to verify that \(\sigma(a_0) = \sigma(a)\), \(\phi(a_0)/a_0 = \phi(a)/a\), and \(\sum_{d | a} \mu(d) = \sum_{d | a_0} \mu(d) \quad (= 0 \text{ when } a > 1)\). Hence \(E_{a_0}(x) = E_a(x)\) and \(F_{a_0}(x) = F_a(x)\), and we may assume in the following that \(a = P_1 \cdots P_s\) is squarefree. (We also assume that \(s \geq 1\).)

We first state six lemmas needed for the proof of the theorem. The first three are Lemmas 1–3 of [CC].

**Lemma 1.** For each natural number \(n\) we have

\[ \frac{\sigma(a)(n)}{n} = \sum_{d | n} \frac{\alpha_a(d)}{d}, \]

where \(\alpha_a(d) := \prod_{p | (a, d)} (1 - p)\).

**Lemma 2.** We have

\[ \sum_{n \leq x} \frac{\alpha_a(n)}{n} = \begin{cases} \log P_1 + O(1/x) & \text{if } s = 1, \\ O(1/x) & \text{if } s > 1. \end{cases} \]

**Lemma 3.** We have

\[ F_a(x) = -\sum_{n=1}^{\infty} \frac{\alpha_a(n)}{n} \left\{ \frac{x}{n} \right\}, \]

where \(\{y\}\) denotes the fractional part of \(y\).
The proof of Lemma 4 is contained in the proof of Lemma 4 in [CC] (put $y = x^{3/4}$ there).

**LEMMA 4.** We have

$$\sum_{x^{3/4} < n \leq x} \frac{\alpha_a(n)}{n} \{\frac{x}{n}\} = O(x^{-1/2}).$$

From Lemmas 2–4, straightforward Abel summations yield

**LEMMA 5.** The error term $F_a$ satisfies

$$F_a(x) = G_a(x) + \begin{cases} \log P_1 + O(1/x) & \text{if } a = 1, \\ O(1/x) & \text{if } s > 1, \end{cases}$$

where

$$G_a(x) := - \sum_{n \leq x} \frac{\alpha_a(n)}{n} \psi\left(\frac{x}{n}\right) = - \sum_{n \leq x^{3/4}} \frac{\alpha_a(n)}{n} \psi\left(\frac{x}{n}\right) + o(1),$$

with $\psi(y) := \{y\} - 1/2$.

Now for every positive integer $M$ we define $N = N(M)$ and $q = q(M)$ by

$$q := \frac{M!}{P_1^{e_1} \cdots P_s^{e_s}} =: N^{1/4} \quad (P_i^{e_i} \parallel M! \text{, } i = 1, \ldots, s).$$

We also put $\beta = 0$ or $\beta = q - 1$, and $u = u(N) = (qN + \beta)^{3/4}$, so that in particular $u \ll N^{15/16}$. Since (again from Lemma 2 with a straightforward Abel summation) we have

$$\sum_{n \leq x} \alpha_a(n) = O(\log x) = 0 \cdot x + o(x),$$

Theorem 1 (with Lemma 6, and with $K = 0$) of [P2] is applicable to $G_a$ and yields the following

**LEMMA 6.** For $G_a$ as defined in Lemma 5, and $\beta = 0$ or $\beta = q - 1$, we have

$$\frac{1}{N} \sum_{n=1}^{N} G_a(nq + \beta) = \sum_{l \leq u} \frac{\alpha_a(l)(q,l)}{l^2} \left(\frac{1}{2} - \frac{\beta}{(q,l)}\right) + O(1).$$

We now proceed to prove the theorem, by evaluating the sum on the right-hand side of this last equation. First note that we may restrict our attention to the case where $\beta = 0$. Indeed, in the case $\beta = q - 1$ we have

$$\frac{1}{2} - \frac{\beta}{(q,l)} = -\frac{1}{2} + \frac{1}{(q,l)} \quad \text{and} \quad \sum_{l \leq u} \frac{\alpha_a(l)}{l^2} = O(1).$$

In Lemma 6 put $l = nm$ with $n \mid q$ and $p \mid m \Rightarrow p \nmid q/n$. Then $(q,l) = n$ and $\alpha_a(l) = \alpha_a(m) = \prod_{P_i\mid m} (1 - P_i).$ Thus if we denote by $\mathcal{P}$ the set of subsets
of \(\{1, \ldots, s\}\) we may write

\[
\frac{1}{N} \sum_{n=1}^{N} G_a(nq) = \frac{1}{2} \sum_{n|q} \frac{1}{n} \sum_{E \in P} \sum_{\substack{m \leq u/n \\ p|m \Rightarrow p|q/n \\ \ \ \ p_i|m \Rightarrow i \in E}} \alpha_a(m) \frac{1}{m^2}.
\]

The error committed by ignoring the condition \(m \leq u/n\) is small. Indeed, with the help of Lemma 2 we see that

\[
\sum_{n|q} \frac{1}{n} \sum_{u/n<m} \frac{\alpha_a(m)}{m^2} \ll \sum_{n|q} \frac{n}{u^2} \ll \frac{q}{u^2} d(q) \ll N^{-13/8+\epsilon} = o(1).
\]

In order to lighten a bit the notation we assume, up to equation (6) below, in sums in which the symbol \(m\) appears, that the condition \(p|m \Rightarrow p|q/n\) is always satisfied. With this convention and the remark just above we may rewrite (3) as

\[
\frac{1}{N} \sum_{n=1}^{N} G_a(nq) = \frac{1}{2} \sum_{n|q} \frac{1}{n} \sum_{E \in P} \prod_{i \in E} (1 - P_i) \sum_{p_i|m \Leftrightarrow i \in E} \frac{1}{m^2} + o(1).
\]

Now if we put \(\bar{E} := P \setminus E\) the last sum on the right-hand side of (4) is

\[
\sum_{p_i|m, \forall i \in E} \frac{1}{m^2} = \sum_{D \subseteq E} (-1)^{|D|} \sum_{p_i|m, \forall i \in E, p_j|m} \frac{1}{m^2} = \sum_{D \subseteq E} (-1)^{|D|} \prod_{i \in D} \frac{1}{P_i^2} \prod_{j \notin D} \left(1 - \frac{1}{P_j^2}\right).
\]

Thus from (4) we have

\[
\frac{1}{N} \sum_{n=1}^{N} G_a(nq) = \frac{1}{2} \sum_{n|q} \frac{1}{n} \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{E \in P} \prod_{i \in E} \frac{1 - P_i}{P_i^2} \prod_{j \notin E} \left(1 - \frac{1}{P_j^2}\right) + o(1).
\]

The last sum on the right-hand side of (5) is

\[
\sum_{E \in P} \prod_{i \in E} \frac{1 - P_i}{P_i^2} \prod_{j \notin E} \left(P_j - 1\right)\left(1 + P_j\right) = \prod_{i=1}^{s} \frac{1 - P_i}{P_i^2} \sum_{E \in P} (-1)^{|E|} \prod_{j \in E} \left(1 + P_j\right) = \prod_{i=1}^{s} \frac{1 - P_i}{P_i^2} \left(1 - (1 + P_i)\right) = \prod_{i=1}^{s} \frac{P_i - 1}{P_i},
\]

and (5) can be rewritten as

\[
\frac{1}{N} \sum_{n=1}^{N} G_a(nq) = \frac{1}{2} \sum_{n|q} \frac{1}{n} \sum_{i=1}^{s} \frac{P_i - 1}{P_i} \sum_{m=1}^{\infty} \frac{1}{m^2} + o(1).
\]
Now the tacit condition \( p \mid m \Rightarrow p \mid q/n \) is less restrictive than \( p \mid m \Rightarrow p = P_i \) for some \( i \) with \( 1 \leq i \leq s \). Hence

\[
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\frac{1}{N} \sum_{n=1}^{N} G_a(nq) \geq \frac{1}{2} \sum_{n \mid q} \frac{1}{n} \prod_{i=1}^{s} \frac{P_i - 1}{P_i} \sum_{j=0}^{\infty} \frac{1}{P_i^{2j}} + o(1)

= \frac{1}{2} \sum_{n \mid q} \frac{1}{n} \prod_{i=1}^{s} \frac{P_i}{P_i + 1} + o(1).
\]

Finally, since \( \log M \sim \log \log N \) we have

\[
\sum_{n \mid q} \frac{1}{n} = \prod_{p \leq M \atop p^{\nu_p || q}} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^{\nu_p}} \right) \sim \prod_{p \leq M \atop p \neq P_i \atop 1 \leq i \leq s} \left( 1 - \frac{1}{P_i} \right)^{-1}

\sim \prod_{i=1}^{s} \left( 1 - \frac{1}{P_i} \right) e^\gamma \log \log N,
\]

whence from (7) we obtain

\[
\frac{1}{N} \sum_{n=1}^{N} G_a(nq) \geq \frac{e^\gamma}{2} \prod_{i=1}^{s} \frac{P_i - 1}{P_i + 1} \log \log N \left( 1 + o(1) \right).
\]

This, in view of Lemma 5 and (2), concludes the proof of the Theorem.

**NOTE.** The referee, to whom I am grateful for this very pertinent question, asked: “Can one expect similar results for the function \( \sum_{d \mid n, (d,a)=1} d^k \)?”

Indeed, with the same method one can derive such estimates for this function—call it \( \sigma(a)_k(n) \)—for every real number \( k > 1 \) (the case \( k < 1 \) appears to be more difficult to handle). I briefly describe how below.

Consider (for \( k > 1 \) ) the remainder terms

\[
E_{a,k}(x) := \sum_{n \leq x} \sigma(a)_k(n) - \frac{\phi(a)}{a} \zeta(k+1) x^{k+1}
\]

and

\[
F_{a,k}(x) := \sum_{n \leq x} \frac{\sigma(a)_k(n)}{n^k} - \frac{\phi(a)}{a} \zeta(k+1)x.
\]

Very similarly to the case \( k = 1 \) (and partly much more easily), one first proves statements corresponding to Lemmas 1 through 5. Mutatis mutandis this yields

\[
F_{a,k}(x) = H_{a,k}(x) + o(1),
\]
where
\[
H_{a,k}(x) := - \sum_{n \leq x^{3/4}} \alpha_{a,k}(n) \frac{\psi\left(\frac{x}{n}\right)}{n^k} \quad \text{and} \quad \alpha_{a,k}(n) := \prod_{p \mid (a,n)} \left(1 - \frac{1}{p^k}\right).
\]

Now with the use of the Euler–Maclaurin sum formula for \(\sum_{n \leq x} n^k\) we obtain, similarly to the proof of Lemma 5 in [CC],
\[
E_{a,k}(x) = o(1).
\]

Then an appeal to [P2] yields
\[
\frac{1}{N} \sum_{n=1}^{N} H_{a,k}(nq + \beta) = \sum_{l \leq u} \alpha_{a,k}(l)(q,l) \left( \frac{1}{2} - \frac{\beta}{(q,l)} \right) + o(1).
\]

As in the case \(k = 1\) we may restrict our attention to the case where \(\beta = 0\), but this time the justification for this requires considering \(\beta = q - \varepsilon\) (instead of simply \(\beta = q - 1\)) for arbitrarily small values of \(\varepsilon\) (this is allowed: see the Addendum of [P2]).

The rest of the argument is straightforward, as it fairly closely mimicks that of the case \(k = 1\), and yields
\[
\frac{1}{N} \sum_{n=1}^{N} H_{a,k}(nq) \geq \frac{1}{2} \zeta(k) \prod_{i=1}^{s} \frac{(P_i - 1)(P_i^k - 1)}{P_i^{k+1} - 1} + o(1).
\]

This implies, in view of (8) and (9), that
\[
\limsup (-1)^i \frac{E_{a,k}(x)}{x^k} \geq \frac{1}{2} \zeta(k) \prod_{i=1}^{s} \frac{(P_i - 1)(P_i^k - 1)}{P_i^{k+1} - 1} \quad (i = 0, 1).
\]

Finally, note that here \(x^k\) is the true order of magnitude of \(E_{a,k}(x)\).

REFERENCES


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