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ON FLUCTUATIONS IN THE MEAN OF A SUM-OF-DIVISORS FUNCTION, II

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Abstract. I give explicit values for the constant implied by an Omega-estimate due to Chen and Chen [CC] on the average of the sum of the divisors of n which are relatively coprime to any given integer a.

Let a be a positive integer, and consider the sum-of-divisors function

$$\sigma_{(a)}(n) := \sum_{\substack{d\mid n\\ (d,a)=1}} d$$

(when $a = 1$ this is the classical $\sigma(n)$). This function is known to be on average of size $\frac{\pi^2\phi(a)}{6a}n$, in the sense that the difference

$$E_a(x) := \sum_{n\le x} \sigma_{(a)}(n) - \frac{\pi^2\phi(a)}{12a}x^2$$

is small compared to x^2. Exactly how small is a difficult problem. In this note I confine myself to establishing explicit lower bounds for the oscillations of $F_a(x)$, where

$$F_a(x) := \sum_{n\le x} \frac{\sigma_{(a)}(n)}{n} - \frac{\pi^2\phi(a)}{6a}x + \frac{1}{2}\log x \sum_{d\mid a} \mu(d).$$

Equivalent bounds for $E_a(x)$ will then follow in view of the relation

$$\frac{E_a(x)}{x} - F_a(x) = O(1),$$

which is proven for $a > 1$ in Lemma 5 of [CC] (the case $a = 1$ is well known, and is for instance easy to obtain by adapting the previous one). In [P1] I proved the two-sided $\Omega-$estimate

$$F_a(x) = \Omega_\pm(\log\log x) \tag{1}$$

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when $a = 1$. In [ACM] Adhikari, Coppola and Mukhopadhyay established (1) in the case where a is a prime number, and finally in [CC] Chen and Chen proved that (1) holds for every positive integer a.

I am now interested in providing explicit values for the constants implied by the estimates (1). Define the positive numbers R_0 and R_1 by

$$R_i := \limsup_{x\to\infty} \frac{(-1)^i F_a(x)}{\log\log x} \qquad (i = 0, 1).$$

In [P2] I proved that

$$R_i \ge \frac{e^\gamma}{2} \qquad (i = 0, 1)$$

when $a = 1$, and in [P3] that

$$R_i \ge \frac{e^\gamma}{2} \cdot \frac{P-1}{P+1} \qquad (i = 0, 1),$$

when $a = P$ is a prime number.

In this note I establish the following generalization. (In what follows, the symbols P, P_i, p will denote prime numbers.)

THEOREM. *For every positive integer a we have*

$$R_i \ge \frac{e^\gamma}{2} \cdot \prod_{P|a} \frac{P-1}{P+1} \qquad (i = 0, 1).$$

Preliminary remark. If a_0 is the squarefree core of a (i.e. if a_0 is squarefree and satisfies $p \mid a_0 \Leftrightarrow p \mid a$), then it is easy to verify that $\sigma_{(a_0)} = \sigma_{(a)}$, $\phi(a_0)/a_0 = \phi(a)/a$, and $\sum_{d|a} \mu(d) = \sum_{d|a_0} \mu(d)$ ($= 0$ when $a > 1$). Hence $E_{a_0}(x) = E_a(x)$ and $F_{a_0}(x) = F_a(x)$, and we may assume in the following that $a = P_1 \cdots P_s$ is sqarefree. (We also assume that $s \ge 1$.)

We first state six lemmas needed for the proof of the theorem. The first three are Lemmas 1–3 of [CC].

LEMMA 1. *For each natural number n we have*

$$\frac{\sigma_{(a)}(n)}{n} = \sum_{d|n} \frac{\alpha_a(d)}{d},$$

where $\alpha_a(d) := \prod_{p|(a,d)} (1-p)$.

LEMMA 2. *We have*

$$\sum_{n\le x} \frac{\alpha_a(n)}{n} = \begin{cases} \log P_1 + O(1/x) & \text{if } s = 1, \\ O(1/x) & \text{if } s > 1. \end{cases}$$

LEMMA 3. *We have*

$$F_a(x) = -\sum_{n=1}^{\infty} \frac{\alpha_a(n)}{n} \left\{ \frac{x}{n} \right\},$$

where $\{y\}$ denotes the fractional part of y.

The proof of Lemma 4 is contained in the proof of Lemma 4 in [CC] (put $y = x^{3/4}$ there).

LEMMA 4. *We have*

$$\sum_{x^{3/4}<n\le x} \frac{\alpha_a(n)}{n}\left\{\frac{x}{n}\right\} = O(x^{-1/2}).$$

From Lemmas 2–4, straightforward Abel summations yield

LEMMA 5. *The error term F_a satisfies*

$$F_a(x) = G_a(x) + \begin{cases} \log P_1 + O(1/x) & \text{if } a = 1,\\ O(1/x) & \text{if } s > 1,\end{cases}$$

where

$$G_a(x) := -\sum_{n\le x}\frac{\alpha_a(n)}{n}\psi\left(\frac{x}{n}\right) = -\sum_{n\le x^{3/4}}\frac{\alpha_a(n)}{n}\psi\left(\frac{x}{n}\right) + o(1),$$

with $\psi(y) := \{y\} - 1/2$.

Now for every positive integer M we define $N = N(M)$ and $q = q(M)$ by

$$q := \frac{M!}{P_1^{e_1}\cdots P_s^{e_s}} =: N^{1/4} \qquad (P_i^{e_i} \,\|\, M!,\ i = 1,\ldots,s).$$

We also put $\beta = 0$ or $\beta = q-1$, and $u = u(N) = (qN+\beta)^{3/4}$, so that in particular $u \ll N^{15/16}$. Since (again from Lemma 2 with a straightforward Abel summation) we have

$$\sum_{n\le x}\alpha_a(n) = O(\log x) = 0\cdot x + o(x),$$

Theorem 1 (with Lemma 6, and with $K = 0$) of [P2] is applicable to G_a and yields the following

LEMMA 6. *For G_a as defined in Lemma* 5, *and $\beta = 0$ or $\beta = q-1$, we have*

$$\frac{1}{N}\sum_{n=1}^{N} G_a(nq+\beta) = \sum_{l\le u}\frac{\alpha_a(l)(q,l)}{l^2}\left(\frac{1}{2} - \frac{\beta}{(q,l)}\right) + O(1).$$

We now proceed to prove the theorem, by evaluating the sum on the right-hand side of this last equation. First note that we may restrict our attention to the case where $\beta = 0$. Indeed, in the case $\beta = q-1$ we have

$$\text{(2)}\qquad \frac{1}{2} - \frac{\beta}{(q,l)} = -\frac{1}{2} + \frac{1}{(q,l)} \quad\text{and}\quad \sum_{l\le u}\frac{\alpha_a(l)}{l^2} = O(1).$$

In Lemma 6 put $l = nm$ with $n \mid q$ and $p \mid m \Rightarrow p \nmid q/n$. Then $(q,l) = n$ and $\alpha_a(l) = \alpha_a(m) = \prod_{P_i\mid m}(1-P_i)$. Thus if we denote by $\mathcal{P}$ the set of subsets of $\{1,\dots,s\}$ we may write

$$\frac{1}{N}\sum_{n=1}^{N} G_a(nq) = \frac{1}{2}\sum_{n\mid q}\frac{1}{n}\sum_{E\in\mathcal{P}}\sum_{\substack{m\le u/n\\ p\mid m\Rightarrow p\nmid q/n\\ P_i\mid m\Leftrightarrow i\in E}}\frac{\alpha_a(m)}{m^2}. \tag{3}$$

The error committed by ignoring the condition $m\le u/n$ is small. Indeed, with the help of Lemma 2 we see that

$$\sum_{n\mid q}\frac{1}{n}\sum_{u/n<m}\frac{\alpha_a(m)}{m^2} \ll \sum_{n\mid q}\frac{n}{u^2} \ll \frac{q}{u^2}\,d(q) \ll N^{-13/8+\epsilon} = o(1).$$

In order to lighten a bit the notation we assume, up to equation (6) below, in sums in which the symbol m appears, that the condition $p\mid m \Rightarrow p\nmid q/n$ is always satisfied. With this convention and the remark just above we may rewrite (3) as

$$\frac{1}{N}\sum_{n=1}^{N} G_a(nq) = \frac{1}{2}\sum_{n\mid q}\frac{1}{n}\sum_{E\in\mathcal{P}}\prod_{i\in E}(1-P_i)\sum_{P_i\mid m\Leftrightarrow i\in E}\frac{1}{m^2} + o(1). \tag{4}$$

Now if we put $\overline{E} := \mathcal{P}\setminus E$ the last sum on the right-hand side of (4) is

$$\sum_{P_i\mid m,\,\forall i\in E}\frac{1}{m^2} - \sum_{\substack{P_i\mid m,\,\forall i\in E\\ \exists j\in\overline{E},\,P_j\mid m}}\frac{1}{m^2} = \sum_{D\subset\overline{E}}(-1)^{|D|}\sum_{\substack{P_i\mid m,\,\forall i\in E\\ P_j\mid m,\,\forall j\in D}}\frac{1}{m^2}$$

$$= \sum_{D\in\overline{E}}\frac{(-1)^{|D|}}{\prod_{i\in E}P_i^2\prod_{j\in D}P_j^2}\sum_{m=1}^{\infty}\frac{1}{m^2} = \sum_{m=1}^{\infty}\frac{1}{m^2}\prod_{i\in E}\frac{1}{P_i^2}\prod_{j\notin E}\left(1-\frac{1}{P_j^2}\right).$$

Thus from (4) we have

$$\frac{1}{N}\sum_{n=1}^{N} G_a(nq) = \frac{1}{2}\sum_{n\mid q}\frac{1}{n}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{E\in\mathcal{P}}\prod_{i\in E}\frac{1-P_i}{P_i^2}\prod_{j\notin E}\left(1-\frac{1}{P_j^2}\right) + o(1). \tag{5}$$

The last sum on the right-hand side of (5) is

$$\sum_{E\in\mathcal{P}}\prod_{i\in E}\frac{1-P_i}{P_i^2}\prod_{j\notin E}\frac{(P_j-1)(1+P_j)}{P_j^2} = \prod_{i=1}^{s}\frac{1-P_i}{P_i^2}\sum_{\overline{E}\in\mathcal{P}}(-1)^{|\overline{E}|}\prod_{j\in\overline{E}}(1+P_j)$$

$$= \prod_{i=1}^{s}\frac{1-P_i}{P_i^2}\,(1-(1+P_i)) = \prod_{i=1}^{s}\frac{P_i-1}{P_i},$$

and (5) can be rewritten as

$$\frac{1}{N}\sum_{n=1}^{N} G_a(nq) = \frac{1}{2}\sum_{n\mid q}\frac{1}{n}\prod_{i=1}^{s}\frac{P_i-1}{P_i}\sum_{m=1}^{\infty}\frac{1}{m^2} + o(1). \tag{6}$$

Now the tacit condition $p \mid m \Rightarrow p \nmid q/n$ is less restrictive than $p \mid m \Rightarrow p = P_i$ for some i with $1 \le i \le s$. Hence

$$\frac{1}{N}\sum_{n=1}^{N} G_a(nq) \ge \frac{1}{2}\sum_{n\mid q}\frac{1}{n}\prod_{i=1}^{s}\frac{P_i-1}{P_i}\sum_{j=0}^{\infty}\frac{1}{P_i^{2j}} + o(1) \tag{7}$$

$$= \frac{1}{2}\sum_{n\mid q}\frac{1}{n}\prod_{i=1}^{s}\frac{P_i}{P_i+1} + o(1).$$

Finally, since $\log M \sim \log\log N$ we have

$$\sum_{n\mid q}\frac{1}{n} = \prod_{\substack{p\le M\\ p^{e_p}\| q}}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{e_p}}\right) \sim \prod_{\substack{p\le M\\ p\ne P_i\ (1\le i\le s)}}\left(1-\frac{1}{p}\right)^{-1}$$

$$\sim \prod_{i=1}^{s}\left(1-\frac{1}{P_i}\right)e^{\gamma}\log\log N,$$

whence from (7) we obtain

$$\frac{1}{N}\sum_{n=1}^{N} G_a(nq) \ge \frac{e^{\gamma}}{2}\prod_{i=1}^{s}\frac{P_i-1}{P_i+1}\log\log N\,(1+o(1)).$$

This, in view of Lemma 5 and (2), concludes the proof of the Theorem.

Note. The referee, to whom I am grateful for this very pertinent question, asked: "Can one expect similar results for the function $\sum_{d\mid n,(d,a)=1} d^k$?"

Indeed, with the same method one can derive such estimates for this function—call it $\sigma_{(a),k}(n)$—for every real number $k > 1$ (the case $k < 1$ appears to be more difficult to handle). I briefly describe how below.

Consider (for $k > 1$) the remainder terms

$$E_{a,k}(x) := \sum_{n\le x}\sigma_{(a),k}(n) - \frac{\phi(a)}{a}\,\frac{\zeta(k+1)}{k+1}\,x^{k+1}$$

and

$$F_{a,k}(x) := \sum_{n\le x}\frac{\sigma_{(a),k}(n)}{n^k} - \frac{\phi(a)}{a}\,\zeta(k+1)x.$$

Very similarly to the case $k = 1$ (and partly much more easily), one first proves statements corresponding to Lemmas 1 through 5. Mutatis mutandis this yields

$$F_{a,k}(x) = H_{a,k}(x) + o(1), \tag{8}$$

where

$$H_{a,k}(x) := -\sum_{n\le x^{3/4}} \frac{\alpha_{a,k}(n)}{n^k}\psi\left(\frac{x}{n}\right) \quad \text{and} \quad \alpha_{a,k}(n) := \prod_{p|(a,n)} (1-p^k).$$

Now with the use of the Euler–Maclaurin sum formula for $\sum_{n\le x} n^k$ we obtain, similarly to the proof of Lemma 5 in [CC],

$$\frac{E_{a,k}(x)}{x^k} - F_{a,k}(x) = o(1). \tag{9}$$

Then an appeal to [P2] yields

$$\frac{1}{N}\sum_{n=1}^{N} H_{a,k}(nq+\beta) = \sum_{l\le u} \frac{\alpha_{a,k}(l)(q,l)}{l^{k+1}}\left(\frac{1}{2} - \frac{\beta}{(q,l)}\right) + o(1).$$

As in the case $k=1$ we may restrict our attention to the case where $\beta=0$, but this time the justification for this requires considering $\beta = q-\varepsilon$ (instead of simply $\beta = q-1$) for arbitrarily small values of ε (this is allowed: see the Addendum of [P2]).

The rest of the argument is straightforward, as it fairly closely mimicks that of the case $k=1$, and yields

$$\frac{1}{N}\sum_{n=1}^{N} H_{a,k}(nq) \ge \frac{1}{2}\,\zeta(k)\prod_{i=1}^{s}\frac{(P_i-1)(P_i^k-1)}{P_i^{k+1}-1} + o(1).$$

This implies, in view of (8) and (9), that

$$\limsup\,(-1)^i\,\frac{E_{a,k}(x)}{x^k} \ge \frac{1}{2}\,\zeta(k)\prod_{i=1}^{s}\frac{(P_i-1)(P_i^k-1)}{P_i^{k+1}-1} \qquad (i=0,1).$$

Finally, note that here x^k is the true order of magnitude of $E_{a,k}(x)$.

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