

SOME INTEGRAL INEQUALITIES OF HÖLDER  
AND MINKOWSKI TYPE

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**Abstract.** A number of integral inequalities of Hölder and Minkowski type involving a class of generalized weighted quasi-arithmetic means in integral form is established. Some well known inequalities and their generalizations are derived as consequences of our results.

**1. Introduction.** The celebrated Hölder and Minkowski inequalities belong to the fundamental and classical inequalities in mathematics. They can be found in many books on real functions, analysis, functional analysis or  $L_p$ -spaces. Their integral analogues are as follows (cf. [6]):

**PROPOSITION 1.1.** *Let  $\gamma$  and  $\delta$  be conjugate exponents, i.e.  $\gamma^{-1} + \delta^{-1} = 1$ , with  $1 < \gamma < \infty$ . Let  $(X, \mathcal{M}, \mu)$  be a measure space and  $f, g : X \rightarrow [0, \infty]$  be measurable functions. Then*

$$\text{(Hölder)} \quad \int_X fg \, d\mu \leq \left( \int_X f^\gamma \, d\mu \right)^{1/\gamma} \left( \int_X g^\delta \, d\mu \right)^{1/\delta}$$

and

$$\text{(Minkowski)} \quad \left( \int_X (f + g)^\gamma \, d\mu \right)^{1/\gamma} \leq \left( \int_X f^\gamma \, d\mu \right)^{1/\gamma} + \left( \int_X g^\gamma \, d\mu \right)^{1/\gamma}.$$

Because of their usefulness in analysis and its applications, these inequalities have received a considerable attention in the past decades and a number of papers have appeared which deal with their various generalizations, extensions and applications. In connection with the theory of special means we can find some extensions and applications of the Hölder and Minkowski inequalities e.g. in [8], [9], [11], [12], and [17].

The main purpose of this paper is to establish some integral inequalities for a class of generalized weighted quasi-arithmetic means in integral form, mainly connected with the classical Hölder and Minkowski inequalities.

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The structure of this article is as follows. In Section 2 we recall the definition of the generalized weighted quasi-arithmetic mean  $M_{[a,b],g}(p, f)$  and state some preliminary results. In Section 3 we give a number of weighted integral inequalities of Hölder type involving  $M_{[a,b],g}(p, f)$  and state a few sufficient conditions for their validity. In the fourth section we give analogous results for Minkowski-type inequalities. These results are natural generalizations of results from [1]. Some applications and generalizations of well known inequalities are given in the last section.

**2. Preliminaries.** Let  $L_1([a, b])$  be the vector space of all real Lebesgue integrable functions defined on the interval  $[a, b] \subset \mathbb{R}$ ,  $a < b$ , with respect to the usual Lebesgue measure. Denote by  $L_1^+([a, b])$  the positive cone of  $L_1([a, b])$ , consisting of the non-negative functions. In what follows  $\|p\|_{[a,b]}$  denotes the  $L_1$ -norm of  $p \in L_1^+([a, b])$ . For the definition below, cf. [4].

**DEFINITION 2.1.** Let  $p \in L_1^+([a, b])$ ,  $f : [a, b] \rightarrow [\alpha, \beta]$  be measurable and  $g : [\alpha, \beta] \rightarrow \mathbb{R}$  be continuous and strictly monotone, where  $-\infty < \alpha < \beta < \infty$ . The *generalized weighted quasi-arithmetic mean* of  $f$  with respect to the weight function  $p$  is the real number

$$(1) \quad M_{[a,b],g}(p, f) = g^{-1} \left( \frac{1}{\|p\|_{[a,b]}} \int_a^b p(x)g(f(x)) dx \right),$$

where  $g^{-1}$  denotes the inverse function to  $g$ .

The means  $M_{[a,b],g}(p, f)$  include many commonly used two-variable integral means as particular cases (cf. [5]). In particular, for  $g(x) = x$  we obtain the classical weighted arithmetic means  $A_{[a,b]}(p, f)$ .

Note that a further possible extension of  $M_{[a,b],g}(p, f)$  could be considered in the case of analytic functions. Indeed, let  $f$  be of the form  $f(\theta) = |h(re^{i\theta})|$ , where  $0 < r < 1$  and  $h$  is an analytic function in the open unit disk  $\mathbb{D} = \{z : |z| < 1\}$  of the complex plane. In that case choosing  $a = 0$ ,  $b = 2\pi$ ,  $g(x) = x^q$  for  $0 < q < \infty$  and  $p(x) \equiv 1$  on  $[0, 2\pi]$  yields the integral mean of order  $q$ ,

$$M_q(r, h) = \left( \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^q d\theta \right)^{1/q}.$$

Much research has been devoted to the dependence of the operator of means on the behavior of the input functions  $p$ ,  $f$  and  $g$ . The following lemma gives a generalization of the well known Jensen inequality to the class of means of Definition 2.1. This enables us to derive various inequalities for the means  $M_{[a,b],g}(p, f)$  depending on the convexity properties of  $f$  and  $g$ .

LEMMA 2.2 (Jensen inequality). *Let  $p \in L_1^+([a, b])$  and  $f : [a, b] \rightarrow [\alpha, \beta]$  be measurable, where  $-\infty < \alpha < \beta < \infty$ . If  $g : [\alpha, \beta] \rightarrow \mathbb{R}$  is convex (resp. concave), then*

$$g(A_{[a,b]}(p, f)) \leq (\text{resp. } \geq) A_{[a,b]}(p, g \circ f).$$

An elementary proof of Lemma 2.2 is given in [7]. Some basic properties of  $M_{[a,b],g}(p, f)$  derived using the weighted integral analogue of the Jensen inequality can be found in [4] and [5]. As an easy consequence of the Jensen inequality we get the following useful result.

COROLLARY 2.3. *Let  $p \in L_1^+([a, b])$  and  $f : [a, b] \rightarrow [\alpha, \beta]$  be measurable, where  $-\infty < \alpha < \beta < \infty$ . If  $g : [\alpha, \beta] \rightarrow \mathbb{R}$  is convex increasing or concave decreasing (resp. convex decreasing or concave increasing), then*

$$A_{[a,b]}(p, f) \leq (\text{resp. } \geq) M_{[a,b],g}(p, f).$$

In the following lemma we summarize some results which will be useful in the rest of this paper.

LEMMA 2.4. *Let  $P \in L_1([a, b])$  and  $F : [a, b] \rightarrow \mathbb{R}$  be measurable. Then the inequality*

$$\int_a^b P(t)F(t) dt \leq 0$$

holds in each of the following cases:

(a)  *$F$  is non-negative and non-increasing and*

$$\int_a^x P(t) dt \leq 0, \quad x \in [a, b];$$

(b)  *$F$  is non-negative and non-decreasing and*

$$\int_x^b P(t) dt \leq 0, \quad x \in [a, b];$$

(c)  *$F \in L_1^+([a, b])$  is symmetrical on  $[a, b]$ , non-increasing on  $[(a + b)/2, b]$  and*

$$\int_{a+x}^{b-x} P(t) dt \leq 0, \quad x \in [0, (b - a)/2];$$

(d)  *$F$  is non-negative and non-increasing on  $[(a + b)/2, b]$  such that  $F(a + x) \geq F(b - x)$  for all  $x \in [0, (b - a)/2]$ , and*

$$P(x) \leq 0, \quad x \in [a, (a + b)/2],$$

and

$$\int_{a+x}^{b-x} P(t) dt \leq 0, \quad x \in [0, (b - a)/2];$$

(e)  $F$  is non-negative and non-decreasing on  $[a, (a+b)/2]$  such that  $F(a+x) \leq F(b-x)$  for  $x \in [0, (b-a)/2]$ , and

$$P(x) \leq 0, \quad x \in [(a+b)/2, b],$$

and

$$\int_{a+x}^{b-x} P(t) dt \leq 0, \quad x \in [0, (b-a)/2].$$

REMARK 2.5. Recall that  $F$  is symmetrical on  $[a, b]$  if

$$F(a+x) = F(b-x) \quad \text{for all } x \in [0, (b-a)/2].$$

The statement of Lemma 2.4 in case (a) was proved in [6] for the interval  $[0, 1]$ . For the proof of the other cases, cf. [1].

**3. Hölder-type inequalities.** In what follows we always consider weight functions  $p_i \in L_1^+([a, b])$  for  $i = 1, \dots, n+1$ , where  $n \in \mathbb{N}$  (the set of all natural numbers). Put

$$P_i(x) = \frac{1}{\|p_i\|_{[a,b]}} \int_a^x p_i(t) dt, \quad x \in [a, b],$$

for  $i = 1, \dots, n+1$ . We establish a few integral inequalities of Hölder and Minkowski type for the means  $M_{[a,b],g}(p, f)$  involving  $P_i$ ,  $i = 1, \dots, n+1$ , and give some sufficient conditions for their validity.

THEOREM 3.1. Let  $p_i \in L_1^+([a, b])$  for  $i = 1, \dots, n+1$  and  $f : [a, b] \rightarrow [\alpha, \beta]$  be a non-negative measurable function, where  $-\infty < \alpha < \beta < \infty$ . Let  $\gamma_i$ ,  $i = 1, \dots, n$ , be positive numbers such that  $\sum_{i=1}^n 1/\gamma_i = 1$  and  $g : [\alpha, \beta] \rightarrow \mathbb{R}$  be continuous.

(a) If  $f$  is non-increasing,  $g$  is either a convex increasing or concave decreasing, and

$$(2) \quad P_{n+1}(x) \leq \prod_{i=1}^n P_i(x)^{1/\gamma_i}, \quad x \in [a, b],$$

then

$$(3) \quad A_{[a,b]}(p_{n+1}, f) \leq \prod_{i=1}^n (M_{[a,b],g}(p_i, f))^{1/\gamma_i}.$$

(b) If  $f$  is non-decreasing,  $g$  is either convex decreasing or concave increasing, and (2) is reversed, then (3) is reversed.

*Proof.* We will prove (a). From Corollary 2.3 we have  $M_{[a,b],g}(p_i, f) \geq A_{[a,b]}(p_i, f)$  for all  $i = 1, \dots, n$ . Then

$$\prod_{i=1}^n (M_{[a,b],g}(p_i, f))^{1/\gamma_i} \geq \prod_{i=1}^n (A_{[a,b]}(p_i, f))^{1/\gamma_i}.$$

Using integration by parts, we have

$$\prod_{i=1}^n \left( \frac{1}{\|p_i\|_{[a,b]}} \int_a^b p_i(x) f(x) dx \right)^{1/\gamma_i} = \prod_{i=1}^n \left( f(b) + \int_a^b P_i(x) d\bar{f}(x) \right)^{1/\gamma_i},$$

where  $\bar{f}(x) = -f(x)$ . From the discrete and integral Hölder inequalities, we obtain

$$\begin{aligned} \prod_{i=1}^n \left( f(b) + \int_a^b P_i(x) d\bar{f}(x) \right)^{1/\gamma_i} &\geq f(b) + \prod_{i=1}^n \left( \int_a^b P_i(x) d\bar{f}(x) \right)^{1/\gamma_i} \\ &\geq f(b) + \int_a^b \prod_{i=1}^n P_i(x)^{1/\gamma_i} d\bar{f}(x), \end{aligned}$$

and by the use of inequality (2), we get

$$\begin{aligned} \prod_{i=1}^n (M_{[a,b],g}(p_i, f))^{1/\gamma_i} &\geq f(b) + \int_a^b P_{n+1}(x) d\bar{f}(x) = \int_a^b P'_{n+1}(x) f(x) dx \\ &= \frac{1}{\|p_{n+1}\|_{[a,b]}} \int_a^b p_{n+1}(x) f(x) dx = A_{[a,b]}(p_{n+1}, f). \end{aligned}$$

The proof of (b) is similar, with the so called Popoviciu inequality from [10] used instead of the discrete Hölder inequality. ■

REMARK 3.2. Observe that the term  $\prod_{i=1}^n P_i(x)^{1/\gamma_i}$  in condition (2) is the weighted (discrete) geometric mean  $\mathcal{G}_{(n)}(P_1(x), \dots, P_n(x))$  of non-negative terms  $P_i(x)$  with weights  $\gamma_i$ ,  $i = 1, \dots, n$ . Therefore, (2) may be rewritten as

$$\mathcal{A}_{(n)}(P_{n+1}(x), \dots, P_{n+1}(x)) \leq \mathcal{G}_{(n)}(P_1(x), \dots, P_n(x)),$$

where  $\mathcal{A}_{(n)}(P_1(x), \dots, P_n(x))$  stands for the (discrete) arithmetic mean.

As a kind of dual to Theorem 3.1 we directly have

THEOREM 3.3. *Let  $p_i, \gamma_i$  and  $f, g$  be as in Theorem 3.1.*

(a) *If  $f$  is non-increasing,  $g$  is either convex decreasing or concave increasing, and the inequality (2) is valid, then*

$$(4) \quad M_{[a,b],g}(p_{n+1}, f) \leq \prod_{i=1}^n (A_{[a,b]}(p_i, f))^{1/\gamma_i}.$$

(b) *If  $f$  is non-decreasing,  $g$  is either convex increasing or concave decreasing, and (2) is reversed, then (4) is reversed.*

*Proof.* From the proof of Theorem 3.1, we have

$$\prod_{i=1}^n (A_{[a,b]}(p_i, f))^{1/\gamma_i} \geq A_{[a,b]}(p_{n+1}, f).$$

If  $g$  is either convex decreasing or concave increasing, then  $A_{[a,b]}(p_{n+1}, f) \geq M_{[a,b],g}(p_{n+1}, f)$ , which completes the proof. ■

Note that Theorems 3.1 and 3.3 seem to be closely related to the comparison problem between means (cf. [5, Theorem 3.1]).

Our purpose now is to weaken the assumption (2) using Lemma 2.4. Therefore, the following theorem involves the derivatives of the weight functions  $P_i$ ,  $i = 1, \dots, n+1$ .

**THEOREM 3.4.** *Let  $p_i, \gamma_i$ , and  $f, g$  be as in Theorem 3.1.*

(a) *If  $f$  is non-increasing,  $g$  is either convex increasing or concave decreasing,*

$$P'_{n+1}(x) \leq \left( \prod_{i=1}^n P_i^{1/\gamma_i} \right)'(x), \quad x \in [a, (a+b)/2],$$

and

$$(5) \quad P_{n+1}(b-x) - P_{n+1}(a+x) \leq \prod_{i=1}^n P_i(b-x)^{1/\gamma_i} - \prod_{i=1}^n P_i(a+x)^{1/\gamma_i}$$

for  $x \in [0, (b-a)/2]$ , then the inequality (3) holds.

(b) *If  $f$  is non-decreasing,  $g$  is either convex decreasing or concave increasing,*

$$P'_{n+1}(x) \geq \left( \prod_{i=1}^n P_i^{1/\gamma_i} \right)'(x), \quad x \in [(a+b)/2, b],$$

and (5) is reversed, then (3) is reversed.

*Proof.* (a) Setting

$$F = f, \quad P = P'_{n+1} - \left( \prod_{i=1}^n P_i^{1/\gamma_i} \right)',$$

and applying Lemma 2.4(d), we get

$$\int_a^b \left( \prod_{i=1}^n P_i^{1/\gamma_i} \right)'(x) f(x) dx \geq \int_a^b P'_{n+1}(x) f(x) dx = A_{[a,b]}(p_{n+1}, f).$$

Since  $g$  is either convex increasing or concave decreasing, using the proof of

Theorem 3.1 we have

$$\begin{aligned} \prod_{i=1}^n (M_{[a,b],g}(p_i, f))^{1/\gamma_i} &\geq \prod_{i=1}^n (A_{[a,b]}(p_i, f))^{1/\gamma_i} \geq f(b) + \int \prod_{a \leq i=1}^b P_i(x)^{1/\gamma_i} d\bar{f}(x) \\ &= \int_a^b \left( \prod_{i=1}^n P_i^{1/\gamma_i} \right)'(x) f(x) dx \geq A_{[a,b]}(p_{n+1}, f). \end{aligned}$$

Item (b) may be proved similarly, by applying Lemma 2.4(e) to  $F = f$  and  $P = \left(\prod_{i=1}^n P_i^{1/\gamma_i}\right)' - P'_{n+1}$ . ■

Obviously, the integral and differential calculus plays a fundamental role when establishing conditions for the inequality (3) to be valid. Thus, it is natural to give the following sufficient conditions.

**THEOREM 3.5.** *Let  $p_i, \gamma_i, f$  and  $g$  be as in Theorem 3.1 and  $f$  be differentiable. Then the inequality (3) holds in each of the following cases:*

- (a)  $f' \leq 0, f$  is convex (or  $f' \geq 0, f$  is concave),  $g$  is either convex increasing or concave decreasing, and

$$(6) \quad \int_a^x P_{n+1}(t) dt \leq \int_a^x \prod_{i=1}^n P_i(t)^{1/\gamma_i} dt, \quad x \in [a, b];$$

- (b)  $f' \leq 0, f$  is concave (or  $f' \geq 0, f$  is convex),  $g$  is either convex increasing or concave decreasing, and

$$(7) \quad \int_x^b P_{n+1}(t) dt \leq \int_x^b \prod_{i=1}^n P_i(t)^{1/\gamma_i} dt, \quad x \in [a, b];$$

- (c)  $f'$  is non-positive and symmetrical on  $[a, b]$ , non-decreasing on  $[(a + b)/2, b]$  (or  $f'$  is non-negative and symmetrical on  $[a, b]$ , non-increasing on  $[(a + b)/2, b]$ ),  $g$  is either convex increasing or concave decreasing, and

$$(8) \quad \int_{a+x}^{b-x} P_{n+1}(t) dt \leq \int_{a+x}^{b-x} \prod_{i=1}^n P_i(t)^{1/\gamma_i} dt$$

for all  $x \in [0, (b - a)/2]$ .

*Proof.* Let us prove (a). Suppose that  $f' \leq 0$ . Put

$$F = -f' \quad \text{and} \quad P = P_{n+1} - \prod_{i=1}^n P_i^{1/\gamma_i}.$$

Since  $f$  is convex,  $f'$  is non-decreasing, and since  $f' \leq 0$ , it follows that  $F$

is a non-negative and non-increasing function on  $[a, b]$ . Then

$$\int_a^x P(t) dt = \int_a^x P_{n+1}(t) dt - \int_a^x \prod_{i=1}^n P_i(t)^{1/\gamma_i} dt \leq 0$$

for all  $x \in [a, b]$ , and using Lemma 2.4(a) we have

$$(9) \quad \int_a^b P_{n+1}(x)F(x) dx \leq \int_a^b \prod_{i=1}^n P_i(x)^{1/\gamma_i} F(x) dx.$$

Replacing  $-f(x)$  by  $\bar{f}(x)$  and adding  $f(b)$  to both sides of (9), we get

$$(10) \quad f(b) + \int_a^b P_{n+1}(x) d\bar{f}(x) \leq f(b) + \int_a^b \prod_{i=1}^n P_i(x)^{1/\gamma_i} d\bar{f}(x).$$

For the left-hand side of (10) we have

$$(11) \quad f(b) + \int_a^b P_{n+1}(x) d\bar{f}(x) = \int_a^b P'_{n+1}(x)f(x) dx = A_{[a,b]}(p_{n+1}, f).$$

For the right-hand side of (10) we use the Hölder inequality to get

$$\begin{aligned} f(b) + \int_a^b \prod_{i=1}^n P_i(x)^{1/\gamma_i} d\bar{f}(x) &\leq f(b) + \prod_{i=1}^n \left( \int_a^b P_i(x) d\bar{f}(x) \right)^{1/\gamma_i} \\ &\leq \prod_{i=1}^n \left( f(b) + \int_a^b P_i(x) d\bar{f}(x) \right)^{1/\gamma_i} \\ &= \prod_{i=1}^n \left( \int_a^b P'_i(x)f(x) dt \right)^{1/\gamma_i} = \prod_{i=1}^n (A_{[a,b]}(p_i, f))^{1/\gamma_i}. \end{aligned}$$

Since  $g$  is either convex increasing or concave decreasing, we have  $A_{[a,b]}(p_i, f) \leq M_{[a,b]}(p_i, f)$  for  $i = 1, \dots, n$ , and therefore

$$(12) \quad \prod_{i=1}^n (A_{[a,b]}(p_i, f))^{1/\gamma_i} \leq \prod_{i=1}^n (M_{[a,b],g}(p_i, f))^{1/\gamma_i}.$$

Substituting (11) and (12) into (10) we obtain the desired inequality. If  $f' \geq 0$ , we replace  $f'$  by  $F$  and use the same method.

Similarly we may prove items (b) and (c) by the use of items (b) and (c) of Lemma 2.4, respectively. ■

The same method yields a kind of dual to Theorem 3.5:

**THEOREM 3.6.** *Let  $p_i, \gamma_i$  and  $f, g$  be as in Theorem 3.5. Then the inequality (4) holds in each of the following cases:*



- (a)  $f' \leq 0$ ,  $f$  is convex (or  $f' \geq 0$ ,  $f$  is concave),  $g$  is either convex decreasing or concave increasing, and (6) is valid;
- (b)  $f' \leq 0$ ,  $f$  is concave (or  $f' \geq 0$ ,  $f$  is convex),  $g$  is either convex decreasing or concave increasing, and (7) is valid;
- (c)  $f'$  is non-positive and symmetrical on  $[a, b]$ , non-decreasing on  $[(a + b)/2, b]$  (or  $f'$  is non-negative and symmetrical on  $[a, b]$ , non-increasing on  $[(a + b)/2, b]$ ),  $g$  is either convex decreasing or concave increasing, and (8) is valid.

**4. Minkowski-type inequalities.** In this section we establish some analogous inequalities of Minkowski type.

**THEOREM 4.1.** *Let  $p_i, f, g$  be as in Theorem 3.1.*

- (a) *Let  $q > 1$  or  $q < 0$ . If  $f$  is non-increasing,  $g$  is either convex increasing or concave decreasing, and*

$$(13) \quad P_{n+1}(x) \leq \left( \sum_{i=1}^n \delta_i P_i(x)^{1/q} \right)^q, \quad x \in [a, b],$$

where  $\delta_i, i = 1, \dots, n$ , are positive numbers such that  $\sum_{i=1}^n \delta_i = 1$ , then

$$(14) \quad A_{[a,b]}(p_{n+1}, f) \leq \left( \sum_{i=1}^n \delta_i (M_{[a,b],g}(p_i, f))^{1/q} \right)^q.$$

If  $f$  is non-decreasing,  $g$  is either convex decreasing or concave increasing, and (13) is valid, then (14) is reversed.

- (b) *Let  $0 < q < 1$ . If  $f$  is non-decreasing,  $g$  is either convex increasing or concave decreasing and (13) is reversed, then (14) holds.*

If  $f$  is non-increasing,  $g$  is either convex decreasing or concave increasing, and (13) is reversed, then (14) is reversed.

*Proof.* Suppose that  $q > 1$ ,  $f$  is non-increasing and (13) is valid. Since  $g$  is either convex increasing or concave decreasing, according to Corollary 2.3 we have  $M_{[a,b],g}(p_i, f) \geq A_{[a,b]}(p_i, f)$  for all  $i = 1, \dots, n$ , and therefore

$$\sum_{i=1}^n \delta_i (M_{[a,b],g}(p_i, f))^{1/q} \geq \sum_{i=1}^n \delta_i (A_{[a,b]}(p_i, f))^{1/q}.$$

Using integration by parts, we get

$$\sum_{i=1}^n \delta_i (A_{[a,b]}(p_i, f))^{1/q} = \sum_{i=1}^n \delta_i \left( f(b) + \int_a^b P_i(t) d\bar{f}(t) \right)^{1/q},$$

where  $\bar{f}(t) = -f(t)$ . Applying the discrete and integral versions of the Min-

kowski inequality, we obtain

$$\begin{aligned} \sum_{i=1}^n \delta_i \left( f(b) + \int_a^b P_i(t) d\bar{f}(t) \right)^{1/q} \\ \geq \left[ \left( \sum_{i=1}^n \delta_i f(b)^{1/q} \right)^q + \left( \sum_{i=1}^n \delta_i \left( \int_a^b P_i(t) d\bar{f}(t) \right)^{1/q} \right)^q \right]^{1/q} \\ \geq \left( f(b) + \int_a^b \left( \sum_{i=1}^n \delta_i P_i^{1/q} \right)^q (t) d\bar{f}(t) \right)^{1/q}. \end{aligned}$$

According to (13), we have

$$\begin{aligned} \left( f(b) + \int_a^b \left( \sum_{i=1}^n \delta_i P_i^{1/q} \right)^q (t) d\bar{f}(t) \right)^{1/q} \geq \left( f(b) + \int_a^b P_{n+1}(t) d\bar{f}(t) \right)^{1/q} \\ = \left( \int_a^b P'_{n+1}(t) f(t) dt \right)^{1/q} = (A_{[a,b]}(p_{n+1}, f))^{1/q}. \end{aligned}$$

In the case  $q < 0$  the Bellman inequality (cf. [10]) is used instead of the discrete Minkowski inequality. ■

REMARK 4.2. Note that the term  $(\sum_{i=1}^n \delta_i P_i^{1/q})^q$  in the previous theorem is, in fact, the weighted (discrete) power mean  $\mathcal{P}_{(n)}^{[1/q]}(P_1(x), \dots, P_n(x))$  of order  $1/q$  for the  $n$ -tuple  $(P_1(x), \dots, P_n(x))$  with weights  $(\delta_1, \dots, \delta_n)$ . Thus, the condition (13) may be equivalently rewritten as

$$\mathcal{A}_{(n)}(P_{n+1}(x), \dots, P_{n+1}(x)) \leq \mathcal{P}_{(n)}^{[1/q]}(P_1(x), \dots, P_n(x)).$$

From the proof of Theorem 4.1(a) and Corollary 2.3 we immediately have the following result.

THEOREM 4.3. *Let  $p_i, f, g$  be as in Theorem 3.1.*

(a) *Let  $q > 1$  or  $q < 0$ . If  $f$  is non-increasing,  $g$  is either convex decreasing or concave increasing, and inequality (13) is valid, then*

$$(15) \quad M_{[a,b],g}(p_{n+1}, f) \leq \left( \sum_{i=1}^n \delta_i (A_{[a,b]}(p_i, f))^{1/q} \right)^q.$$

*If  $f$  is non-decreasing,  $g$  is either convex increasing or concave decreasing, and (13) is valid, then (15) is reversed.*

(b) *Let  $0 < q < 1$ . If  $f$  is non-decreasing,  $g$  is either convex decreasing or concave increasing, and (13) is reversed, then (15) holds.*

*If  $f$  is non-increasing,  $g$  is either convex increasing or concave decreasing, and (13) is reversed, then (15) is reversed.*

As in the theorems stated in the previous section, the requirement (13) could be given in a weaker form. In what follows we will consider only the case when  $q > 1$  or  $q < 0$ . Similar results hold for  $0 < q < 1$ .

**THEOREM 4.4.** *Let  $p_i, \delta_i, f$  and  $g$  be as in Theorem 4.1 and  $f$  be differentiable.*

- (a) *If  $f$  is non-increasing,  $g$  is either convex increasing or concave decreasing,*

$$P'_{n+1}(x) \leq \left( \left( \sum_{i=1}^n \delta_i P_i(x)^{1/q} \right)^q \right)' \quad \text{for } x \in [a, (a+b)/2],$$

and

$$(16) \quad P_{n+1}(b-x) - P_{n+1}(a+x) \leq \left( \sum_{i=1}^n \delta_i P_i^{1/q} \right)^q (b-x) - \left( \sum_{i=1}^n \delta_i P_i^{1/q} \right)^q (a+x)$$

for  $x \in [0, (b-a)/2]$ , then (14) holds.

- (b) *If  $f$  is non-decreasing,  $g$  is either convex decreasing or concave increasing,*

$$P'_{n+1}(x) \geq \left( \left( \sum_{i=1}^n \delta_i P_i(x)^{1/q} \right)^q \right)' \quad \text{for } x \in [(a+b)/2, b],$$

and (16) is reversed, then (14) is reversed.

The proof is analogous to the proof of Theorem 3.4. The following result may be proved similarly to Theorem 3.5.

**THEOREM 4.5.** *Let  $p_i, \delta_i, f$  and  $g$  be as in Theorem 4.1 and  $f$  be differentiable. Then the inequality (14) holds in each of the following cases:*

- (a)  *$f' \leq 0$ ,  $f$  is convex (or  $f' \geq 0$ ,  $f$  is concave),  $g$  is either convex increasing or concave decreasing, and*

$$(17) \quad \int_a^x P_{n+1}(t) dt \leq \int_a^x \left( \sum_{i=1}^n \delta_i P_i(t)^{1/q} \right)^q dt, \quad x \in [a, b];$$

- (b)  *$f' \leq 0$ ,  $f$  is concave (or  $f' \geq 0$ ,  $f$  is convex),  $g$  is either convex increasing or concave decreasing, and*

$$(18) \quad \int_x^b P_{n+1}(t) dt \leq \int_x^b \left( \sum_{i=1}^n \delta_i P_i(t)^{1/q} \right)^q dt, \quad x \in [a, b];$$

- (c)  *$f'$  is non-positive and symmetrical on  $[a, b]$ , non-decreasing on  $[(a+b)/2, b]$  (or  $f'$  is non-negative and symmetrical on  $[a, b]$ , non-*

increasing on  $[(a + b)/2, b]$ ,  $g$  is either convex increasing or concave decreasing, and

$$(19) \quad \int_{a+x}^{b-x} P_{n+1}(t) dt \leq \int_{a+x}^{b-x} \left( \sum_{i=1}^n \delta_i P_i(t)^{1/q} \right)^q dt$$

for all  $x \in [0, (b - a)/2]$ .

**5. Applications.** In this section we deduce some inequalities from the integral inequalities stated in Sections 3 and 4. Since the means  $M_{[a,b],g}(p, f)$  cover many known two-variable integral means, the inequalities obtained are generalizations of some well known ones.

**COROLLARY 5.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be non-negative and non-increasing, and  $h_i : [a, b] \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$ , be non-negative non-decreasing functions with continuous first derivatives and  $h_i(a) = 0$  for all  $i = 1, \dots, n$ . If  $\gamma_i$ ,  $i = 1, \dots, n$ , are positive numbers such that  $\sum_{i=1}^n 1/\gamma_i = 1$ , then*

$$(20) \quad \int_a^b \left( \prod_{i=1}^n h_i(t)^{1/\gamma_i} \right)' f(t) dt \leq \prod_{i=1}^n \left( \int_a^b h_i'(t) f(t) dt \right)^{1/\gamma_i}.$$

*Proof.* Put

$$p_i = h_i', \quad i = 1, \dots, n, \quad p_{n+1} = \left( \prod_{i=1}^n h_i^{1/\gamma_i} \right)'.$$

Then

$$P_i(x) = \frac{h_i(x)}{h_i(b)}, \quad i = 1, \dots, n, \quad P_{n+1}(x) = \frac{\prod_{i=1}^n h_i(x)^{1/\gamma_i}}{\prod_{i=1}^n h_i(b)^{1/\gamma_i}}.$$

Thus  $P_{n+1}(x) = \prod_{i=1}^n P_i(x)^{1/\gamma_i}$  for  $x \in [a, b]$ . If we now choose  $g(x) = x$ , then the result follows from Theorem 3.1(a). ■

**REMARK 5.2.** Note that the inequality (20) (cf. [15]) is a generalization of the so-called *Gauss–Pólya inequality* (cf. [16]). Namely, for  $n = 2$ ,  $a = 0$ ,  $\gamma_1 = \gamma_2 = 2$ ,  $h_1(t) = t^{2u+1}$ ,  $h_2(t) = t^{2v+1}$  for  $u, v > -1/2$  and  $f$  a non-negative non-increasing function, we have

$$\left( \int_0^b t^{u+v} f(t) dt \right)^2 \leq \left( 1 - \left( \frac{u - v}{u + v + 1} \right)^2 \right) \int_0^b t^{2u} f(t) dt \int_0^b t^{2v} f(t) dt,$$

whenever the integrals exist. Putting  $u = 0$ ,  $v = 2$  and letting  $b \rightarrow \infty$  we obtain the result of C. F. Gauss on the second and fourth moments (cf. [6]):

$$\left( \int_0^\infty t^2 f(t) dt \right)^2 \leq \frac{5}{9} \int_0^\infty f(t) dt \cdot \int_0^\infty t^4 f(t) dt.$$

REMARK 5.3. Alzer [2] derived the inequality

$$\int_a^b (\mathcal{G}_{(2)}(h_1(t), h_2(t)))' f(t) dt \leq \mathcal{G}_{(2)} \left( \int_a^b h_1'(t) f(t) dt, \int_a^b h_2'(t) f(t) dt \right),$$

which holds for non-negative decreasing functions  $h_1, h_2, f : [a, b] \rightarrow \mathbb{R}$  such that  $h_1, h_2$  and  $\sqrt{h_1 h_2}$  are continuously differentiable with  $h_1(a) = h_2(a)$  and  $h_1(b) = h_2(b)$ . Obviously, this is a special case of (20) for  $n = 2$ .

An analogous result connected with the weighted power mean  $\mathcal{P}_{(2)}^{[r]}$  is as follows (cf. [14]):

COROLLARY 5.4. *Let  $h_1, h_2 : [a, b] \rightarrow \mathbb{R}$  be non-negative non-decreasing functions with continuous first derivatives and  $h_1(a) = h_2(a), h_1(b) = h_2(b)$ . Let  $\gamma_1, \gamma_2 > 0$  with  $\gamma_1 + \gamma_2 = 1$ .*

(a) *If  $f : [a, b] \rightarrow \mathbb{R}$  is non-negative and non-decreasing, then*

$$(21) \quad \mathcal{P}_{(2)}^{[r]} \left( \int_a^b h_1'(t) f(t) dt, \int_a^b h_2'(t) f(t) dt \right) \leq \int_a^b (\mathcal{P}_{(2)}^{[s]}(h_1(t), h_2(t)))' f(t) dt$$

*for  $r, s < 1$ , and for  $r, s > 1$  the inequality is reversed.*

(b) *If  $f : [a, b] \rightarrow \mathbb{R}$  is non-negative and non-increasing, then for  $r < 1 < s$  the inequality (21) holds and for  $r > 1 > s$  the inequality is reversed.*

For some analogous results related to the Gauss–Pólya inequality involving quasi-arithmetic means and logarithmic means, see [14] and [15]. A generalization of the Pólya inequality for the Stolarsky and Gini means is given in [13].

REMARK 5.5. Similarly, if  $f : [0, 1] \rightarrow \mathbb{R}$  is non-negative and non-decreasing, then

$$\left( \int_0^1 t^{u+v} f(t) dt \right)^2 \geq \left( 1 - \left( \frac{u-v}{u+v+1} \right)^2 \right) \int_0^1 t^{2u} f(t) dt \int_0^1 t^{2v} f(t) dt.$$

In the following corollary we give a generalization of the above inequality.

COROLLARY 5.6. *Let  $f : [0, 1] \rightarrow \mathbb{R}$  be non-negative and non-decreasing, and  $\gamma_i, i = 1, \dots, n$ , be positive numbers such that  $\sum_{i=1}^n 1/\gamma_i = 1$ . If  $\lambda_i > -1/\gamma_i$  for  $i = 1, \dots, n$ , then*

$$\int_0^1 t^{\sum_{i=1}^n \lambda_i} f(t) dt \geq \frac{\prod_{i=1}^n (1 + \lambda_i \gamma_i)^{1/\gamma_i}}{1 + \sum_{i=1}^n \lambda_i} \prod_{i=1}^n \left( \int_0^1 t^{\lambda_i \gamma_i} f(t) dt \right)^{1/\gamma_i}.$$

*Proof.* Since  $\lambda_i > -1/\gamma_i$  for  $i = 1, \dots, n$ , we put

$$p_i(t) = (t^{1+\lambda_i\gamma_i})', \quad i = 1, \dots, n, \quad p_{n+1}(t) = \left( \prod_{i=1}^n t^{\lambda_i+1/\gamma_i} \right)'.$$

Then for  $a = 0$ ,  $b = 1$ , we have

$$P_i(x) = x^{1+\lambda_i\gamma_i}, \quad i = 1, \dots, n, \quad P_{n+1}(x) = x^{1+\sum_{i=1}^n \lambda_i},$$

and therefore

$$P_{n+1}(x) = \prod_{i=1}^n P_i(x)^{1/\gamma_i}.$$

Choosing  $g$  to be the identity and applying Theorem 3.1(b), we obtain

$$A_{[0,1]}(p_{n+1}, f) = \left( 1 + \sum_{i=1}^n \lambda_i \right) \int_0^1 t^{\sum_{i=1}^n \lambda_i} f(t) dt,$$

and

$$\prod_{i=1}^n (M_{[0,1],g}(p_i, f))^{1/\gamma_i} = \prod_{i=1}^n (1 + \lambda_i\gamma_i)^{1/\gamma_i} \cdot \prod_{i=1}^n \left( \int_0^1 t^{\lambda_i\gamma_i} f(t) dt \right)^{1/\gamma_i}.$$

Hence the result. ■

The following corollary is a consequence of Theorem 4.1 and may be found in [17].

**COROLLARY 5.7.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be non-negative and non-increasing, and  $h_i : [a, b] \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$ , be non-negative non-decreasing functions with continuous first derivatives. If  $q > 1$ , then*

$$(22) \quad \left( \int_a^b \left( \left( \sum_{i=1}^n h_i(t) \right)^q \right)' f(t) dt \right)^{1/q} \leq \sum_{i=1}^n \left( \int_a^b (h_i(t)^q)' f(t) dt \right)^{1/q}.$$

*Proof.* Put

$$\delta_i = \frac{h_i(b)}{\sum_{i=1}^n h_i(b)}$$

and

$$p_i(x) = (h_i(x)^{1/q})', \quad i = 1, \dots, n, \quad p_{n+1}(x) = \left( \left( \frac{\sum_{i=1}^n h_i(x)}{\sum_{i=1}^n h_i(b)} \right)^q \right)'.$$

If  $g$  is the identity, then the functions  $f$ ,  $g$ ,  $p_i$ , and numbers  $\delta_i$ ,  $i = 1, \dots, n$ , satisfy the assumptions of Theorem 4.1(a), which yields (22). ■

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