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PRE-TANGO STRUCTURES AND UNIRULED VARIETIES

BY

YOSHIFUMI TAKEDA (Nara)

Abstract. The pre-Tango structure is an ample invertible sheaf of locally exact differentials on a variety of positive characteristic. It is well known that pre-Tango structures on curves often induce pathological uniruled surfaces. We show that almost all pre-Tango structures on varieties induce higher-dimensional pathological uniruled varieties, and that each of these uniruled varieties also has a pre-Tango structure. For this purpose, we first consider the p-closed rational vector field induced by a pre-Tango structure, and the smoothness of the fibration induced by the p-closed rational vector field.

Moreover, we give two examples: of a 3-dimensional variety of general type whose automorphism group scheme is not reduced, and of a non-uniruled variety which has a pre-Tango structure inducing a higher-dimensional pathological uniruled variety.

1. Introduction. Let X be a projective algebraic variety over an algebraically closed field k of characteristic $p > 0$. Let $F : \widetilde{X} \to X$ be the relative Frobenius morphism over k. We then have a short exact sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_{\widetilde{X}} \to F_*\mathcal{B}^1_{\widetilde{X}} \to 0,$$

where $\mathcal{B}^1_{\widetilde{X}}$ is the first sheaf of coboundaries of the de Rham complex of $\widetilde{X}$. Suppose that there exists an ample invertible subsheaf $\mathcal{L}$ of $F_*\mathcal{B}^1_{\widetilde{X}}$ regarded as an $\mathcal{O}_X$-module. We then call $\mathcal{L}$ a *pre-Tango structure*. Consider the exact sequence

$$0 \to \mathcal{L}^{-1} \to F_*\mathcal{O}_{\widetilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \to F_*\mathcal{B}^1_{\widetilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \to 0.$$

By taking cohomology, we have

$$0 \to H^0(X, \mathcal{L}^{-1}) \to H^0(X, F_*\mathcal{O}_{\widetilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) \to H^0(X, F_*\mathcal{B}^1_{\widetilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) \to H^1(X, \mathcal{L}^{-1}) \to \cdots$$

Since $H^0(X, F_*\mathcal{O}_{\widetilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) = 0$ and $H^0(X, F_*\mathcal{B}^1_{\widetilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) \neq 0$, we know that $H^1(X, \mathcal{L}^{-1}) \neq 0$. Hence, if X is a smooth variety of dimension greater than one, then the pair $(X, \mathcal{L})$ is a counter-example to the Kodaira vanishing theorem in positive characteristic. However, it is hard to find such a pair in dimension greater than one.

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On the other hand, in dimension one, almost all smooth projective curves have pre-Tango structures (see Takeda and Yokogawa [14]). In fact, Raynaud's famous counter-example ([6]) is constructed as a uniruled surface over a smooth projective curve with a Tango structure, which is a pre-Tango structure satisfying an additional condition (see Section 3). Furthermore, Russell ([8]) gave an example of a smooth surface of general type with non-trivial global vector fields (see also Mukai [4]). He constructed the example from a curve with a certain Tango structure. That example is also a uniruled surface.

Raynaud obtained his uniruled surface by taking a separable double or triple covering of a ruled surface over a curve with a Tango structure, and he showed that the uniruled surface has a pre-Tango structure. By generalizing Raynaud's method, Mukai ([4]) constructed some $(d+1)$-dimensional uniruled varieties with pre-Tango structures from d-dimensional varieties with pre-Tango structures. In particular, Mukai's 2-dimensional examples include not only Raynaud's but also Russell's examples. Moreover, for each natural number d greater than one, Mukai has given an example of a curve with a Tango structure which induces a smooth uniruled variety of dimension d, with a Tango structure (see the diagram below).

$$\begin{array}{ccc} \begin{array}{c}\text{separable finite covering} \\ \text{with a pre-Tango structure}\end{array} & \Leftarrow & (d+1)\text{-dimensional ruled variety} \\ & {\scriptstyle\text{fibration}}\ \searrow & \Uparrow \\ & & \begin{array}{c} d\text{-dimensional original variety} \\ \text{with a pre-Tango structure}\end{array} \end{array}$$

Every example of a higher-dimensional variety with a pre-Tango structure mentioned above has a fibration onto the lower-dimensional original variety (see the diagram above). All fibres of such a fibration are rational curves which have one cuspidal point each. Further, the locus of the cuspidal points is a p-section of the fibration. More precisely, that locus is a purely inseparable covering of degree p of the base variety. In addition, after normalizing the fibre product of the fibration and the purely inseparable covering over the original variety, we obtain a ruled variety. Indeed, Russell obtained his example by taking the quotient of a ruled surface over (a purely inseparable covering of) a curve with a Tango structure, by a p-closed rational vector field.

It is the primary purpose of the present article to comprehend the relations among pre-Tango structures, purely inseparable coverings, and p-closed rational vector fields. Furthermore, as an application, we shall construct a $(d+1)$-dimensional uniruled variety with a pre-Tango structure by taking the quotient by a certain p-closed rational vector field on an appropriate ruled variety over the induced purely inseparable covering of a d-dimensional variety with a pre-Tango structure (see the diagram below and see also Section 4). This is a generalization of Russell's method and the resulting uniruled variety has a fibration onto the original variety.

$$\begin{array}{ccc} \text{ruled variety} & \Rightarrow & \begin{array}{c}(d+1)\text{-dimensional uniruled variety}\\ \text{with a pre-Tango structure}\end{array} \\ \Uparrow & & \downarrow \text{ fibration} \\ \text{purely inseparable covering} & \Leftarrow & \begin{array}{c}d\text{-dimensional original variety}\\ \text{with a pre-Tango structure}\end{array} \end{array}$$

In Section 2, we shall consider p-closed rational vector fields on a smooth variety with a smooth fibration onto a lower-dimensional smooth variety. It is the secondary purpose of the present article to investigate the smoothness of the induced fibration between the quotient varieties by a p-closed rational vector field. The results in Section 2 are extensions of those of the author's earlier article [10] and they will be used in Section 4.

Section 3 is devoted to our primary purpose mentioned above. We shall introduce the notion of pre-Tango and Tango structure on a variety. That is a generalization of the notion introduced in [14]. We shall show that each pre-Tango structure induces a purely inseparable covering with a p-closed rational vector field. In particular, in the case of Tango structure, we shall observe that the invertible sheaf associated to the divisor of the induced vector field is isomorphic to the dual of the pull-back of the Tango structure.

In Section 4, we shall construct higher-dimensional uniruled varieties from varieties with pre-Tango structures by the above-mentioned generalization of Russell's method, that is to say, the reverse of the process described in Section 3. More precisely, taking a suitable ruled variety over the purely inseparable covering induced from a pre-Tango structure, we shall naturally obtain a p-closed rational vector field whose divisor is an anti-ample divisor on the ruled variety, and we shall obtain the quotient variety, which is a uniruled variety. Moreover, on the quotient variety, we shall find an invertible sheaf of local exact differentials whose pull-back is isomorphic to the dual of the invertible sheaf associated to the divisor of the above-mentioned p-closed rational vector field, that is, a pre-Tango structure. The resulting uniruled varieties are the same as Mukai's and most of the results in Sections 3 and 4 can be found in [4].

If a smooth projective variety of dimension greater than one has a pre-Tango structure, then it gives a counter-example to the Kodaira vanishing theorem in positive characteristic, as mentioned at the beginning. Here, we are interested in the question whether every projective variety of dimension greater than one with a pre-Tango structure induces a counter-example to the Kodaira vanishing theorem. In this connection we may pose the following question:

> Suppose that a projective variety X of dimension greater than one has a pre-Tango structure. Does then the Hodge–de Rham spectral sequence of X degenerate at E_1?

Regrettably, the author does not know what the complete answer is. However, it is known that, in the case of the uniruled surfaces obtained from pre-Tango structures on curves, frequently, the spectral sequences do not degenerate at E_1 (see [13] and [14]). In Section 5, we shall prove that the above-mentioned higher-dimensional uniruled variety constructed from a pre-Tango structure satisfying an extra condition on a projective variety has non-closed global differential 1-forms.

Through our construction of higher-dimensional uniruled varieties, we have explicit local equations which determine the varieties. Hence it is easy to consider their automorphisms. In fact, we shall prove in Section 5 that the uniruled variety constructed from a specific Tango structure on a projective variety is a variety of general type whose automorphism group scheme is not reduced.

The uniruled varieties which are constructed by several applications of the above-mentioned processes from certain Tango structures on projective curves are the only known examples of higher-dimensional smooth varieties which have pre-Tango structures. Therefore, we pose the following question:

> Suppose that a smooth projective variety X of dimension greater than one has a pre-Tango structure. Is then X a uniruled variety?

The author regrets once again that he does not know the answer to this question. However, it is known that, if a smooth projective variety which is not uniruled has an ample invertible sheaf $\mathcal{L}$ such that $\mathcal{L}^{p-1} \otimes_{\mathcal{O}_X} \omega_X^{-1}$ is ample, then $H^1(X, \mathcal{L}^{-1}) = 0$ (Corollary II.6.3 in Kollár [2]). On the other hand, in the case of a normal projective variety, the answer is negative. Indeed, Mumford gave an example of a pre-Tango structure on a normal projective surface which is not uniruled ([5]). However, it is unknown whether or not the desingularization of that surface has a pre-Tango structure. In Section 6, we shall give a generalization of Mumford's example and, by considering it precisely, we shall show that the uniruled variety which is constructed from a pre-Tango structure on our non-uniruled variety has non-closed global differential 1-forms. This suggests that the uniruled varieties are not the only varieties which trigger off pathological phenomena in positive characteristic.

Further, in Section 6, we shall construct another example, which is a 3-dimensional smooth variety of general type whose automorphism group scheme is not reduced.

2. p-closed rational vector fields and fibrations. In this section, we use the notion of quotient by a rational vector field. We follow the terminology of Rudakov–Shafarevich [7]. For the precise definition and basic properties of p-closed rational vector fields on an algebraic variety, see [7] and Seshadri [9].

Let k be an algebraically closed field of characteristic $p > 0$ and let V be a smooth irreducible variety of dimension m over k. We consider a p-closed rational vector field D on V. At an arbitrary point P of V we can write

$$D = g_P\left(f_1 \frac{\partial}{\partial x_1} + \cdots + f_m \frac{\partial}{\partial x_m}\right),$$

where $x_1, \ldots, x_m$ are local coordinates at P, where g_P is a rational function and $f_1, \ldots, f_m$ are regular functions at P without common factors. The functions $\{g_P\}_{P\in V}$ determine a divisor on V, which we call the *divisor of* D and denote by (D). Furthermore, we have a natural injection

$$\mathcal{O}_V((D))D \to \Theta_V,$$

where Θ_V is the tangent bundle of V. Let C be an irreducible subvariety of codimension one. We call C an *integral subvariety* for D provided that, for the general point P of C, the vector $(1/g_P)D|_P$ is tangent to C.

Let W be a smooth irreducible variety of dimension n over k and let $f : V \to W$ be a smooth morphism over k. Assume that $D(k(W)) \subset k(W)$ and set $\Delta = D|_{k(W)}$. It is easy to verify that if D is p-closed, then so is Δ. We have a diagram of sheaves on V:

$$\begin{array}{ccccc} 0 & \to & \mathcal{O}_V((D))D & \to & \Theta_V \\ & & & & \downarrow \\ 0 & \to & f^*\mathcal{O}_W((\Delta))\Delta & \to & f^*\Theta_W \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

Moreover, we have

LEMMA 2.1. *There exists a natural injection*

$$\mathcal{O}_V((D))D \to f^*\mathcal{O}_W((\Delta))\Delta.$$

Proof. Take an arbitrary point P of V and set $Q = f(P)$. Let $x_1, \ldots,$ x_n be a regular system of parameters of the local ring $\mathcal{O}_{W,Q}$. We can write

$$\Delta = \frac{g}{h}\left(f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}\right),$$

where $f_1, \ldots, f_n$, g, and h are regular functions at Q such that $f_1, \ldots, f_n$ have no common factors and g and h have no common factors. Then the divisor (Δ) is locally determined by the function g/h. Take regular functions

$x_{n+1}, \dots, x_m$ at P so that $x_1, \dots, x_n, x_{n+1}, \dots, x_m$ form a regular system of parameters of the local ring $\mathcal{O}_{V,P}$. We can write

$$D = \frac{g}{h} f_1 \frac{\partial}{\partial x_1} + \cdots + \frac{g}{h} f_n \frac{\partial}{\partial x_n} + D(x_{n+1}) \frac{\partial}{\partial x_{n+1}} + \cdots + D(x_m) \frac{\partial}{\partial x_m}.$$

Take regular functions h', $f_{n+1}, \dots, f_m$ at P such that

$$D(x_{n+1}) = \frac{f_{n+1}}{h'}, \quad \dots, \quad D(x_m) = \frac{f_m}{h'}.$$

We then have

$$\begin{aligned} D &= \frac{g}{h} f_1 \frac{\partial}{\partial x_1} + \cdots + \frac{g}{h} f_n \frac{\partial}{\partial x_n} + \frac{f_{n+1}}{h'} \frac{\partial}{\partial x_{n+1}} + \cdots + \frac{f_m}{h'} \frac{\partial}{\partial x_m} \\ &= \frac{1}{hh'} \left(h' g f_1 \frac{\partial}{\partial x_1} + \cdots + h' g f_n \frac{\partial}{\partial x_n} + h f_{n+1} \frac{\partial}{\partial x_{n+1}} + \cdots + h f_m \frac{\partial}{\partial x_m} \right). \end{aligned}$$

Let g' be the greatest common factor of $h'gf_1, \dots, h'gf_n, hf_{n+1}, \dots, hf_m$. The divisor (D) is thus locally determined by the function g'/hh'.

Now we use the following sublemma:

Sublemma 2.2. *Let $\varphi : A \hookrightarrow B$ be a flat extension of unique factorization domains. If $\alpha, \beta \in A$ have no common factors in A, then they have none in B.*

By the sublemma, $f_1, \dots, f_n$ have no common factors in $\mathcal{O}_{V,P}$. Hence g' divides $h'g$. So, $h'g/g'$ is regular at P. Since g'/hh' and g/h locally determine the divisors (D) and $f^*(\Delta)$ respectively, we have an injection $\mathcal{O}_V((D)) \to f^*\mathcal{O}_W((\Delta))$. Therefore, we only have to show the sublemma. ∎

Proof of Sublemma. Consider the multiplication by α in the residue ring of A by the ideal βA, i.e., $\alpha \cdot * : A/\beta A \to A/\beta A$. Since α and β have no common factors in A, the mapping $\alpha \cdot *$ is injective. By tensoring with B, we obtain the mapping $\alpha \cdot * \otimes_A B : A/\beta A \otimes_A B \to A/\beta A \otimes_A B$, that is, multiplication by α in the residue ring $B/\beta B$. Since B is flat over A, we know that $\alpha \cdot * \otimes_A B$ is injective. Hence α and β have no common factors in B.

Let us return to the subject and consider the quotient of a variety by a rational vector field. We retain the same references [7] and [9]. Let P be a closed point of V and set $Q = f(P)$. Assume that D and Δ have only divisorial singularities at P and at Q, respectively. By the previous lemma, we have an injection $\mathcal{O}_V((D))D \to f^*\mathcal{O}_W((\Delta))\Delta$. Let $\mathcal{R}$ be its cokernel. Consider the quotients V^D and W^Δ. We then have a morphism $g : V^D \to W^\Delta$. Let P' and Q' be the images of P and Q, respectively the canonical epimorphisms. We then have

Theorem 2.3. *Under the above notation and assumptions, g is smooth at P' if and only if P lies outside the support of $\mathcal{R}$.*

Proof. We take a regular system of parameters $(\xi_1, \ldots, \xi_m)$ of the completion $\widehat{\mathcal{O}}_{V,P}$ of $\mathcal{O}_{V,P}$ such that

$$D = h \frac{\partial}{\partial \xi_1},$$

where h determines the divisor (D). In addition, we take a regular system of parameters $(\tau_1, \ldots, \tau_n)$ of the completion $\widehat{\mathcal{O}}_{W,Q}$ of $\mathcal{O}_{W,Q}$ such that

$$\varDelta = h' \frac{\partial}{\partial \tau_1},$$

where h' determines the divisor $(\varDelta)$. Since $D|_{k(W)} = \varDelta$, we have $D(\tau_1) = \varDelta(\tau_1)$ and so

$$\frac{\partial \tau_1}{\partial \xi_1} = \frac{h'}{h}.$$

In particular, the function h'/h is in $\widehat{\mathcal{O}}_{V,P}$. Note that $\mathcal{R}$ is the cokernel of the injection $\mathcal{O}_V((D))D \to f^*\mathcal{O}_W((\varDelta))\varDelta$ and that h and h' are local equations of the divisors (D) and $(\varDelta)$, respectively. If P lies in the support of $\mathcal{R}$, then

$$\frac{h'}{h}(P) = 0$$

and so $\partial\tau_1/\partial\xi_1(P) = 0$. Otherwise, $h'/h(P) \neq 0$ and so $\partial\tau_1/\partial\xi_1(P) \neq 0$. Consider the Jacobian matrix

$$\begin{pmatrix} \frac{\partial \tau_1}{\partial \xi_1} & \cdots & \frac{\partial \tau_n}{\partial \xi_1} \\ \vdots & & \vdots \\ \frac{\partial \tau_1}{\partial \xi_m} & \cdots & \frac{\partial \tau_n}{\partial \xi_m} \end{pmatrix}.$$

Its rank is n everywhere since f is smooth. On the other hand , since $D(\tau_2) = \cdots = D(\tau_n) = 0$, we have

$$\frac{\partial \tau_2}{\partial \xi_1} = \cdots = \frac{\partial \tau_n}{\partial \xi_1} = 0.$$

Hence the rank of

$$\begin{pmatrix} \frac{\partial \tau_2}{\partial \xi_2}(P) & \cdots & \frac{\partial \tau_n}{\partial \xi_2}(P) \\ \vdots & & \vdots \\ \frac{\partial \tau_2}{\partial \xi_m}(P) & \cdots & \frac{\partial \tau_n}{\partial \xi_m}(P) \end{pmatrix}.$$

is $n-1$. Next, consider the Jacobian matrix of the fibration $g : V^D \to W^{\varDelta}$. Since $D = h\partial/\partial\xi_1$ and $\varDelta = h'\partial/\partial\tau_1$, we know that $(\xi_1^p, \xi_2, \ldots, \xi_m)$ and $(\tau_1^p, \tau_2, \ldots, \tau_n)$ form regular systems of parameters of the completions $\widehat{\mathcal{O}}_{V^D,P'}$ and $\widehat{\mathcal{O}}_{W^{\varDelta},Q'}$, respectively. Therefore, the Jacobian matrix of g :

$V^D \to W^\Delta$ is

$$\begin{pmatrix} \frac{\partial \tau_1^p}{\partial \xi_1^p}(P') & \dots & \frac{\partial \tau_n}{\partial \xi_1^p}(P') \\ \vdots & & \vdots \\ \frac{\partial \tau_1^p}{\partial \xi_m}(P') & \dots & \frac{\partial \tau_n}{\partial \xi_m}(P') \end{pmatrix}.$$

Here, except for the first column and the first row, each entry coincides with that of the earlier Jacobian matrix. Moreover, we have

$$\frac{\partial \tau_1^p}{\partial \xi_2} = \dots = \frac{\partial \tau_1^p}{\partial \xi_m} = 0.$$

So, the rank of the last Jacobian matrix is n if and only if

$$\frac{\partial \tau_1^p}{\partial \xi_1^p}(P') \neq 0.$$

Moreover, since

$$\frac{\partial \tau_1^p}{\partial \xi_1^p} = \left(\frac{\partial \tau_1}{\partial \xi_1}\right)^p,$$

the fibration g is smooth at P' if and only if

$$\frac{\partial \tau_1}{\partial \xi_1}(P) \neq 0.$$

This yields the stated assertion. ∎

REMARK 2.4. The author has introduced in [10] the concept of tangent locus, which is the support of the cokernel of $\mathcal{O}_V((D))D \to f^*\Theta_W$ in the case where V is a smooth surface and W is a smooth curve. In addition, the author has shown that the fibre of g is singular only at the points lying on the image of the tangent locus. It is easy to verify that the tangent sheaf Θ_W is isomorphic to $\mathcal{O}_W((\Delta))$. Therefore, the previous theorem is an extension of the result in [10].

3. Pre-Tango structures on varieties. In this section, we introduce the notion of pre-Tango structure on a variety. Let k be an algebraically closed field of characteristic $p > 0$ and let X be a projective variety over k, i.e., a projective integral separated scheme of finite type over k. Consider the relative Frobenius morphism $F : \widetilde{X} \to X$ over k. We then have a short exact sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_{\widetilde{X}} \to F_*\mathcal{B}^1_{\widetilde{X}} \to 0$$

on X, where $\mathcal{B}^1_{\widetilde{X}}$ is the first sheaf of coboundaries of the de Rham complex of $\widetilde{X}$. Throughout the present article, $F_*\mathcal{O}_{\widetilde{X}}$ and $F_*\mathcal{B}^1_{\widetilde{X}}$ are often abbreviated as $\mathcal{O}_{\widetilde{X}}$ and $\mathcal{B}^1_{\widetilde{X}}$ for the sake of simplicity.

DEFINITION 3.1. We call any ample invertible subsheaf of the $\mathcal{O}_X$-module $F_*\mathcal{B}^1_{\widetilde{X}}$ a *pre-Tango structure* on X.

Suppose that we have a pre-Tango structure $\mathcal{L} \subset F_*\mathcal{B}^1_{\widetilde{X}}$. We then have an extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{L} \to 0,$$

where $\mathcal{E}$ is the inverse image of $\mathcal{L}$ in $F_*\mathcal{O}_{\widetilde{X}}$. We also have a canonical homomorphism $\mathrm{Symm}(\mathcal{E}) \to \mathcal{O}_{\widetilde{X}}$, where $\mathrm{Symm}(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$. Consider the scheme $\widehat{X}$ whose underlying topological space is the same as that of X and whose structure sheaf is the image of the above-mentioned canonical homomorphism. Then $\widehat{X}$ is a variety. Moreover, there is a purely inseparable morphism $G : \widehat{X} \to X$ of degree p which splits the Frobenius morphism F.

DEFINITION 3.2. In the above notation, we call the pre-Tango structure $\mathcal{L}$ a *Tango structure* if $\widehat{X}$ is nonsingular.

Keep the same notation as above. Let $\{U_i\}_{i\in I}$ be an affine open covering of X. Pick local sections $\{q_i\}_{i\in I}$ of $\mathcal{O}_{\widetilde{X}}$ whose images $\{dq_i\}_{i\in I}$ in $\mathcal{B}^1_{\widetilde{X}}$ are local generators of the pre-Tango structure $\mathcal{L}$. Then the restriction of the structure sheaf $\mathcal{O}_{\widehat{X}}$ to U_i coincides with $\mathcal{O}_{U_i}[q_i]$. For each $i \in I$, we consider a derivation $\Delta_i^* : \mathcal{O}_{\widehat{X}}|_{U_i} \to G^*\mathcal{L}|_{U_i}$ such that

$$\Delta_i^*(fq_i^e) = efq_i^{e-1}dq_i,$$

where f is a local section of $\mathcal{O}_X$ and e is a nonnegative integer. We then have a local homomorphism $\Omega^1_{\widehat{X}/k}|_{U_i} \to G^*\mathcal{L}|_{U_i}$ for each $i \in I$. We can glue those homomorphisms together to obtain a surjective homomorphism

$$\Omega^1_{\widehat{X}/k} \to G^*\mathcal{L}. \tag{1}$$

Hence we have an injection

$$G^*\mathcal{L}^{-1} \to \Theta_{\widehat{X}} \tag{2}$$

and a p-closed rational vector field Δ on $\widehat{X}$ obtained by gluing $\{\partial/\partial q_i\}_{i\in I}$ together. Furthermore, the structure sheaf $\mathcal{O}_X$ coincides with the sheaf $\mathcal{O}^{\Delta}_{\widehat{X}}$ of germs killed by Δ. In other words, X is the quotient of $\widehat{X}$ by Δ.

LEMMA 3.3 (cf. [4]). *Under the above notation and assumptions, suppose furthermore, that $\mathcal{L}$ is a Tango structure. Then X is nonsingular and the invertible sheaf $\mathcal{O}_{\widehat{X}}((\Delta))$ associated to the divisor of Δ is isomorphic to $G^*\mathcal{L}^{-1}$.*

Proof. Since $\mathcal{L}$ is a Tango structure, $\widehat{X}$ is nonsingular. Therefore, $\Omega^1_{\widehat{X}/k}$ and $\Theta_{\widehat{X}}$ are locally free $\mathcal{O}_{\widehat{X}}$-modules. By considering the homomorphisms (1) and (2), we see that each q_i can be one of coordinates on U_i and that $G^*\mathcal{L}^{-1}$

is a line subbundle of $\Theta_{\widehat{X}}$. Hence it follows immediately that $\mathcal{O}_{\widehat{X}}((\Delta)) \cong G^*\mathcal{L}^{-1}$. Furthermore, the p-closed rational vector field Δ has only divisorial singularities. Hence the quotient X is nonsingular. ∎

Remark 3.4 (cf. [4]). Keep the notation and assumptions as in the preceding lemma. Recall that the rational vector field Δ on $\widehat{X}$ is obtained by gluing $\{\partial/\partial q_i\}_{i\in I}$ together. Therefore, we know that, if $dq_i, d\tau_2, \dots, d\tau_d$ locally generate $\Omega^1_{\widehat{X}/k}$, then $da_i, d\tau_2, \dots, d\tau_d$ locally generate $\Omega^1_{X/k}$, where $\tau_2, \dots, \tau_d$ are regular fuctions on $\widehat{X}$ and $a_i = q_i^p$. On the other hand, we have an injection $\mathcal{L}^p \to \Omega^1_{X/k}$ which is the adjoint mapping of the injection $\mathcal{L} \to F_*\mathcal{B}^1_{\widetilde{X}} \to F_*\Omega^1_{\widetilde{X}/k}$. We know that $\{da_i\}_{i\in I}$ are local generators of the image of $\mathcal{L}^p$ in $\Omega^1_{X/k}$ and that the quotient $\Omega^1_{X/k}/\mathcal{L}^p$ is a locally free $\mathcal{O}_X$-module. Moreover, we have the following exact sequence of locally free $\mathcal{O}_{\widehat{X}}$-modules:

$$0 \to G^*\mathcal{L}^p \to G^*\Omega^1_{X/k} \to \Omega^1_{\widehat{X}/k} \to G^*\mathcal{L} \to 0$$

on $\widehat{X}$. Furthermore, we obtain an isomorphism

$$G^*\omega_{X/k} \otimes_{\mathcal{O}_{\widehat{X}}} G^*\mathcal{L} \cong \omega_{\widehat{X}/k} \otimes_{\mathcal{O}_{\widehat{X}}} G^*\mathcal{L}^p.$$

Remark 3.5. Assume, in addition, that the dimension of X is one. Since X is nonsingular, $G : \widehat{X} \to X$ coincides with $F : \widetilde{X} \to X$ and the surjection $\Omega^1_{\widehat{X}/k} \to G^*\mathcal{L}$ is an isomorphism. In that case, $\Omega^1_{X/k} \cong \mathcal{L}^p$. So, $\mathcal{L}$ is a Tango structure in the sense of [14] (see also Corollary 3.7 below).

By Lemma 3.3 and Remark 3.4, in the case where $\mathcal{L}$ is a Tango structure on X, we infer that X is nonsingular and that the cokernel of the injection $\mathcal{L}^p \to \Omega^1_{X/k}$ is locally free. Conversely, we have the following:

Lemma 3.6 ([4]). *Suppose that X is nonsingular and that $\mathcal{L}$ is a pre-Tango structure on X. If the cokernel of the injection $\mathcal{L}^p \to \Omega^1_{X/k}$ is locally free, then $\mathcal{L}$ is a Tango structure.*

Proof. Similarly to the argument above, take an affine open covering $\{U_i\}_{i\in I}$ and choose local sections $\{q_i\}_{i\in I}$ of $\mathcal{O}_{\widetilde{X}}$ such that $\{dq_i\}_{i\in I}$ are local generators of $\mathcal{L}$ in $\mathcal{B}^1_{\widetilde{X}}$. Set $a_i = q_i^p$ for each $i \in I$. Then $\{da_i\}_{i\in I}$ are local generators of the image of $\mathcal{L}^p$ in $\Omega^1_{X/k}$. Take local sections $\eta_{i1}, \dots, \eta_{id}$ in $\mathcal{O}_X(U_i)$ such that $\Omega^1_{X/k}$ is locally generated by $d\eta_{i1}, \dots, d\eta_{id}$. Consider the differential 1-form

$$da_i = \frac{\partial a_i}{\partial \eta_{i1}}\, d\eta_{i1} + \dots + \frac{\partial a_i}{\partial \eta_{id}}\, d\eta_{id}. \tag{3}$$

It is a local generator of $\mathcal{L}^p$. By the assumption, at least one of the partial derivatives $\partial a_i/\partial\eta_{i1}, \dots, \partial a_i/\partial\eta_{id}$ appearing in (3) is not zero. On the other hand, $\widehat{X}$ locally coincides with $\operatorname{Spec}\mathcal{O}_X(U_i)[q_i]$ subjected to the relation

$q_i^p = a_i$. By applying the Jacobian criterion, we find that $\widehat{X}$ is nonsingular. Hence we conclude that $\mathcal{L}$ is Tango structure. ∎

From the previous lemma, we immediately obtain the following corollary:

COROLLARY 3.7. *Under the above notation and assumptions, assume in addition that the dimension of X is one. If the injection $\mathcal{L}^p \to \Omega^1_{X/k}$ is surjective, then $\mathcal{L}$ is a Tango structure.*

4. Uniruled varieties induced from pre-Tango structures. In the preceding section, we have defined the notion of pre-Tango structure on a variety. Now we show that if there exists a pre-Tango structure on a d-dimensional variety, then we almost always have a $(d+1)$-dimensional uniruled variety with a pre-Tango structure.

Suppose that there exists a pre-Tango structure $\mathcal{L}$ on a projective variety X over an algebraically closed field k of characteristic $p > 0$. Suppose, furthermore, that there exist a natural number $n > 1$ and an invertible sheaf $\mathcal{N}$ on X such that $\mathcal{N}^n \cong \mathcal{L}$ and n is prime to p. Then $\mathcal{N}$ is ample. Let $G : \widehat{X} \to X$ be a morphism as in the previous section. Set

$$\mathfrak{P} = \mathrm{Proj}(\mathrm{Symm}(G^*\mathcal{N}^{-1} \oplus \mathcal{O}_{\widehat{X}}))$$

and let $\psi : \mathfrak{P} \to \widehat{X}$ be the canonical morphism. We have two sections S and T such that

$$\mathcal{O}_S(S) \cong G^*\mathcal{N}, \quad \mathcal{O}_T(T) \cong G^*\mathcal{N}^{-1}, \quad \mathcal{O}_{\mathfrak{P}}(S) \cong \mathcal{O}_{\mathfrak{P}}(T) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{N}.$$

Furthermore, we know that

$$\omega_{\mathfrak{P}/\widehat{X}} \cong \mathcal{O}_{\mathfrak{P}}(-2S) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{N} \cong \mathcal{O}_{\mathfrak{P}}(-2T) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{N}^{-1}.$$

Take an affine open covering $\{U_i\}_{i\in I}$ and transition functions $\{d_{ij}\}_{i,j\in I}$ of $\mathcal{N}$ such that $dq_i = d_{ij}^n dq_j$, where $\{q_i\}_{i\in I}$ are sections of $\mathcal{O}_{\widetilde{X}}$ as in the previous section, i.e., dq_i is a local generator of $\mathcal{L}|_{U_i}$ in $\mathcal{B}^1_{\widetilde{X}}$. Let $\{t_i\}_{i\in I}$ be local inhomogeneous coordinates of $\mathfrak{P}$ such that $t_i = d_{ij}t_j$ for $i, j \in I$. Then T is locally defined by $t_i = 0$. Set $s_i = t_i^{-1}$. Then S is locally defined by $s_i = 0$. Consider a p-closed rational vector field

$$D_i = \frac{\partial}{\partial q_i} + \frac{1}{nt_i^{n-1}}\frac{\partial}{\partial t_i}$$

on $\psi^{-1}(U_i)$. We can glue $\{D_i\}_{i\in I}$ together to obtain a p-closed rational vector field D on $\mathfrak{P}$. Since

$$D_i = \frac{1}{t_i^{n-1}}\left(t_i^{n-1}\frac{\partial}{\partial q_i} + \frac{1}{n}\frac{\partial}{\partial t_i}\right) = \frac{\partial}{\partial q_i} - \frac{s_i^{n+1}}{n}\frac{\partial}{\partial s_i}$$

and

$$D_i = \frac{1}{d_{ij}^n} D_j,$$

we deduce that the rational vector field D corresponds to a global section of $\Theta_{\mathfrak{P}} \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathcal{O}_{\mathfrak{P}}((n-1)T) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^* G^* \mathcal{N}^n$. Recall that $\mathcal{O}_{\widehat{X}}((\Delta)) \cong G^* \mathcal{L}^{-1}$ if $\mathcal{L}$ is a Tango structure. Hence, in the case of a Tango structure, the invertible sheaf $\mathcal{O}_{\mathfrak{P}}((D))$ associated to the divisor of D is isomorphic to

$$\mathcal{O}_{\mathfrak{P}}(-(n-1)T) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^* G^* \mathcal{N}^{-n}.$$

Consider the invariant field $k(\mathfrak{P})^D$, and denote the quotient $(\mathfrak{P}, \mathcal{O}_{\mathfrak{P}} \cap k(\mathfrak{P})^D)$ by $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. We then also have a purely inseparable finite morphism $\pi : \mathfrak{P} \to \mathfrak{X}$ of degree p, and so $\mathfrak{X}$ is a uniruled variety. On the other hand, $D|_{k(\widehat{X})} = \Delta$, where Δ is the rational vector field on $\widehat{X}$ induced from the pre-Tango structure $\mathcal{L}$ in the previous section. Since $\widehat{X}^{\Delta} = X$, we have a morphism $\varphi : \mathfrak{X} \to X$ making the following diagram commutative:

$$\begin{array}{ccc} \mathfrak{P} & \xrightarrow{\pi} & \mathfrak{X} \\ \psi \downarrow & & \downarrow \varphi \\ \widehat{X} & \xrightarrow{G} & X \end{array}$$

Let $\widehat{X}_0$ be the set of smooth points of $\widehat{X}$. Set $X_0 = G(\widehat{X}_0)$, $\mathfrak{P}_0 = \psi^{-1}(\widehat{X}_0)$, and $\mathfrak{X}_0 = \pi(\mathfrak{P}_0)$. By Lemma 3.3 and its proof, X_0 is nonsingular and the rational vector field Δ has only divisorial singularities on $\widehat{X}_0$. Therefore, the rational vector field D has only divisorial singularities on $\mathfrak{P}_0$, and so the quotient $\mathfrak{X}_0$ is nonsingular. Moreover, we have

THEOREM 4.1. *Under the above notation and assumptions*:

(1) *The morphism φ is smooth at a point P of $\mathfrak{X}_0$ if and only if P is not in the image of T under π.*

(2) *Each fibre of $\varphi|_{\mathfrak{X}_0}$ is a rational curve with one cusp of type $x^n + y^p = 0$.*

Proof. (1) Since $\widehat{X}_0$ and $\mathfrak{P}_0$ are nonsingular, we can define the divisors $(\Delta|_{\widehat{X}_0})$ on $\widehat{X}_0$ and $(D|_{\mathfrak{P}_0})$ on $\mathfrak{P}_0$. Recall that the rational vector field D is locally described as

$$\frac{1}{t_i^{n-1}}\left(t_i^{n-1}\frac{\partial}{\partial q_i} + \frac{1}{n}\frac{\partial}{\partial t_i}\right)$$

and also as

$$\frac{\partial}{\partial q_i} - \frac{s_i^{n+1}}{n}\frac{\partial}{\partial s_i}.$$

Since $\{t_i = 0\}_{i \in I}$ define the divisor T, the support of the cokernel of the injection

$$\mathcal{O}_{\mathfrak{P}_0}((D|_{\mathfrak{P}_0}))D|_{\mathfrak{P}_0} \to \psi^* \mathcal{O}_{\widehat{X}_0}((\Delta|_{\widehat{X}_0}))\Delta|_{\widehat{X}_0}$$

considered in Section 2 coincides with $T|_{\mathfrak{P}_0}$. So, by Theorem 2.3, we have the stated assertion.

(2) Take a point $\widehat{P}$ in $\mathfrak{P}_0$ lying on T. Set $P = \pi(\widehat{P})$, $\widehat{Q} = \psi(\widehat{P})$ and $Q = G(\widehat{Q})$. Note that q_i is a local coordinate at $\widehat{Q}$. Set $a_i = q_i^p$. Then a_i is a local coordinate at Q. Set $x_i = t_i^p$. Then x_i is a regular function at P. Furthermore, set $y_i = q_i - t_i^n$. By a simple computation, $D(y_i) = 0$. Hence y_i is a regular function at P. Moreover, we have a relation

$$y_i^p - a_i + x_i^n = 0.$$

Since the rank of the Jacobian matrix of this equation is one, we can regard the equation as a local defining equation of $\mathfrak{X}$. Furthermore, the fibre $\varphi^{-1}(Q)$ is defined by $y_i^p + x_i^n = 0$ at P. ∎

Next we consider the short exact sequence

$$0 \to \mathcal{O}_{\mathfrak{X}} \to \mathfrak{F}_* \mathcal{O}_{\widetilde{\mathfrak{X}}} \to \mathfrak{F}_* \mathcal{B}^1_{\widetilde{\mathfrak{X}}} \to 0,$$

where $\mathfrak{F} : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ is the relative Frobenius morphism over k. Let Γ and Σ denote the images under π of S and T, respectively. We consider them each with the reduced induced structure. Let q_i, a_i, t_i, s_i, x_i, y_i be as above. Since p and n are prime to each other, there exist two integers α and β such that $\alpha p + \beta n = 1$. We may assume that β is positive. Set $w_i = s_i(q_i s_i^n - 1)^{-\beta}$. Then, by a simple computation, we have $D(w_i) = 0$ for the above-mentioned vector field D. So, w_i is a regular function on $\mathfrak{X}|_{U_i}$ near $\Gamma|_{U_i}$. Set $z_i = s_i^p$. Then z_i is a regular function on $\mathfrak{X}|_{U_i}$ near $\Gamma|_{U_i}$. Furthermore, we have a relation

$$w_i^p = z_i(a_i z_i^n - 1)^{-\beta}. \tag{4}$$

On the other hand, $y_i^p = a_i - x_i^n$ near $\Sigma|_{U_i}$. Since $x_i = z_i^{-1}$, we obtain

$$y_i z_i^{n-1} = a_i z_i^{n-1} - x_i. \tag{5}$$

For simplicity, we omit the index i for the time being. From (4), it follows that $z = w^p(az^n - 1)^\beta$. By combining this relation and (5), we have

$$y^p w^{p(n-1)}(az^n - 1)^{\beta(n-1)} = a w^{p(n-1)}(az^n - 1)^{\beta(n-1)} - x.$$

This can be rewritten as

$$y^p w^{p(n-1)}\left((-1)^{\beta(n-1)} + \sum_{l=1}^{\beta(n-1)} \binom{\beta(n-1)}{l} (az^n)^l (-1)^{\beta(n-1)-l}\right)$$
$$= a w^{p(n-1)}(az^n - 1)^{\beta(n-1)} - x.$$

Hence we obtain

$$y^p w^{p(n-1)}(-1)^{\beta(n-1)} = a w^{p(n-1)}$$
$$\times \left(-y^p z^n \sum_{l=1}^{\beta(n-1)} \binom{\beta(n-1)}{l} (az^n)^{l-1}(-1)^{\beta(n-1)-l} - (az^n - 1)^{\beta(n-1)}\right) - x.$$

Since $y^p z^n = az^n - 1$, we know that $y^p z^n$ is regular near Γ and hence so is the function

$$(6)\quad -y^p z^n \sum_{l=1}^{\beta(n-1)} \binom{\beta(n-1)}{l} (az^n)^{l-1}(-1)^{\beta(n-1)-l} - (az^n-1)^{\beta(n-1)}.$$

Moreover, by using $\alpha p + \beta n = 1$, we have

$$\binom{\beta(n-1)}{1}(-1)^{\beta(n-1)-1} - (-1)^{\beta(n-1)} = (-1)^{\beta(n-1)}.$$

Therefore, the regular function (6) is invertible near Γ. Set

$$a_i^* = a_i\left(-y_i^p z_i^n \sum_{l=1}^{\beta(n-1)} \binom{\beta(n-1)}{l} (a_i z_i^n)^{l-1}(-1)^{\beta(n-1)-l} - (a_i z_i^n - 1)^{\beta(n-1)}\right)$$

for each $i \in I$. We then have

$$y_i^p w_i^{p(n-1)}(-1)^{\beta(n-1)} = w_i^{p(n-1)} a_i^* - x_i.$$

Set $q_i^* = \sqrt[p]{a_i^*}$, i.e.,

$$(7)\quad q_i^* = q_i\left(-y_i s_i^n \sum_{l=1}^{\beta(n-1)} \binom{\beta(n-1)}{l} (q_i s_i^n)^{l-1}(-1)^{\beta(n-1)-l} - (q_i s_i^n - 1)^{\beta(n-1)}\right).$$

Then q_i^* is a local section of $\mathcal{O}_{\widetilde{\mathfrak{X}}}$ and

$$y_i w_i^{n-1}(-1)^{\beta(n-1)} = w_i^{n-1} q_i^* - t_i.$$

Consider the images of t_i and q_i^* in $\mathcal{B}^1_{\widetilde{\mathfrak{X}}}$. We have

$$dt_i = w_i^{n-1} dq_i^*, \qquad dt_i = d_{ij} dt_j.$$

Recall that $\{d_{ij}\}_{i,j\in I}$ are transition functions of $\mathcal{N}$. Since $w_i = 0$ is a local equation of Γ, we know that $\{dt_i\}_{i\in I}$ and $\{dq_i^*\}_{i\in I}$ generate an invertible subsheaf of $\mathcal{B}^1_{\widetilde{\mathfrak{X}}}$, which is isomorphic to $\mathcal{O}_{\mathfrak{X}}((n-1)\Gamma) \otimes_{\mathcal{O}_{\mathfrak{X}}} \varphi^*\mathcal{N}$.

Now we state the following

THEOREM 4.2 ([4]). *Under the above notation and assumptions, there exists a pre-Tango sturcture on $\mathfrak{X}$ which is isomorphic to $\mathcal{O}_{\mathfrak{X}}((n-1)\Gamma) \otimes_{\mathcal{O}_{\mathfrak{X}}} \varphi^*\mathcal{N}$.*

Proof. We only have to show the ampleness of $\mathcal{O}_{\mathfrak{X}}((n-1)\Gamma) \otimes_{\mathcal{O}_{\mathfrak{X}}} \varphi^*\mathcal{N}$. Since S is an integral subvariety for the rational vector field D, we have $\pi^*\mathcal{O}_{\mathfrak{X}}(\Gamma) \cong \mathcal{O}_{\mathfrak{P}}(S)$ (see [7]). Hence it suffices to show that $\mathcal{O}_{\mathfrak{P}}((n-1)S) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^* G^* \mathcal{N}$ is ample on $\mathfrak{P}$. Recall that $\mathcal{N}$ is ample and that S is a section such that $\mathcal{O}_S(S) \cong G^*\mathcal{N}$. Now the Nakai–Moishezon criterion yields the required assertion immediately. ∎

In the case where $\mathcal{L}$ is a Tango structure, we have

THEOREM 4.3 ([4]). *Under the above notation and assumptions, suppose furthermore that $\mathcal{L}$ is a Tango structure. Then the pre-Tango structure of the preceding theorem is a Tango structure.*

Proof. Since $\mathcal{L}$ is a Tango structure on X, we know that $\widehat{X}$ is nonsingular, and so is $\mathfrak{P}$. Furthermore, $\{q_i\}_{i\in I}$ are local coordinates on $\widehat{X}$ (see proof of Lemma 3.3) and $\{a_i\}_{i\in I}$ are local coordinates on X. By Theorem 4.1(1), φ is smooth near Γ. Hence $\{a_i\}_{i\in I}$ are local coordinates near Γ, and hence so are $\{a_i^*\}_{i\in I}$. On the other hand, $\mathfrak{X}$ is locally determined by the equation $y_i^p - a_i + x_i^n = 0$ near Σ. Hence x_i are local coordinates near Σ. Recall that $\{dt_i,\, dq_i^*\}_{i\in I}$ are local generators of the pre-Tango structure in question. Therefore, $\{dx_i,\, da_i^*\}_{i\in I}$ in $\Omega^1_{\mathfrak{X}/k}$ are local generators of the image of the p-fold tensor power of the pre-Tango structure. By Lemma 3.6, our pre-Tango structure is a Tango structure on $\mathfrak{X}$. ∎

REMARK 4.4. From the local equation $y_i^p - a_i + x_i^n = 0$, we obtain a relation $nx_i^{n-1}dx_i = da_i$. So, we have

$$\frac{1}{x_i^{n-1}}\, da_i = ndx_i.$$

Recall that $\{da_i\}_{i\in I}$ are local generators of $\mathcal{L}^p$ on X. Since $\{x_i = 0\}_{i\in I}$ are local equations of Σ, we have an injection

$$\mathcal{O}_{\mathfrak{X}}((n-1)\Sigma) \otimes_{\mathcal{O}_{\mathfrak{X}}} \varphi^*\mathcal{L}^p \to \Omega^1_{\mathfrak{X}/k}.$$

On the other hand, it is easy to verify that $\mathcal{O}_{\mathfrak{X}}(\Sigma) \cong \mathcal{O}_{\mathfrak{X}}(p\Gamma) \otimes_{\mathcal{O}_{\mathfrak{X}}} \varphi^*\mathcal{N}^{-p}$. Indeed, by identifying q_i with y_i on Σ, there exists a closed immersion $\widehat{X} \to \mathfrak{X}$ whose image coincides with Σ, and so Σ is a p-section:

$$\begin{array}{ccc} \Sigma & \hookrightarrow & \mathfrak{X} \\ \uparrow & & \downarrow \varphi \\ \widehat{X} & \xrightarrow{G} & X \end{array}$$

Therefore, we have an injection

$$\mathcal{O}_{\mathfrak{X}}(p(n-1)\Gamma) \otimes_{\mathcal{O}_{\mathfrak{X}}} \varphi^*\mathcal{N}^p \to \Omega^1_{\mathfrak{X}/k}.$$

Its image coincides with that of the p-fold tensor power of the pre-Tango structure mentioned in the previous proof. Furthermore, this injection is also considered in the proof of the first theorem in the next section.

REMARK 4.5. Assume that $\mathcal{L}$ is a Tango structure. Then $\mathfrak{X}$ is nonsingular and there exists the above-mentioned Tango structure on $\mathfrak{X}$. By using it, we obtain a finite covering $\widehat{\mathfrak{X}} \to \mathfrak{X}$ such that $\widehat{\mathfrak{X}}$ is nonsingular. The function field $k(\widehat{\mathfrak{X}})$ coincides with $k(\mathfrak{X})(q_i^*)$. Recall the relation (7). Since q_i, s_i and y_i are in $k(\mathfrak{P})$, so also are the q_i^*. Therefore, $k(\widehat{\mathfrak{X}}) = k(\mathfrak{P})$, and so $\widehat{\mathfrak{X}} = \mathfrak{P}$.

By combining Remark 3.4 and the previous remark, we have the exact sequence

$$\begin{aligned}0 \to \pi^*(\mathcal{O}_{\mathfrak{X}}(p(n-1)\Gamma) \otimes_{\mathcal{O}_{\mathfrak{X}}} \varphi^*\mathcal{N}^p) \to \pi^*\Omega^1_{\mathfrak{X}/k} \\ \to \Omega^1_{\mathfrak{P}/k} \to \pi^*(\mathcal{O}_{\mathfrak{X}}((n-1)\Gamma) \otimes_{\mathcal{O}_{\mathfrak{X}}} \varphi^*\mathcal{N}) \to 0.\end{aligned}$$

We can rewrite it as

$$\begin{aligned}0 \to \mathcal{O}_{\mathfrak{P}}(p(n-1)S) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{N}^p \to \pi^*\Omega^1_{\mathfrak{X}/k} \\ \to \Omega^1_{\mathfrak{P}/k} \to \mathcal{O}_{\mathfrak{P}}((n-1)S) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{N} \to 0.\end{aligned}$$

By applying the isomorphism

$$\mathcal{O}_{\mathfrak{P}}(S) \cong \mathcal{O}_{\mathfrak{P}}(T) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{N},$$

we can furthermore rewrite it as

$$\begin{aligned}0 \to \mathcal{O}_{\mathfrak{P}}(p(n-1)T) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{L}^p \to \pi^*\Omega^1_{\mathfrak{X}/k} \\ \to \Omega^1_{\mathfrak{P}/k} \to \mathcal{O}_{\mathfrak{P}}((n-1)T) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{L} \to 0.\end{aligned}$$

On the other hand, on $\widehat{X}$, we have the exact sequence

$$0 \to G^*\mathcal{L}^p \to G^*\Omega^1_{X/k} \to \Omega^1_{\widehat{X}/k} \to G^*\mathcal{L} \to 0$$

induced from the Tango structure $\mathcal{L}$. Therefore, we obtain a diagram of locally free $\mathcal{O}_{\mathfrak{P}}$-modules:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \to & \psi^*G^*\mathcal{L}^p & \to & \mathcal{O}_{\mathfrak{P}}(p(n-1)T) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{L}^p \\
 & & \downarrow & & \downarrow \\
0 & \to & \psi^*G^*\Omega^1_{X/k} & \to & \pi^*\Omega^1_{\mathfrak{X}/k} \\
 & & \downarrow & & \downarrow \\
0 & \to & \psi^*\Omega^1_{\widehat{X}/k} & \to & \Omega^1_{\mathfrak{P}/k} \\
 & & \downarrow & & \downarrow \\
0 & \to & \psi^*G^*\mathcal{L} & \to & \mathcal{O}_{\mathfrak{P}}((n-1)T) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{L} \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

Here, note that we naturally deduce the top and bottom horizontal injections. Furthermore, the dual of the bottom injection

$$\mathcal{O}_{\mathfrak{P}}(-(n-1)T) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^*G^*\mathcal{L}^{-1} \to \psi^*G^*\mathcal{L}^{-1}$$

corresponds to the injection considered in Section 2, i.e.,

$$\mathcal{O}_{\mathfrak{P}}((D))D \to \psi^*\mathcal{O}_{\widehat{X}}((\Delta))\Delta.$$

5. Properties of the induced uniruled varieties. In this section, we shall deduce some pathological properties of the uniruled varieties induced from pre-Tango structures. These phenomena are well known in dimension two (see Lang [3], [4], and [8]). It seems that such properties are ascribed to the cuspidal points of the general fibres. The arguments mentioned below are analogous to those in the author's earlier articles [12]–[14] apart from changes relating to dimension. However, we restate them for the reader's convenience. In the present section, we employ the previous notation X, $\widehat{X}$, $\mathfrak{P}$, $\mathfrak{X}$ etc.

First, we consider the case where there exists a pre-Tango structure $\mathcal{L}$ on X. Suppose that $\mathcal{N}$ is an invertible sheaf such that $\mathcal{N}^n \cong \mathcal{L}$ with n prime to p. Recall that we have an extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{L} \to 0.$$

Tensoring by $\mathcal{N}^p$, we obtain

$$0 \to \mathcal{N}^p \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{N}^p \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}^p \to 0.$$

We prove the following theorem:

THEOREM 5.1. *Under the above notation and assumptions, if*

$$H^0(X, \mathcal{N}^p) \subsetneq H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{N}^p),$$

then there exist nonclosed global differential 1*-forms on* $\mathfrak{X}$.

Proof. Let x_i, y_i, a_i be regular functions on $\mathfrak{X}$ as in Section 4. Then $y_i^p - a_i + x_i^n = 0$. Set $u_i = x_i^{-1}$ and $v_i = y_i u_i^m$ where m is the smallest integer greater than n/p. Then u_i, v_i and a_i are regular near Γ, the image of S under π. Moreover, $v_i^p - u_i^{mp-n}(a_i u_i^n - 1) = 0$. By exterior differentiation, $da_i = n x_i^{n-1} dx_i$. Hence

$$n dx_i = u_i^{n-1} da_i,$$

and so

$$n y_i dx_i = v_i u_i^{n-1-m} da_i.$$

Since n is a natural number greater than one, $n-1-m$ is non-negative. Hence the right hand sides of the previous two equations show that the differential forms are regular on $(\mathfrak{X} - \Gamma)|_{\varphi^{-1}(U_i)}$, while the left hand sides show that they are regular on $(\mathfrak{X} - \Sigma)|_{\varphi^{-1}(U_i)}$. Therefore, the two differential forms are regular on $\mathfrak{X}|_{\varphi^{-1}(U_i)}$. Now consider the sub-$\mathcal{O}_X$-module $\mathcal{P}$ of the direct image $\varphi_* \Omega^1_{\mathfrak{X}/k}$ which is locally generated by $\{dx_i\}_{i \in I}$, and the sub-$\mathcal{O}_X$-module $\mathcal{Q}$ of $\varphi_* \Omega^1_{\mathfrak{X}/k}$ which is locally generated by $\{dx_i\}_{i \in I}$ and $\{y_i dx_i\}_{i \in I}$. Since $dx_i = d_{ij}^p dx_j$ for $i, j \in I$, the invertible sheaf $\mathcal{P}$ is isomorphic to $\mathcal{N}^p$. Moreover, since $\{y_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ have the same transition matrices, the locally free $\mathcal{O}_X$-module $\mathcal{Q}$ is isomorphic to $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{N}^p$ and the inclusion $\mathcal{P} \hookrightarrow \mathcal{Q}$ corresponds to the injection $\mathcal{N}^p \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{N}^p$ of the above-mentioned short exact sequence. Note that, on $\mathfrak{X}$, the differential 1-form dx_i is closed but $y_i dx_i$ is not. Hence we obtain the stated assertion. ∎

REMARK 5.2. If $\mathfrak{X}$ has nonclosed global differential 1-forms, then so does the desingularization of $\mathfrak{X}$. On the other hand, by Corollaire 2.4 in Deligne and Illusie [1], for any smooth proper variety over k of dimension $\leq p$ which lifts over the ring of Witt vectors of length two, its Hodge–de Rham spectral sequence degenerates at E_1, and so it has no nonclosed global differential forms.

As proved in the preceding section, if $\mathcal{L}$ is a Tango structure, then the induced uniruled variety $\mathfrak{X}$ is a nonsingular variety which has a Tango structure. The latter nonsingular uniruled variety clearly gives a counter-example to the Kodaira vanishing theorem in positive characteristic. Here we mention another pathological phenomenon arising from certain Tango structures. That phenomenon strictly depends on the type of the cusps of the fibration $\varphi : \mathfrak{X} \to X$. In order to state it, we consider the ring of dual numbers, i.e., $k[\varepsilon]$ with $\varepsilon^2 = 0$.

THEOREM 5.3. *Suppose that there exist a pre-Tango structure $\mathcal{L}$ on X and an invertible sheaf $\mathcal{N}$ on X such that $\mathcal{N}^n \cong \mathcal{L}$ and $n \equiv p-1$ mod p. Then we have a natural injection*

$$H^0(X, \mathcal{N}) \to \mathrm{Aut}_{X\otimes k[\varepsilon]}(\mathfrak{X} \otimes k[\varepsilon]).$$

Proof. We employ the same notation as in the proof of the preceding theorem. Since $n \equiv p-1$ mod p, we know that $mp - n = 1$. Hence the equation

$$v_i^p = u_i(a_i u_i^n - 1)$$

locally determines $\mathfrak{X}$. Take an arbitrary global section γ of $\mathcal{N}$. Assume that γ is represented by local sections $\{\gamma_i\}_{i\in I}$ of $\mathcal{O}_X$ subject to the relation $\gamma_j = d_{ij}\gamma_i$. We define an automorphism σ_i of $\mathfrak{X}\otimes k[\varepsilon]|_{U_i}$ such that $\sigma_i^*(x_i) = x_i$ and $\sigma_i^*(y_i) = y_i + \gamma_i x_i^m \varepsilon$. Since $\sigma_i^*(u_i) = u_i$ and $\sigma_i^*(v_i) = v_i + \gamma_i\varepsilon$, we know that σ_i^* is compatible with the above-mentioned local equation of $\mathfrak{X}$, and so σ_i is well defined. Moreover, we have

$$\begin{aligned}\sigma_j(y_i) &= \sigma_j(d_{ij}^n y_j + b_{ij}) = d_{ij}^n \sigma_j(y_j) + b_{ij} = d_{ij}^n y_j + d_{ij}^n \gamma_j x_j^m \varepsilon + b_{ij} \\ &= d_{ij}^n y_j + d_{ij}^{mp-1} \gamma_j x_j^m \varepsilon + b_{ij} = d_{ij}^n y_j + \gamma_i x_i^m \varepsilon + b_{ij} \\ &= y_i + \gamma_i x_i^m \varepsilon = \sigma_i(y_i).\end{aligned}$$

Therefore, we can glue $\{\sigma_i\}_{i\in I}$ together to obtain an automorphism of $\mathfrak{X} \otimes k[\varepsilon]$ induced from the global section γ. ∎

The assertion of the previous theorem is interesting in view of the following:

PROPOSITION 5.4. *Suppose $\mathcal{L}$ is a Tango structure on X. Then*

$$\pi^*\omega_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{P}}(((p-1)(n-1)-2)S) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^* G^*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}^{-np+n+p}).$$

Suppose in addition that $(p-1)(n-1)-2>0$ and $\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}^{-np+n+p}$ is ample. Then $\omega_{\mathfrak{X}}$ is ample.

Proof. Recall that X is the quotient of $\widehat{X}$ by Δ and $\mathfrak{X}$ is the quotient of $\mathfrak{P}$ by D. By [7], we have

$$G^*\omega_X = \omega_{\widehat{X}} \otimes_{\mathcal{O}_{\widehat{X}}} \mathcal{O}_{\widehat{X}}(-(p-1)(\Delta)),$$
$$\pi^*\omega_{\mathfrak{X}} = \omega_{\mathfrak{P}} \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathcal{O}_{\mathfrak{P}}(-(p-1)(D)).$$

Recall that

$$\mathcal{O}_{\widehat{X}}((\Delta)) = G^*\mathcal{L}^{-1}, \qquad \mathcal{O}_{\mathfrak{P}}((D)) = \mathcal{O}_{\mathfrak{P}}(-(n-1)S) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^* G^* \mathcal{N}^{-1}.$$

We then have

$$G^*\omega_X = \omega_{\widehat{X}} \otimes_{\mathcal{O}_{\widehat{X}}} G^*\mathcal{L}^{p-1},$$
$$\pi^*\omega_{\mathfrak{X}} = \omega_{\mathfrak{P}} \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathcal{O}_{\mathfrak{P}}((p-1)(n-1)S) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^* G^* \mathcal{N}^{p-1}.$$

(These equations can also be obtained by using Remarks 3.4 and 4.5.) Since

$$\omega_{\mathfrak{P}} = \mathcal{O}_{\mathfrak{P}}(-2S) \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^* G^* \mathcal{N} \otimes_{\mathcal{O}_{\mathfrak{P}}} \psi^* \omega_{\widehat{X}},$$

we get the required formula. Moreover, the Nakai–Moishezon criterion yields the last assertion immediately. ∎

REMARK 5.5. By the previous proposition, most of the induced uniruled varieties are of general type. Hence their automorphism groups are finite. On the other hand, by Theorem 5.3, for certain induced uniruled varieties, the tangent spaces of the automorphism group schemes are not trivial. Hence those group schemes are not reduced.

6. Examples. To close the present article, we give two examples. The first is a typical one. Namely, we construct a smooth projective 3-dimensional variety of general type with a Tango structure, whose automorphism group scheme is not reduced.

EXAMPLE 6.1. Choose a natural number h such that $(p-1)(hp-2)-2 > 0$. Let C be the affine plane curve determined by the equation

$$\eta^{p(hp-1)(p(hp-1)+1)} - \eta = \xi^{p(hp-1)(p(hp-1)+1)-1}.$$

Set $\zeta = \xi^{-1}$ and $\upsilon = \eta\zeta$. Then

$$\upsilon^{p(hp-1)(p(hp-1)+1)} - \upsilon\zeta^{p(hp-1)(p(hp-1)+1)-1} = \zeta. \tag{8}$$

Hence

$$-\zeta^{p(hp-1)(p(hp-1)+1)-1}d\upsilon + \upsilon\zeta^{p(hp-1)(p(hp-1)+1)-2}d\zeta = d\zeta.$$

So, we obtain

$$\zeta^{p(hp-1)(p(hp-1)+1)-1}d\upsilon = (\upsilon\zeta^{p(hp-1)(p(hp-1)+1)-2} - 1)d\zeta. \tag{9}$$

On the other hand, from (8) we obtain

$$\zeta = (1+\upsilon\zeta^{p(hp-1)(p(hp-1)+1)-2})^{-1}\upsilon^{p(hp-1)(p(hp-1)+1)}.$$

Since $d\xi = -\zeta^{-2}d\zeta$, we have

$$d\xi = -(1+\upsilon\zeta^{p(hp-1)(p(hp-1)+1)-2})^{2}\upsilon^{-2p(hp-1)(p(hp-1)+1)}d\zeta.$$

By using (9), we conclude that

$$d\xi = \alpha\upsilon^{p(hp-1)(p(hp-1)+1)(p(hp-1)(p(hp-1)+1)-3)}d\upsilon,$$

where α is an invertible function at the point P_∞ at infinity. Hence

$$(d\xi) = p(hp-1)(p(hp-1)+1)(p(hp-1)(p(hp-1)+1)-3)P_\infty$$

as divisors. So, we have the isomorphism

$$\mathcal{O}_C(p(hp-1)(p(hp-1)+1)(p(hp-1)(p(hp-1)+1)-3)P_\infty) \xrightarrow{d\xi} \Omega^1_{C/k}.$$

Consider its adjoint mapping

$$\mathcal{O}_C((hp-1)(p(hp-1)+1)(p(hp-1)(p(hp-1)+1)-3)P_\infty) \to F_*\Omega^1_{\widetilde{C}/k}.$$

Since the mapping is obtained from the exact form $d\xi$, its image is contained in $F_*\mathcal{B}^1{}_{\widetilde{C}}$. By Corollary 3.7, we obtain a Tango structure $\mathcal{L}$ on C which is isomorphic to

$$\mathcal{O}_C((hp-1)(p(hp-1)+1)(p(hp-1)(p(hp-1)+1)-3)P_\infty).$$

Consider two invertible sheaves

$$\mathcal{N} = \mathcal{O}_C((hp-1)(p(hp-1)(p(hp-1)+1)-3)P_\infty),$$
$$\mathcal{M} = \mathcal{O}_C((p(hp-1)(p(hp-1)+1)-3)P_\infty).$$

We then have $\mathcal{N}^{p(hp-1)+1} \cong \mathcal{L}$ and $\mathcal{M}^{hp-1} \cong \mathcal{N}$.

Now, first, from the Tango structure $\mathcal{L}$ and the invertible sheaf $\mathcal{N}$, we obtain a uniruled surface $f : X \to C$ which has a Tango structure isomorphic to

$$\mathcal{O}_X(p(hp-1)E) \otimes_{\mathcal{O}_X} f^*\mathcal{N},$$

where E is a section of f such that $\mathcal{O}_E(E) \cong \mathcal{N}$. Note that $\omega_C \cong \mathcal{N}^{p(p(hp-1)+1)}$ and E is the image of an integral subvariety. Hence, by Proposition 5.4, it is easy to verify that the canonical sheaf ω_X of X is numerically equivalent to

$$\mathcal{O}_X(((p-1)p(hp-1)-2)E) \otimes_{\mathcal{O}_X} f^*\mathcal{N}^{hp^2+1}.$$

In particular, by applying the Nakai–Moishezon criterion, we see that this sheaf is ample, and so X is a surface of general type.

Next, we can write the above-mentioned Tango structure on X as

$$\mathcal{O}_X(p(hp-1)E) \otimes_{\mathcal{O}_X} f^*\mathcal{N} \cong (\mathcal{O}_X(pE) \otimes_{\mathcal{O}_X} f^*\mathcal{M})^{hp-1}.$$

Therefore, we also obtain a uniruled 3-dimensional variety $\varphi : \mathfrak{X} \to X$, which has a Tango structure isomorphic to

$$\mathcal{O}_{\mathfrak{X}}((hp-2)\varGamma) \otimes_{\mathcal{O}_{\mathfrak{X}}} \varphi^*(\mathcal{O}_X(pE) \otimes_{\mathcal{O}_X} f^*\mathcal{M}).$$

Moreover, we have the following numerical equivalence:

$$\begin{aligned}\omega_X \otimes_{\mathcal{O}_X} (\mathcal{O}_X(pE) \otimes_{\mathcal{O}_X} f^*\mathcal{M})^{-p(hp-1)+(hp-1)+p} \\ &\equiv \mathcal{O}_X((hp^3-hp^2-p^2+p-2)E) \otimes_{\mathcal{O}_X} f^*\mathcal{N}^{hp^2+1} \\ &\quad \otimes_{\mathcal{O}_X} \mathcal{O}_X((-hp^3+hp^2+2p^2-p)E) \otimes_{\mathcal{O}_X} f^*\mathcal{M}^{-hp^2+hp+2p-1} \\ &\equiv \mathcal{O}_X((p^2-2)E) \otimes_{\mathcal{O}_X} f^*\mathcal{M}^{h^2p^3-2hp^2+2hp+2p-2}.\end{aligned}$$

By the Nakai–Moishezon criterion, this sheaf is ample. Hence, by Proposition 5.4, the canonical divisor of $\mathfrak{X}$ is ample since $(p-1)(hp-1-1)-2>0$ by assumption. So, $\mathfrak{X}$ is of general type. On the other hand, since $\mathcal{M} = \mathcal{O}_C((p(hp-1)(p(hp-1)+1)-3)P_\infty)$, we have $H^0(C,\mathcal{M}) \neq 0$. Therefore, $H^0(X, \mathcal{O}_X(pE) \otimes_{\mathcal{O}_X} f^*\mathcal{M}) \neq 0$. Hence, by Theorem 5.3 and Remark 5.5, $\mathrm{Aut}_X(\mathfrak{X})$ is not reduced.

Remark 6.2 (cf. [4]). Consider an affine plane curve C determined by the equation

$$\eta^{pl} - \eta = \xi^{pl-1},$$

where l is a natural number. Then, in exactly the same way as above, we have

$$(d\xi) = pl(pl-3)P_\infty.$$

Therefore, when l is prime to p and greater than one, there exists a Tango structure $\mathcal{L}$ isomorphic to

$$\mathcal{O}_C(l(pl-3)P_\infty).$$

Furthermore, if l is divisible by d, then

$$\mathcal{O}_C\left(\frac{l}{d}(pl-3)P_\infty\right)^d \cong \mathcal{L}.$$

Hence provided that we choose l with suitable factors, we can construct a smooth uniruled variety of arbitrary dimension which has a Tango structure. In particular, for each natural number n prime to p, we can construct a uniruled variety such that each fibre has one cusp of type $x^p + y^n = 0$.

Mumford constructed in [5] a normal surface which has a pre-Tango structure in our terminology. It seems hard to decide whether or not its desingularization gives a counter-example to the Kodaira vanishing theorem. However, Mumford's example is remarkable in that it is not uniruled (see the remark below). In the following example, we generalize his example and consider the induced higher-dimensional uniruled varieties. In particular, we observe that some generalized examples induce higher-dimensional varieties with nonclosed global differential 1-forms. Hence we surmise that the uniruled varieties are not the only varieties in higher dimensions which trigger off pathological phenomena in positive characteristic.

EXAMPLE 6.3. Let X' be a projective variety. Consider the short exact sequence

$$0 \to \mathcal{O}_{X'} \to F_*\mathcal{O}_{\widetilde{X}'} \to F_*\mathcal{B}^1_{\widetilde{X}'} \to 0.$$

By taking cohomology, we have

$$H^0(X', F_*\mathcal{B}^1_{\widetilde{X}'}) \to H^1(X', \mathcal{O}_{X'}) \to H^1(X', F_*\mathcal{O}_{\widetilde{X}'}).$$

Suppose that the kernel of the second homomorphism is not zero. Then there exists a nonzero differential 1-from ω in $H^0(X', F_*\mathcal{B}^1_{\widetilde{X}'})$. So, we have an injection $\mathcal{O}_{X'}\omega \to F_*\mathcal{B}^1_{\widetilde{X}'}$. (This is not a pre-Tango structure.) Furthermore, we have an extension

$$0 \to \mathcal{O}_{X'} \to \mathcal{E}' \to \mathcal{O}_{X'} \to 0,$$

where $\mathcal{E}'$ is a locally free sub-$\mathcal{O}_{X'}$-module of rank two of $F_*\mathcal{O}_{\widetilde{X}'}$. Take an affine open covering $\{U_i\}_{i\in I}$ and let $\{b_{ij}\}_{i,j\in I}$ be the 1-cocycle of $\mathcal{O}_{X'}$ which coincides with the image of ω in $H^1(X', \mathcal{O}_{X'})$. Note that the above extension corresponds to the cocycle $\{b_{ij}\}$. Since $\{b_{ij}\}$ maps to zero in $H^1(X', F_*\mathcal{O}_{\widetilde{X}'})$, there is a 0-cochain $\{e_i\}_{i\in I}$ of $F_*\mathcal{O}_{\widetilde{X}'}$ such that $e_i - e_j = b_{ij}$ and $\omega = de_i$. Set $c_i = e_i^p$. Then each c_i is a section of $\mathcal{O}_{X'}(U_i)$. Now take an effective ample divisor H on X' such that $H^1(X', \mathcal{O}_{X'}(pH)) = 0$. By applying $\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{X'}(pH)$ to the above extension, we have

$$0 \to \mathcal{O}_{X'}(pH) \to \mathcal{E}' \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(pH) \to \mathcal{O}_{X'}(pH) \to 0.$$

Since $H^0(X, \mathcal{O}_{X'}(pH)) \neq 0$ and $H^1(X, \mathcal{O}_{X'}(pH)) = 0$, we have

$$H^0(X', \mathcal{O}_{X'}(pH)) \subsetneq H^0(X', \mathcal{E}' \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(pH)). \tag{10}$$

Let $\{h_i\}_{i\in I}$ be local equations of H. Note that each h_i is regular on U_i. Consider the field extension $k(X')(\{\theta_i \mid i \in I\})/k(X')$, where

$$\theta_i^p - h_i^{np}\theta_i = c_i$$

for each i in I with n prime to p and greater than one. Let $\sigma : X \to X'$ be the normalization of X' in the extension field $k(X')(\{\theta_i \mid i \in I\})$. On X, we have a relation

$$-\frac{1}{h_i^{np}}\, dc_i = d\theta_i.$$

Set $\eta = dc_i$. We then have an injection

$$\mathcal{O}_X(np\sigma^* H)\eta \to \mathcal{B}^1_X.$$

By taking its adjoint, we obtain an injection

$$\mathcal{O}_X(n\sigma^* H)\omega \to F_*\mathcal{B}^1_{\widetilde{X}}.$$

Its image is a pre-Tango structure on X. We denote it by $\mathcal{L}$. Set $\mathcal{N} = \mathcal{O}_X(\sigma^* H)$. Then $\mathcal{N}^n \cong \mathcal{L}$. Moreover, by using $\mathcal{N}$, we obtain a uniruled variety $\mathfrak{X}$.

Now, we have the inclusions

$$\mathcal{O}_X\omega \subset \mathcal{L} \subset F_*\mathcal{B}^1_{\widetilde{X}}.$$

On the other hand, we obtain an extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{L} \to 0,$$

which is associated with the pre-Tango structure $\mathcal{L}$. Moreover, we deduce a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \sigma^*\mathcal{O}_{X'} & \to & \sigma^*\mathcal{E}' & \to & \sigma^*\mathcal{O}_{X'} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \omega & & \\ 0 & \to & \mathcal{O}_X & \to & \mathcal{E} & \to & \mathcal{L} & \to & 0 \end{array}$$

By applying $\otimes_{\mathcal{O}_X} \mathcal{N}^p$, we obtain

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \sigma^*\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \mathcal{N}^p & \to & \sigma^*\mathcal{E}' \otimes_{\mathcal{O}_X} \mathcal{N}^p & \to & \sigma^*\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \mathcal{N}^p & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{N}^p & \to & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{N}^p & \to & \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}^p & \to & 0 \end{array}$$

Note that the first vertical injection is an isomorphism. From (10) and from the flatness of σ, it follows that

$$H^0(X, \mathcal{N}^p) \subsetneq H^0(X', \sigma^*\mathcal{E}' \otimes_{\mathcal{O}_X} \mathcal{N}^p).$$

Hence

$$H^0(X, \mathcal{N}^p) \subsetneq H^0(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{N}^p).$$

By Theorem 5.1, $\mathfrak{X}$ has nonclosed global differential 1-forms.

REMARK 6.4. Take an arbitrary natural number d and a supersingular elliptic curve. Let X' be the d-fold product of copies of that curve. Then X' is a d-dimensional abelian variety satisfying the assumption of the previous example. Moreover, no finite covering of X' can be a uniruled variety.

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Department of Mathematics
Nara Women's University
Nara 6308506, Japan
E-mail: y-takeda@cc.nara-wu.ac.jp

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