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SOME REMARKS ON HILBERT–SPEISER AND LEOPOLDT FIELDS OF GIVEN TYPE

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Abstract. Let p be a rational prime, G a group of order p, and K a number field containing a primitive pth root of unity. We show that every tamely ramified Galois extension of K with Galois group isomorphic to G has a normal integral basis if and only if for every Galois extension L/K with Galois group isomorphic to G, the ring of integers O_L in L is free as a module over the associated order $\mathcal{A}_{L/K}$. We also give examples, some of which show that this result can still hold without the assumption that K contains a primitive pth root of unity.

1. Introduction. Throughout the present article p is a rational prime, the ring of integers in a number field F is denoted by O_F, and $Cl(O_F)$ denotes the ideal class group of F of order h_F. If F is a finite extension of the p-adic numbers $\mathbb{Q}_p$, then O_F denotes the valuation ring in F, and O_N denotes the integral closure of O_F in a finite extension N/F of F.

Now let G be a finite group and let K be a number field. If L/K is a Galois extension with Galois group G then O_L is a module over the integral group ring $O_K G$ by way of the Galois action of G on L. If O_L is free as an $O_K G$-module, necessarily of rank one, we say L/K has a *normal integral basis*. It is well known that L/K has such a basis only if L/K is *tame*, that is, at most tamely ramified. If L/K is not tame, we can still ask for a freeness result. To do this we consider the associated order $\mathcal{A}_{L/K}$ contained in the K-algebra KG. It consists of all elements α of KG such that $\alpha O_L \subseteq O_L$. Of course $O_K G \subseteq \mathcal{A}_{L/K}$ and, as is well known, L/K is tame if and only if $O_K G = \mathcal{A}_{L/K}$. Moreover, for L/K tame or otherwise, it may happen that O_L is a free $\mathcal{A}_{L/K}$-module.

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Let us now consider all finite abelian extensions of K. If for each such extension L/K, O_L is free as a module over $\mathcal{A}_{L/K}$, then we call K a *Leopoldt field*. In [10] Leopoldt showed that the rational field $\mathbb{Q}$ is such a field. A simplified version of the proof of this result can be found in [11]. Note that if K is a Leopoldt field then it has the property that for any finite abelian group G and any tame Galois extension L/K with Galois group G, O_L is a free $O_K G$-module. Thus we recover the famous result of Hilbert and Speiser: Every tame finite abelian extension of $\mathbb{Q}$ has a normal integral basis. Any number field sharing this property with $\mathbb{Q}$ is called a *Hilbert–Speiser field*. From [7] we know that $\mathbb{Q}$ is the only such field. In other words, we have the following theorem.

THEOREM 1.1. *Let K be a number field. Then K is a Hilbert–Speiser field if and only if K is a Leopoldt field.*

Evidently, freeness for all tame finite abelian extensions is enough to guarantee freeness for all finite abelian extensions. This result suggests a conjecture regarding a restricted case of its statement which we next explain.

Let G be a finite abelian group. A number field K is called a *Leopoldt field of type G* if O_L is a free $\mathcal{A}_{L/K}$-module whenever L/K is a Galois extension with Galois group isomorphic to G. If K satisfies the condition that all of its tame Galois extensions with Galois group isomorphic to G have a normal integral basis, then we call K a *Hilbert–Speiser field of type G*. These fields have been studied, for instance, in [3], [4], [8], [9] and [15].

CONJECTURE 1.1. *Let G be a finite abelian group and let K be a number field. Then K is a Hilbert–Speiser field of type G if and only if K is a Leopoldt field of type G.*

We will provide some limited evidence in support of Conjecture 1.1 in the form of the following theorem and some examples in Section 4.

THEOREM 1.2. *If G is a finite group of order p and K is a number field which contains a primitive pth root of unity, then K is a Hilbert–Speiser field of type G if and only if K is a Leopoldt field of type G.*

The nontrivial implication of Theorem 1.1 follows from the fact that $\mathbb{Q}$ is a Leopoldt field, and the fact proved in [7] that $\mathbb{Q}$ is the only Hilbert–Speiser field. Using results of [7], the following result is proved in [8] (see [8, Proposition 1]).

PROPOSITION 1.1. *Let G be a group of order p and let K be a number field containing a primitive pth root of unity. If $p \geq 5$ then K is not a Hilbert–Speiser field of type G.*

It follows from Proposition 1.1 that Theorem 1.2 is true for all p such that $p \geq 5$. In what follows we will show that it is true in the remaining two cases as well.

2. Realizable classes. Let G be a finite group and let K be any number field. Let L/K vary over all tame Galois extensions of K with Galois group isomorphic to G. Then the class of O_L in the locally free class group $Cl(O_KG)$ varies over a subset $R(O_KG)$ of realizable classes of $Cl(O_KG)$. In [14] it is shown that when G is abelian then $R(O_KG)$ is a subgroup of $Cl(O_KG)$. Hence, for a finite abelian group G we deduce that K is a Hilbert–Speiser field of type G if and only if $R(O_KG)$ is the trivial subgroup of $Cl(O_KG)$.

Now suppose G is an elementary abelian group and K is any number field. In [13], $R(O_KG)$ is determined in terms of the kernel of a certain map defined on $Cl(O_KG)$. When G has order 2 or 3 this result is the following proposition, which is Theorem 1 of [3].

PROPOSITION 2.1. *Let G be a group of order 2 or 3. Then*

$$R(O_KG) = Cl'(O_KG)$$

where $Cl'(O_KG)$ is the kernel of the map $\varepsilon_* : Cl(O_KG) \to Cl(O_K)$ which is induced by the augmentation map $\varepsilon : O_KG \to O_K$.

From now on C_p is a group of order p. Let K be a number field and let $\mathfrak{M}$ be the maximal O_K-order in KC_p. The inclusion map $O_KC_p \to \mathfrak{M}$ induces a map from $Cl(O_KC_p)$ onto the locally free class group $Cl(\mathfrak{M})$ giving rise to the well-known exact sequence

$$0 \to D(O_KC_p) \to Cl(O_KC_p) \to Cl(\mathfrak{M}) \to 0. \tag{1}$$

The following result due to C. Greither is presented on pp. 268–269 of [3]. We slightly modify its statement and proof here in order to adapt them to our present needs.

PROPOSITION 2.2. *Let K be a number field which contains a primitive pth root of unity. If p equals 2 or 3 then there is an exact sequence*

$$0 \to D(O_KC_p) \to R(O_KC_p) \to \bigoplus_{i=1}^{p-1} Cl(O_K) \to 0.$$

Proof. Since K contains a primitive pth root of unity we have $Cl(\mathfrak{M}) \simeq \bigoplus_{i=1}^{p} Cl(O_K)$. From this and (1) we obtain an exact sequence

$$0 \to D(O_KC_p) \to Cl(O_KC_p) \to \bigoplus_{i=1}^{p} Cl(O_K) \to 0. \tag{2}$$

Also, in the notation of Proposition 2.1, there is an exact sequence

$$0 \to Cl'(O_K C_p) \to Cl(O_K C_p) \to Cl(O_K) \to 0. \tag{3}$$

Finally, we have the exact sequence

$$0 \to \bigoplus_{i=1}^{p-1} Cl(O_K) \to \bigoplus_{i=1}^{p} Cl(O_K) \to Cl(O_K) \to 0 \tag{4}$$

where the maps are the appropriate inclusion and projection maps. The sequences (2), (3), and (4) yield the following diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & Cl'(O_K C_p) & & \bigoplus_{i=1}^{p-1} Cl(O_K) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & D(O_K C_p) & \to & Cl(O_K C_p) & \to & \bigoplus_{i=1}^{p} Cl(O_K) & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & Cl(O_K) & = & Cl(O_K) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{5}$$

One easily verifies that (5) is commutative. Hence there is a unique map $\alpha : Cl'(O_K C_p) \to \bigoplus_{i=1}^{p-1} Cl(O_K)$ completing the diagram. Applying the snake lemma to the two vertical exact sequences and maps between them gives an exact sequence

$$0 \to \ker(\alpha) \to D(O_K C_p) \to 0 \to \operatorname{coker}(\alpha) \to 0.$$

Hence, α is surjective with kernel $D(O_K C_p)$. Finally, if $p = 2$ or $p = 3$ we have $Cl'(O_K C_p) = R(O_K C_p)$ by Proposition 2.1. ∎

COROLLARY 2.1 (cf. [8, Proposition 2]). *Let K be a number field which contains a primitive pth root of unity. If p equals 2 or 3 then the following are equivalent*:

(i) *K is a Hilbert–Speiser field of type C_p.*
(ii) *$h_K = 1$ and $D(O_K C_p)$ is trivial.*
(iii) *$Cl(O_K C_p)$ is trivial.*

Proof. This is an immediate consequence of Proposition 2.2 and (2). ∎

3. Main result. Let G be a finite abelian group and K a number field, or a finite extension of the field of p-adic numbers $\mathbb{Q}_p$. Let L/K be a Galois extension with Galois group G. Many authors have considered the problem of determining when O_L is free as a module over $\mathcal{A}_{L/K}$, or, in the global case,

at least locally free over $\mathcal{A}_{L/K}$. In addition to the references already cited, see, for instance, [5] and [12] and the appropriate references listed in these papers. Some of these results lead to a proof of the following proposition.

PROPOSITION 3.1. *Let K be a number field which contains a primitive pth root of unity. Suppose L/K is a Galois extension with Galois group C_p. If p equals 2 or 3 then O_L is a locally free $\mathcal{A}_{L/K}$-module.*

Proof. If $p = 2$ then O_L is a locally free $\mathcal{A}_{L/K}$-module by [5, Theorems 2.1 and 17.3].

To finish the proof we consider the following situation. Let M be a finite extension of the field of 3-adic numbers $\mathbb{Q}_3$, and assume M contains a primitive cube root of unity. Let $\mathfrak{p}$ be the prime ideal of M with corresponding valuation ring O_M. Let N/M be a Galois extension with Galois group C_3 and assume $\mathfrak{p}$ ramifies in N/M. The proposition will follow if we can show that the integral closure O_N of O_M in N is a free $\mathcal{A}_{N/M}$-module. To this end let e be the absolute ramification index of M, and let t be the ramification number of N/M. Since M contains a primitive cube root of unity we have $e = 2e_1$ for some positive rational integer e_1. It is well known that $1 \leq t \leq 3e_1$. If $t \equiv 0 \pmod 3$ (resp. $1 \leq t < 3e_1 - 1$ and $t \not\equiv 0 \pmod 3$), then O_N is a free $\mathcal{A}_{N/M}$-module by part a (resp. part b) of the theorem appearing on p. 1333 of [2]. Finally, if $t = 3e_1 - 1$ then O_N is a free $\mathcal{A}_{N/M}$-module by [1, Theorem 1]. ∎

We can now prove our main result.

Proof of Theorem 1.2. As already noted, Theorem 1.2 is true if $p \geq 5$ by Proposition 1.1. Now suppose either $p = 2$ or $p = 3$. Let K be a number field containing a primitive pth root of unity and assume K is a Hilbert–Speiser field of type C_p. Let L/K be any Galois extension with Galois group isomorphic to C_p. By Proposition 3.1, O_L is a locally free $\mathcal{A}_{L/K}$-module. Since $Cl(O_K C_p)$ is trivial by Corollary 2.1 and maps onto $Cl(\mathcal{A}_{L/K})$ by [6, 49.25(iii)], it follows that $Cl(\mathcal{A}_{L/K})$ is trivial. So the class of O_L in $Cl(\mathcal{A}_{L/K})$ is trivial, which shows that O_L is a free $\mathcal{A}_{L/K}$-module. Hence, K is a Leopoldt field of type C_p. Since the other implication of Theorem 1.2 is clear this concludes the proof. ∎

4. Examples

EXAMPLE 4.1. Among all imaginary quadratic fields there are exactly three Hilbert–Speiser fields of type C_2 by [3, Corollary 3]. They are the fields $\mathbb{Q}(\sqrt{m})$ where $m \in \{-1, -3, -7\}$. Hence, by Theorem 1.2 these fields are Leopoldt fields of type C_2 as well. The fact that among all imaginary quadratic fields these fields are precisely the Leopoldt fields of type C_2 is also proved in [15].

EXAMPLE 4.2. Let $\mathbb{Z}$ be the ring of rational integers and let $m \in \mathbb{Z}$ with $m > 1$ and square free. Let ε_m be the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{m})$. Then either $\varepsilon_m = a + b\sqrt{m}$ or $\varepsilon_m = (a + b\sqrt{m})/2$ where $a, b \in \mathbb{Z}$, and the greatest common divisor $(2, ab)$ is 1. By [3, Corollary 4], $\mathbb{Q}(\sqrt{m})$ is a Hilbert–Speiser field of type C_2 exactly when its class number equals 1 and one of the following holds: (i) $m \equiv 1 \pmod 8$; (ii) $m \equiv 5 \pmod 8$ and $\varepsilon_m \notin \mathbb{Z}[\sqrt{m}]$; (iii) $m \equiv 2$ or $3 \pmod 4$ and $(2, b) = 1$. For $1 < m < 100$ such that the class number of $\mathbb{Q}(\sqrt{m})$ is 1 we find: m satisfies (i) if $m \in \{17, 33, 41, 57, 73, 89, 97\}$; m satisfies (ii) if $m \in \{5, 13, 21, 29, 37, 53, 61, 69, 77, 93\}$; m satisfies (iii) if $m \in \{2, 3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 83\}$. Hence, for these values of m, $\mathbb{Q}(\sqrt{m})$ is a Leopoldt field of type C_2 by Theorem 1.2.

EXAMPLE 4.3. Among all quadratic fields there are exactly twelve Hilbert–Speiser fields of type C_3 by [4, Corollary 5] or [9, 5.3]. They are the fields $\mathbb{Q}(\sqrt{m})$ where $m \in \{-11, -3, -2, -1, 2, 3, 5, 6, 17, 33, 41, 89\}$. By Theorem 1.2 the field $\mathbb{Q}(\sqrt{-3})$ is a Leopoldt field of type C_3. We next show that the remaining eleven fields are also Leopoldt fields of type C_3.

Let ω be a primitive cube root of unity and assume K is a number field satisfying $\omega \notin K$. After some routine changes, the proof of the $p = 3$ case of Proposition 2.2 becomes the argument on p. 268 of [3]. As shown there, that argument gives the exact sequences

$$0 \to D(O_K C_3) \to Cl(O_K C_3) \to Cl(O_K) \oplus Cl(O_{K(\omega)}) \to 0 \tag{6}$$

and

$$0 \to D(O_K C_3) \to R(O_K C_3) \to Cl(O_{K(\omega)}) \to 0. \tag{7}$$

Now suppose K is one of our eleven remaining fields. Since K is a Hilbert–Speiser field of type C_3 we see from (7) that $h_{K(\omega)} = 1$ and $D(O_K C_3)$ is trivial. Hence, $Cl(O_K C_3) \simeq Cl(O_K)$ by (6). Since $h_K = 1$ it follows that $Cl(O_K C_3)$ is trivial. So if K is one of our eleven remaining fields and L/K is any Galois extension with Galois group isomorphic to C_3,then $Cl(\mathcal{A}_{L/K})$ is trivial by [6, 49.25(iii)]. Therefore, the example will be complete once we prove the following proposition.

PROPOSITION 4.1. *Let K be a quadratic field and let L/K be a Galois extension with Galois group isomorphic to C_3. Then O_L is a locally free $\mathcal{A}_{L/K}$-module.*

Proof. The proof is similar to the proof of the $p = 3$ case of Proposition 3.1. Let M be a quadratic extension of the field of 3-adic numbers $\mathbb{Q}_3$, and let e be the absolute ramification index of M. Let $\mathfrak{p}$ be the prime ideal of M with corresponding valuation ring O_M. Let N/M be a Galois extension with Galois group isomorphic to C_3. Let O_N be the integral closure of O_M in N.

Assume $\mathfrak{p}$ ramifies in N/M and let t be the ramification number of N/M. We know that $1 \leq t \leq 3e/2$. If $e = 1$ then $t = 1$. Hence, O_N is a free $\mathcal{A}_{N/M}$-module by [1, Theorem 1]. If $e = 2$ then $t \in \{1, 2, 3\}$. If $t = 3$ (resp. $t = 1$) then O_N is a free $\mathcal{A}_{N/M}$-module by part a (resp. part b) of the theorem appearing on p. 1333 of [2]. If $t = 2$ then O_N is a free $\mathcal{A}_{N/M}$-module by [1, Theorem 1]. ∎

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