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ON THE FOURIER TRANSFORM, BOEHMIANS, AND DISTRIBUTIONS

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Abstract. We introduce some spaces of generalized functions that are defined as generalized quotients and Boehmians. The spaces provide simple and natural frameworks for extensions of the Fourier transform.

1. Introduction. The Fourier transform is one of the most important tools in analysis. When using the Fourier transform on a space of functions or generalized functions, knowing the range is usually essential. For this reason, the space of square integrable functions and the space of tempered distributions are very useful. However, the space of square integrable functions is often too small, while the space of tempered distributions requires substantial machinery from functional analysis.

In this paper we consider a number of spaces of generalized functions for which the Fourier transform can be defined in a simple manner and the range can be easily characterized. The constructions have algebraic character, which means that the definitions do not require topological considerations. On the other hand, the spaces have natural topologies that have desirable properties.

Some of the relevant objects are examples of the so-called "generalized quotients" (see [6] and [3]), other are examples of Boehmians. One of the spaces of Boehmians is isomorphic to the space of all Schwartz distributions.

The space $\mathcal{B}(L^2(\mathbb{R}^N),\mathcal{G})$ is an example of the so-called generalized quotients. We start by recalling essential details of the construction of generalized quotients.

Let X be a nonempty set and let G be a commutative semigroup acting on X injectively. This means that every $\varphi\in G$ is an injective map $\varphi : X\to X$ and $(\varphi\psi)x=\varphi(\psi x)$ for all $\varphi,\psi\in G$ and $x\in X$.

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Let $\mathcal{A} = X \times G$. For $(x, \varphi), (y, \psi) \in \mathcal{A}$ we write

$$(x, \varphi) \sim (y, \psi) \quad \text{if} \quad \psi x = \varphi y.$$

It is easy to check that this is an equivalence relation in $\mathcal{A}$. Finally, we define

$$\mathcal{B}(X, G) = \mathcal{A}/\!\sim,$$

the set of generalized quotients.

The equivalence class of (x, φ) will be denoted by $\frac{x}{\varphi}$. This is a slight abuse of notation, but we follow here the tradition of denoting rational numbers by $\frac{p}{q}$ even though the same formal problem is present there.

Elements of X can be identified with elements of $\mathcal{B}(X, G)$ via the embedding $\iota : X \to \mathcal{B}(X, G)$ defined by

$$\iota(x) = \frac{\varphi x}{\varphi},$$

where φ is an arbitrary element of G. Clearly, ι is well defined, that is, it is independent of φ. Action of G can be extended to $\mathcal{B}(X, G)$ via

$$\varphi \frac{x}{\psi} = \frac{\varphi x}{\psi}.$$

If $\varphi \frac{x}{\psi} = \iota(y)$ for some $y \in X$, we will write $\varphi \frac{x}{\psi} \in X$ and $\varphi \frac{x}{\psi} = y$, which is formally incorrect, but convenient and harmless. For instance, we have $\varphi \frac{x}{\varphi} = x$.

The construction of Boehmians is similar to the construction of generalized quotients. We start with a nonempty set X and a commutative semigroup G acting on X. However, we do not assume that G acts on X injectively. Instead, we assume that there is a $\Delta \subset G^{\mathbb{N}}$ such that the following two conditions are satisfied:

1. If $(\varphi_n), (\psi_n) \in \Delta$, then $(\varphi_n \psi_n) \in \Delta$.
2. If $x, y \in X$, $(\varphi_n) \in \Delta$, and $\varphi_n x = \varphi_n y$ for every $n \in \mathbb{N}$, then $x = y$.

Members of Δ are called *delta sequences.*

Now we introduce an equivalence relation on a subset of $X^{\mathbb{N}} \times \Delta$:

$$\mathcal{A} = \{(x_n, \varphi_n) : x_n \in X,\ (\varphi_n) \in \Delta,\ \varphi_n x_m = \varphi_m x_n \text{ for all } m, n \in \mathbb{N}\}.$$

If $(x_n, \varphi_n), (y_n, \psi_n) \in \mathcal{A}$ and $\varphi_n y_m = \psi_m x_n$ for all $m, n \in \mathbb{N}$, then we write $(x_n, \varphi_n) \sim (y_n, \psi_n)$. It is easy to verify that this defines an equivalence relation in $\mathcal{A}$. The equivalence classes are called *Boehmians.* To simplify notation, the equivalence class of (x_n, φ_n) will be denoted by $\frac{x_n}{\varphi_n}$. Hence,

$$\frac{x_n}{\varphi_n} = \frac{y_n}{\psi_n} \quad \text{means} \quad \varphi_n y_m = \psi_m x_n \text{ for all } m, n \in \mathbb{N}.$$

The space of Boehmians will be denoted by $\mathcal{B}(X, \Delta)$. Elements of X can be identified with elements of $\mathcal{B}(X, \Delta)$ via the embedding $\iota : X \to \mathcal{B}(X, \Delta)$

defined by

$$\iota(x) = \frac{\varphi_n x}{\varphi_n},$$

where $(\varphi_n) \in \Delta$ is arbitrary. It is easy to check that ι is independent of (φ_n).

All functions considered in this paper are complex-valued, unless otherwise stated. An infinitely differentiable function $f : \mathbb{R}^N \to \mathbb{C}$ is called *rapidly decreasing* if

$$\sup_{|\alpha|\le m} \sup_{x\in\mathbb{R}^N} (1 + x_1^2 + \cdots + x_N^2)^m |D^\alpha f(x)| < \infty$$

for every nonnegative integer m, where $x = (x_1, \ldots, x_N)$, $\alpha = (\alpha_1, \ldots, \alpha_N)$, α_n's are nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}.$$

The space of all rapidly decreasing functions is denoted by $\mathcal{S}(\mathbb{R}^N)$. The space of all smooth functions with compact support is denoted by $\mathcal{D}(\mathbb{R}^N)$.

The Fourier transform will play a major role in our considerations. The Fourier transform of f will be denoted by either $\mathcal{F}f$ or $\widehat{f}$ and defined by

$$\mathcal{F}f(x) = \widehat{f}(x) = \int_{\mathbb{R}^N} f(y) e^{-2\pi i x\cdot y}\, dy.$$

2. The space $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$. In this section we consider the space of generalized quotients where $X = L^2(\mathbb{R}^N)$ and G is the family of functions

$$\mathcal{G} = \{\varphi \in \mathcal{S}(\mathbb{R}^N) : \operatorname{supp} \widehat{\varphi} = \mathbb{R}^N\}$$

acting on $L^2(\mathbb{R}^N)$ by convolution. Here supp denotes the closed support, so the Fourier transform of a function from $\mathcal{G}$ may vanish at some points.

Clearly, $\mathcal{G}$ is a semigroup with respect to convolution. Moreover, since for any $f, g \in L^2(\mathbb{R}^N)$ and any $\varphi \in \mathcal{G}$, $\widehat{f}\widehat{\varphi} = \widehat{g}\widehat{\varphi}$ implies $f = g$ a.e., $\mathcal{G}$ acts on $L^2(\mathbb{R}^N)$ injectively. Note that the space of all smooth functions with compact support is a proper subspace of $\mathcal{G}$.

The following operations can be defined in a natural way for elements of $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$:

(a) $\dfrac{f}{\varphi} + \dfrac{g}{\psi} = \dfrac{f * \psi + g * \varphi}{\varphi * \psi}$ and $\alpha \dfrac{f}{\varphi} = \dfrac{\alpha f}{\varphi}$, $\alpha \in \mathbb{C}$.

(b) $T_z \dfrac{f}{\varphi} = \dfrac{T_z f}{\varphi}$, where T_z is the shift operator.

(c) $\dfrac{f}{\varphi} * \dfrac{g}{\psi} = \dfrac{f * g}{\varphi * \psi}$, where $g \in \mathcal{S}(\mathbb{R}^N)$.

(d) $D^\alpha \dfrac{f}{\varphi} = \dfrac{f * D^\alpha \varphi}{\varphi * \varphi}$.

The Fourier transform of $F = \frac{f}{\varphi} \in \mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$ can be defined as $\widehat{F} = \mathcal{F}\big(\frac{f}{\varphi}\big) = \frac{\widehat{f}}{\widehat{\varphi}}$. Since $\frac{f}{\varphi} = \frac{g}{\psi}$ implies $\widehat{f}\widehat{\psi} = \widehat{g}\widehat{\varphi}$, $\widehat{F}$ is well defined as a function defined a.e. It is easy to see that the Fourier transform thus defined has the usual properties:

(a) $\mathcal{F}$ is linear.
(b) $\mathcal{F}(T_z F)(x) = e^{-2\pi i x\cdot z}\widehat{F}(x)$.
(c) $\mathcal{F}(F * g) = \widehat{F}\widehat{g}$, where $g \in L^1(\mathbb{R}^N)$.
(d) $\mathcal{F}\big(\frac{\partial}{\partial x_k} F\big) = 2\pi i x_k \widehat{F}$.

A function is called *locally square integrable* if its restriction to any compact set is square integrable over that set. The space of all locally square integrable functions will be denoted by $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Note that

$$\mathcal{C}(\mathbb{R}^N) \subset L^2_{\mathrm{loc}}(\mathbb{R}^N) \subset L_{\mathrm{loc}}(\mathbb{R}^N),$$

where $\mathcal{C}(\mathbb{R}^N)$ denotes the space of all continuous functions and $L_{\mathrm{loc}}(\mathbb{R}^N)$ denotes the space of all locally integrable functions.

THEOREM 2.1. *Every locally square integrable function is the Fourier transform of some $F \in \mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$.*

Proof. Let $\psi_1, \psi_2, \ldots \in \mathcal{D}(\mathbb{R}^N)$ be a partition of unity. If $g \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$, then there are positive constants $\alpha_1, \alpha_2, \ldots$ such that the series $\sum_{n=1}^{\infty} \alpha_n \psi_n g$ converges in $L^2(\mathbb{R}^N)$ and the series $\sum_{n=1}^{\infty} \alpha_n \psi_n$ converges in $\mathcal{S}(\mathbb{R}^N)$. Then $\sum_{n=1}^{\infty} \alpha_n \psi_n = \widehat{\varphi}$ for some $\varphi \in \mathcal{G}$ and $g\widehat{\varphi} = \widehat{f}$ for some $f \in L^2(\mathbb{R}^N)$. Hence

$$\mathcal{F}\left(\frac{f}{\varphi}\right) = \frac{\widehat{f}}{\widehat{\varphi}} = \frac{g\widehat{\varphi}}{\widehat{\varphi}} = g. \ \blacksquare$$

The range of the Fourier transform on $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$ contains functions that are not locally integrable. For example, consider the function $g(t) = te^{-t^2}$. Then $\frac{g}{g*g} \in \mathcal{B}(L^2(\mathbb{R}), \mathcal{G})$, but $\mathcal{F}\big(\frac{g}{g*g}\big) = \frac{1}{\widehat{g}}$ is not locally integrable.

3. Integrable and square integrable Boehmians. In this section we compare two spaces of Boehmians with the space $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$. First we consider integrable Boehmians, introduced in [7]. To obtain them we take $X = L^1(\mathbb{R}^N)$ and $G = L^1(\mathbb{R}^N)$ acting on X by convolution. For Δ we take the family Δ_{L^1} of all sequences of functions $\varphi_n \in L^1(\mathbb{R}^N)$ satisfying the following three conditions:

(a) $\int_{\mathbb{R}^N} \varphi_n(x)\,dx = 1$ for all $n \in \mathbb{N}$.
(b) $\int_{\mathbb{R}^N} |\varphi_n(x)|\,dx \le M$ for all $n \in \mathbb{N}$ and some constant M.

(c) For every $\varepsilon > 0$, $\lim_{n\to\infty} \int_{\|x\|\geq\varepsilon} |\varphi_n(x)|\, dx = 0$.

It is shown in [7] that for every Boehmian $F = \frac{f_n}{\varphi_n} \in \mathcal{B}(L^1(\mathbb{R}^N), \Delta_{L^1})$ the sequence $(\widehat{f_n})$ converges uniformly on compact sets and the limit does not depend on the representation of F. Hence, one can define the Fourier transform of F as $\widehat{F} = \lim_{n\to\infty} \widehat{f_n}$. It follows that the Fourier transform of an integrable Boehmian is a continuous function. Every distribution with compact support can be identified with an integrable Boehmian. There are integrable Boehmians that are not distributions, for example, $\sum_{n=0}^{\infty} \delta^{(n)}/(2n)!$, where δ is the Dirac delta distribution (see [2]).

Now we define the space of square integrable Boehmians $\mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$. The family of delta sequences $\Delta_{\mathcal{S}}$ is defined as the collection of all sequences $\varphi_1, \varphi_2, \ldots \in \mathcal{S}$ such that $\widehat{\varphi}_n \to 1$ uniformly on compact sets. It is easy to see that the conditions necessary for the construction of the space of Boehmians $\mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$ are satisfied. The Fourier transform of $\frac{f_n}{\varphi_n} \in \mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$ is defined as

$$\mathcal{F}\left(\frac{f_n}{\varphi_n}\right) = \frac{\widehat{f_n}}{\widehat{\varphi}_n}.$$

The expression $\frac{\widehat{f_n}}{\widehat{\varphi}_n}$ should be interpreted as a Boehmian with respect to pointwise multiplication. Since $\widehat{f_n}\widehat{\varphi}_m = \widehat{f_m}\widehat{\varphi}_n$ a.e. for all $m, n \in \mathbb{N}$, the quotient $\frac{\widehat{f_n}}{\widehat{\varphi}_n}$ is well defined. Moreover, $\frac{\widehat{f_n}}{\widehat{\varphi}_n}$ defines a locally square integrable function. Indeed, for any compact $K \subset \mathbb{R}^N$ there exists an $n \in \mathbb{N}$ such that $\widehat{\varphi}_n \neq 0$ on K. Then we can define $f(x) = \frac{\widehat{f_n}(x)}{\widehat{\varphi}_n(x)}$ for $x \in K$.

THEOREM 3.1. *The Fourier transform is a vector space isomorphism between* $\mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$ *and* $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

Proof. It suffices to show that the Fourier transform is surjective. Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and let $(\varphi_n) \in \Delta_{\mathcal{S}}$ be such that $\widehat{\varphi}_n \in \mathcal{D}(\mathbb{R}^N)$. Then, for every $n \in \mathbb{N}$, there exists a $g_n \in L^2(\mathbb{R}^N)$ such that $\widehat{g}_n = f\widehat{\varphi}_n$. Then $\frac{g_n}{\varphi_n} \in \mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$ and

$$\mathcal{F}\left(\frac{g_n}{\varphi_n}\right) = \frac{f\widehat{\varphi}_n}{\widehat{\varphi}_n} = f. \blacksquare$$

THEOREM 3.2. $\mathcal{B}(L^1(\mathbb{R}^N), \Delta_{L^1}) \subset \mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}}) \subset \mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$.

Proof. The first inclusion follows from the fact that the Fourier transform of an integrable Boehmian is a continuous function and from Theorem 3.1. The second inclusion follows from Theorem 2.1. ∎

Since $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$ contains both integrable and square integrable Boehmians and the construction is simpler, it may seem that there is no point in considering the two spaces of Boehmians. This is not necessarily true. For example, continuity of the Fourier trasform of an integrable Boehmian is a desirable property. Moreover, the simple description of the range of the Fourier transform on $\mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$ makes it a convenient framework. Integrable and square integrable Boehmians are local objects. It makes sense to compare two Boehmians on an open set. This is not the case with elements of $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$.

4. Schwartz distributions as a space of Boehmians. First we consider a subspace of $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$:

$$\mathcal{K} = \Big\{ \sum_{n=1}^{m} \lambda_n D^{\alpha_n} f_n : \lambda_n \in \mathbb{C},\ f_n \in L^2(\mathbb{R}^N),\ \text{and } \alpha_n \in \mathbb{N}_0^N \Big\},$$

where $\mathbb{N}_0 = \{0, 1, \dots\}$ and $D^{\alpha_n} f_n$ is defined as an element of $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$, that is,

$$D^{\alpha_n} f_n = \frac{f_n * D^{\alpha_n}\varphi}{\varphi},$$

where $\varphi \in \mathcal{G}$ is arbitrary.

Since $\mathcal{K} \subset \mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$, the Fourier transform is defined for every F in $\mathcal{K}$. It can be written in the following explicit form:

$$\mathcal{F}\Big(\sum_{n=1}^{m} \lambda_n D^{\alpha_n} f_n \Big) = \sum_{n=1}^{m} (2\pi i)^{|\alpha_n|} \lambda_n M_{\alpha_n} \widehat{f_n}, \tag{4.1}$$

where M_α is the multiplication operator defined by $M_\alpha f(x) = x^\alpha f(x)$.

The range of the Fourier transform on $\mathcal{K}$ is the space of *moderate functions.* A function $f : \mathbb{R}^N \to \mathbb{C}$ will be called moderate if $f = pg$ where p is a polynomial and $g \in L^2(\mathbb{R}^N)$. The space of all moderate functions will be denoted by $\mathcal{M}$. Note that $\mathcal{M}$ is a vector space that is closed under multiplication by polynomials. $\mathcal{M}$ contains all slowly increasing functions as a proper subspace. (A locally integrable function $f : \mathbb{R}^N \to \mathbb{C}$ is called *slowly increasing* if $|f| \le p$ for some polynomial p.)

THEOREM 4.1. *The Fourier transform is a vector space isomorphism between $\mathcal{K}$ and $\mathcal{M}$.*

This follows easily from (4.1).

LEMMA 4.2. *If $\varphi \in \mathcal{S}$ and $f \in L^2(\mathbb{R}^N)$, then $\varphi * (pf) \in \mathcal{M}$ for any polynomial p.*

Proof. It suffices to prove the lemma for $p(x) = x^\alpha = x_1^{\alpha_1} \dots x_N^{\alpha_N}$. Then

$$\varphi * (pf)(y) = \int_{\mathbb{R}^N} \varphi(y - x) x^\alpha f(x)\, dx = \sum_{\beta \le \alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} y^{\alpha-\beta} (y^\beta \varphi(y) * f(y)).$$

Since $y^\beta \varphi(y) * f(y)$ is in $L^2(\mathbb{R}^N)$ for every β, we conclude $\varphi * (pf) \in \mathcal{M}$. ∎

In view of the above lemma and Theorem 4.1, for any $\varphi \in \mathcal{S}$ and $F \in \mathcal{K}$ we can define the product

$$\varphi F = \mathcal{F}^{-1}(\mathcal{F}\varphi * \mathcal{F}F). \tag{4.2}$$

Since $\mathcal{F}$ is a bijection from $\mathcal{K}$ onto $\mathcal{M}$, $\mathcal{F}^{-1}f$ is well defined for all $f \in \mathcal{M}$. $\mathcal{F}^{-1}$ should not be interpreted as the usual inverse Fourier transform. Note that for any $\varphi \in \mathcal{S}$ and $f \in L^2(\mathbb{R}^N)$ we have

$$\varphi D^\alpha f = \sum_{\beta \le \alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} D^{\alpha-\beta}(D^\beta \varphi f). \tag{4.3}$$

Now we are going to construct a space of Boehmians that is isomorphic to the space of all Schwartz distributions $\mathcal{D}'$. Before we describe that construction, it is essential to observe that $\mathcal{K}$ can be identified with a subspace of the space of tempered distributions $\mathcal{S}'$. Indeed, elements of $\mathcal{K}$ are defined in terms of derivatives of square integrable functions. If these derivatives are interpreted in the distributional sense, then $\mathcal{K}$ becomes a space of distributions. Formally, the identification map $\iota : \mathcal{K} \to \mathcal{S}'$ can be defined by $\iota(F) = \mathcal{F}_{\mathcal{S}'}^{-1}(\mathcal{F}F)$, where $\mathcal{F}$ denotes the Fourier transform $\mathcal{F} : \mathcal{K} \to \mathcal{M}$ defined above and $\mathcal{F}_{\mathcal{S}'}^{-1} : \mathcal{M} \to \mathcal{S}'$ is the distributional inverse transform. It is possible to define ι directly: if

$$F = \sum_{n=1}^{m} \lambda_n D^{\alpha_n} f_n = \frac{\sum_{n=1}^{m} f_n * D^{\alpha_n}\varphi}{\varphi} \in \mathcal{K},$$

then

$$\langle \iota(F), \psi \rangle = \sum_{n=1}^{m} \lambda_n (-1)^{\alpha_n} \int_{\mathbb{R}^N} f_n D^{\alpha_n} \psi,$$

for any $\psi \in \mathcal{S}$. Moreover, we have the following properties:

(a) ι is linear,
(b) $D^\alpha \iota(F) = \iota(D^\alpha F)$,
(c) $\iota(F)\varphi = \iota(F\varphi)$ for any $\varphi \in \mathcal{S}$,
(d) $\iota(F) * \varphi = \iota(F * \varphi)$ for any $\varphi \in \mathcal{S}$.

These properties follow from the definition of ι and some elementary properties of distributions and the Fourier transform. Note that all distributions with compact support are in $\iota(\mathcal{K})$.

Now we describe the space $\mathcal{B}(\mathcal{K}, \Delta_{\widehat{\mathcal{S}}})$. First note that $\mathcal{S}$ with pointwise multiplication is a commutative semigroup acting on $\mathcal{K}$ by multiplication defined in (4.2). By a *delta sequence* in $\mathcal{S}$ we mean any sequence $\varphi_1, \varphi_2, \ldots \in \mathcal{S}$ convergent to the constant function 1 uniformly on compact sets. The family of all such delta sequences is denoted by $\Delta_{\widehat{\mathcal{S}}}$. It is easy to see that the conditions necessary for the construction of the space of Boehmians $\mathcal{B}(\mathcal{K}, \Delta_{\widehat{\mathcal{S}}})$ are satisfied.

THEOREM 4.3. *$\mathcal{D}'$ and $\mathcal{B}(\mathcal{K}, \Delta_{\widehat{\mathcal{S}}})$ are isomorphic vector spaces.*

Proof. Let $f \in \mathcal{D}'$ and let $(\varphi_n) \in \Delta_{\widehat{\mathcal{S}}}$ be such that $\varphi_n \in \mathcal{D}$ for all $n \in \mathbb{N}$. Then $f\varphi_n$ is a distribution with compact support, and thus $f\varphi_n = \iota(F_n)$ for some $F_n \in \mathcal{K}$. We define an isomorphism $\sigma : \mathcal{D}' \to \mathcal{B}(\mathcal{K}, \Delta_{\widehat{\mathcal{S}}})$ via

$$\sigma(f) = \frac{\iota^{-1}(f\varphi_n)}{\varphi_n}.$$

This is clearly a linear map. Moreover, if

$$\frac{\iota^{-1}(f\varphi_n)}{\varphi_n} = \frac{\iota^{-1}(g\varphi_n)}{\varphi_n},$$

then $f\varphi_n\varphi_m = g\varphi_m\varphi_n$ for every $m, n \in \mathbb{N}$, and hence $f = g$.

Now, let $\frac{F_n}{\psi_n} \in \mathcal{B}(\mathcal{K}, \Delta_{\widehat{\mathcal{S}}})$ and $\varphi \in \mathcal{D}$. We define a distribution $f \in \mathcal{D}'$ by

$$\langle f, \varphi \rangle = \left\langle \iota(F_n), \frac{\varphi}{\psi_n} \right\rangle,$$

where $n \in \mathbb{N}$ is any index for which $\psi_n \neq 0$ on the support of ϕ. This definition is independent of n. Indeed, if $\psi_n \neq 0$ and $\psi_m \neq 0$ on the support of φ, then

$$\begin{aligned}\left\langle \iota(F_n), \frac{\varphi}{\psi_n} \right\rangle &= \left\langle \iota(F_n)\psi_m, \frac{\varphi}{\psi_n\psi_m} \right\rangle = \left\langle \iota(F_n\psi_m), \frac{\varphi}{\psi_n\psi_m} \right\rangle \\ &= \left\langle \iota(F_m\psi_n), \frac{\varphi}{\psi_n\psi_m} \right\rangle = \left\langle \iota(F_m)\psi_n, \frac{\varphi}{\psi_n\psi_m} \right\rangle = \left\langle \iota(F_m), \frac{\varphi}{\psi_m} \right\rangle.\end{aligned}$$

Since $\langle f, \varphi \rangle$ is clearly linear and continuous in φ, f is a distribution. ∎

5. Convergence and continuity. In this section we examine continuity of maps considered in the previous sections. Since we are only dealing with sequential topologies, all the proofs are formulated in terms of sequential continuity.

Let X be a topological space and let G be a commutative semigroup of continuous injections acting on X. We assume that G is equipped with the discrete topology. Now we can define the product topology on $\mathcal{A}$ and then the quotient topology in $\mathcal{B}(X, G)$. We will refer to this topology as the natural topology. Properties of such topologies are discussed in [3].

The following theorem describes convergence in the natural topology of $\mathcal{B}(X, G)$.

THEOREM 5.1 ([3]). *If $X \times G$ is first countable, then $F_n \to F$ in the natural topology in $\mathcal{B}(X, G)$ if and only if every subsequence of (F_n) has a subsequence (F_{p_n}) such that $\varphi F_{p_n} \to \varphi F$ in X for some $\varphi \in G$.*

Let $\widehat{\mathcal{G}} = \{\widehat{\varphi} : \varphi \in \mathcal{G}\}$. The range of the Fourier transform on $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$ is the space of all functions $f : \mathbb{R}^N \to \mathbb{C}$ such that $f\psi \in L^2(\mathbb{R}^N)$ for some

$\psi \in \widehat{\mathcal{G}}$. Note that this space can be described as the space $\mathcal{B}(L^2(\mathbb{R}^N), \widehat{\mathcal{G}})$ of generalized quotients with $X = L^2(\mathbb{R}^N)$ and $G = \widehat{\mathcal{G}}$ acting on X by multiplication. As a space of generalized quotients, it can be equipped with the natural topology. The following theorem is an obvious consequence of the definitions and the Plancherel theorem.

THEOREM 5.2. *The Fourier transform is a topological isomorphism between* $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$ *and* $\mathcal{B}(L^2(\mathbb{R}^N), \widehat{\mathcal{G}})$.

The natural topology in spaces of Boehmians is the topology of the so-called Δ-convergence [5]. Let $F, F_n \in \mathcal{B}(X, \Delta)$, $n \in \mathbb{N}$. The sequence (F_n) is Δ-*convergent* to F, denoted by $F_n \xrightarrow{\Delta} F$, if there exists a delta sequence $(\varphi_n) \in \Delta$ such that $\varphi_n(F_n - F) \in X$ for every $n \in \mathbb{N}$ and $\varphi_n(F_n - F) \to 0$ in the topology of X. A second type of convergence, called δ-convergence, is also used. The sequence (F_n) is δ-*convergent* to F, denoted by $F_n \xrightarrow{\delta} F$, if there exists a delta sequence $(\varphi_n) \in \Delta$ such that $\varphi_k(F_n - F) \in X$ for every $k, n \in \mathbb{N}$ and, for every $k \in \mathbb{N}$, $\varphi_k(F_n - F) \to 0$ in the topology of X as $n \to \infty$. Under some natural conditions on the topology of X and the family of delta sequences Δ, we can prove the following relationship between these two types of convergence (see [5]): $F_n \xrightarrow{\Delta} F$ if and only if every subsequence of (F_n) has a subsequence (F_{p_n}) such that $F_{p_n} \xrightarrow{\delta} F$.

Now we consider continuity of the isomorphism $\varrho : \mathcal{B}(L^1(\mathbb{R}^N), \Delta_{L^1}) \to \mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$ defined by

$$\varrho\left(\frac{f_n}{\varphi_n}\right) = \mathcal{F}^{-1}(\lim_{n\to\infty} \widehat{f}_n),$$

where $\mathcal{F}^{-1} : \mathcal{B}(L^2(\mathbb{R}^N), \widehat{\mathcal{G}}) \to \mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$. In other words, ϱ is the composition of the Fourier transform $\mathcal{F} : \mathcal{B}(L^1(\mathbb{R}^N), \Delta_{L^1}) \to \mathcal{C}(\mathbb{R}^N)$ with the inverse of the Fourier transform $\mathcal{F} : \mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G}) \to \mathcal{B}(L^2(\mathbb{R}^N), \widehat{\mathcal{G}})$.

THEOREM 5.3. $\varrho : \mathcal{B}(L^1(\mathbb{R}^N), \Delta_{L^1}) \to \mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$ *is continuous.*

Proof. The Fourier transform $\mathcal{F} : \mathcal{B}(L^1(\mathbb{R}^N), \Delta_{L^1}) \to \mathcal{C}(\mathbb{R}^N)$ is continuous with respect to the topology of uniform convergence on compact sets in $\mathcal{C}(\mathbb{R}^N)$ (see [7]). Since uniform convergence on compact sets is stronger than the convergence in $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$, the conclusion follows from Theorem 5.2. ∎

THEOREM 5.4. $\omega : \mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}}) \to \mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$ *is continuous.*

Proof. It suffices to show that $F_n \xrightarrow{\delta} 0$ in $\mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$ implies $F_n \to 0$ in $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$. If $F_n \xrightarrow{\delta} 0$ in $\mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$, then there exists a delta sequence $(\varphi_n) \in \Delta_{\mathcal{S}}$ such that, for every $k \in \mathbb{N}$, $F_n * \varphi_k \to 0$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. Consequently, $F_n \to 0$ in $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$. ∎

THEOREM 5.5. *$\mathcal{F} : \mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}}) \to L^2_{\mathrm{loc}}(\mathbb{R}^N)$ is a topological isomorphism.*

Proof. If $F_n \overset{\Delta}{\to} 0$ in $\mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$, then $F_n * \varphi_n \to 0$ in $L^2(\mathbb{R}^N)$ for some $(\varphi_n) \in \Delta_{\mathcal{S}}$. Hence $\widehat{F}_n \widehat{\varphi}_n \to 0$ in $L^2(\mathbb{R}^N)$ and $|\widehat{F}_n|^2 |\widehat{\varphi}_n|^2 \to 0$ in $L^1(\mathbb{R}^N)$. If $K \subset \mathbb{R}^N$ is compact, then there exists $n_0 \in \mathbb{N}$ such that $|\widehat{\varphi}_n|^2 > 1/2$ for all $n > n_0$. Then

$$\int_K |\widehat{F}_n|^2 = \int_K \frac{|\widehat{F}_n|^2 |\widehat{\varphi}_n|^2}{|\widehat{\varphi}_n|^2} < 2 \int_K |\widehat{F}_n|^2 |\widehat{\varphi}_n|^2$$

for all $n > n_0$. Hence $\int_K |\widehat{F}_n|^2 \to 0$. Since K is arbitrary, we conclude that $\widehat{F}_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

Now assume that $F_1, F_2, \ldots \in \mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$ and $\widehat{F}_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Let $(\varphi_n) \in \Delta_{\mathcal{S}}$ be such that $\widehat{\varphi}_n \in \mathcal{D}$ for all $n \in \mathbb{N}$. Then $\widehat{F}_n \widehat{\varphi}_k \to 0$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$, for every $k \in \mathbb{N}$. Applying the diagonal method, we can find a sequence of positive integers $p_n \to \infty$ such that $\widehat{F}_n \widehat{\varphi}_{p_n} \to 0$ in $L^2(\mathbb{R}^N)$, and hence $F_n * \varphi_{p_n} \to 0$ in $L^2(\mathbb{R}^N)$. Since $(\varphi_{p_n}) \in \Delta_{\mathcal{S}}$, we have $F_n \overset{\Delta}{\to} 0$ in $\mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$. ∎

The above theorem also implies that the embedding of $\mathcal{B}(L^1(\mathbb{R}^N), \Delta_{L^1})$ into $\mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$ is continuous. Indeed, if $F_n \overset{\Delta}{\to} 0$ in $\mathcal{B}(L^1(\mathbb{R}^N), \Delta_{L^1})$, then $\widehat{F}_n \to 0$ uniformly on compact sets. Hence $\widehat{F}_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. This implies $F_n \overset{\Delta}{\to} 0$ in $\mathcal{B}(L^2(\mathbb{R}^N), \Delta_{\mathcal{S}})$, by Theorem 5.5.

Now we consider the question of continuity of the Fourier transform from $\mathcal{K}(\mathbb{R}^N)$ to $\mathcal{M}(\mathbb{R}^N)$. The space $\mathcal{M}(\mathbb{R}^N)$ has a natural convergence: a sequence of functions $f_n \in \mathcal{M}(\mathbb{R}^N)$ converges to 0 if there exist functions $g_n \in L^2(\mathbb{R}^N)$ and a polynomial p such that $f_n = pg_n$, for all $n \in \mathbb{N}$, and $g_n \to 0$ in $L^2(\mathbb{R}^N)$. Then we define $f_n \to f$ in $\mathcal{M}(\mathbb{R}^N)$ as $f_n - f \to 0$ in $\mathcal{M}(\mathbb{R}^N)$. In order to define convergence in $\mathcal{K}(\mathbb{R}^N)$ we first note that every $F \in \mathcal{K}(\mathbb{R}^N)$ can be written in the form $F = Af$ where $f \in L^2(\mathbb{R}^N)$ and A is a linear diferential operator with constant coefficients. A sequence $F_1, F_2, \ldots \in \mathcal{K}(\mathbb{R}^N)$ converges to 0 if there are $f_n \in L^2(\mathbb{R}^N)$ and a linear diferential operator A with constant coefficients such that $F_n = Af_n$, for all $n \in \mathbb{N}$, and $f_n \to 0$ in $L^2(\mathbb{R}^N)$. Convergence to an arbitrary element of $\mathcal{K}(\mathbb{R}^N)$ is defined by linearity. With these definitions the following theorem is obvious.

THEOREM 5.6. *$\mathcal{F} : \mathcal{K}(\mathbb{R}^N) \to \mathcal{M}(\mathbb{R}^N)$ is a topological isomorphism.*

Finally, we address the question of continuity of the identification map in Theorem 4.3. In the following theorem, the topology of $\mathcal{B}(\mathcal{K}, \Delta_{\mathcal{S}})$ is defined as the topology of δ-convergence: $F_n \overset{\delta}{\to} F$ in $\mathcal{B}(\mathcal{K}, \Delta_{\mathcal{S}})$ if there exists $(\varphi_n) \in \Delta_{\mathcal{S}}$ such that $F_n \varphi_k \to F \varphi_k$ in $\mathcal{K}$ for every $k \in \mathbb{N}$.

THEOREM 5.7. *$\varrho : \mathcal{D}' \to \mathcal{B}(\mathcal{K}, \Delta_{\mathcal{S}})$ is a topological isomorphism.*

Proof. Let $f_n \to 0$ in $\mathcal{D}'$ and let $(\varphi_n) \in \Delta_{\mathcal{S}}$ be such that $\varphi_n \in \mathcal{D}$ for all $n \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, there exist continuous functions $g_1, g_2, \ldots$ and a multi-index α such that $f_n\varphi_k = D^\alpha g_n\varphi_k$ and $g_n \to 0$ uniformly. Since, for every $k \in \mathbb{N}$, $g_n\chi_{\operatorname{supp}\varphi_k} \to 0$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$, we have $\varrho f_n \xrightarrow{\delta} 0$ in $\mathcal{B}(\mathcal{K}, \Delta_{\mathcal{S}})$.

Conversely, if $f_n \in \mathcal{D}'$ and $\varrho f_n \xrightarrow{\delta} 0$ in $\mathcal{B}(\mathcal{K}, \Delta_{\mathcal{S}})$, then $f_n\varphi_k \to 0$ in $\mathcal{K}$ for every $k \in \mathbb{N}$ and $(\varphi_n) \in \Delta_{\mathcal{S}}$. Let $\psi \in \mathcal{D}$ and let $k \in \mathbb{N}$ be such that $\varphi_k \neq 0$ on the support of ψ. Then there exist functions $g_1, g_2, \ldots \in L^2(\mathbb{R}^N)$ such that $g_n \to 0$ in $L^2(\mathbb{R}^N)$ and $f_n\varphi_k = D^\alpha g_n$ for some multi-index α and all $n \in \mathbb{N}$. Since

$$\langle f_n, \psi\rangle = \left\langle f_n\varphi_k, \frac{\psi}{\varphi_k}\right\rangle = \left\langle D^\alpha g_n, \frac{\psi}{\varphi_k}\right\rangle = (-1)^{|\alpha|}\left\langle g_n, D^\alpha\frac{\psi}{\varphi_k}\right\rangle \to 0,$$

it follows that $f_n \to 0$ in $\mathcal{D}'$. ■

6. Applications. In this section we consider three different applications of the introduced framework. Our goal is to show that it provides convenient and effective tools.

6.1. *Sobolev space.* First we present a simplified definition of Sobolev spaces. If $F \in \mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$ can be written in the form $F = \frac{f*\varphi}{\varphi}$ for some $f \in L^2(\mathbb{R}^N)$ and $\varphi \in \mathcal{G}$, then we write $F \in L^2(\mathbb{R}^N)$ and $F = f$. This convention is used in the following definition of the Sobolev space $W^m(\mathbb{R}^N)$:

$$W^m(\mathbb{R}^N) = \{f \in L^2(\mathbb{R}^N) : D^\alpha f \in L^2(\mathbb{R}^N) \text{ for all } |\alpha| \le m\}.$$

The inner product in $W^m(\mathbb{R}^N)$ is defined in the usual way:

$$\langle f, g\rangle = \sum_{|\alpha|\le m} \int_{\mathbb{R}^N} D^\alpha f\, \overline{D^\alpha g}.$$

To illustrate the simplicity of this approach, we will show that $W^m(\mathbb{R}^N)$ is complete.

Let (f_n) be a Cauchy sequence in $W^m(\mathbb{R}^N)$ and let $|\alpha| \le m$. Then $(D^\alpha f_n)$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$, and hence $D^\alpha f_n \to g_\alpha$ in $L^2(\mathbb{R}^N)$ for some $g_\alpha \in L^2(\mathbb{R}^N)$. To complete the proof it suffices to show that $D^\alpha g_0 = g_\alpha$ in $\mathcal{B}(L^2(\mathbb{R}^N), \mathcal{G})$. Indeed, since $D^\alpha f_n \to g_\alpha$, we have

$$f_n * D^\alpha\varphi = D^\alpha f_n * \varphi \to g_\alpha * \varphi$$

for any $\varphi \in \mathcal{G}$. On the other hand,

$$f_n * D^\alpha\varphi \to g_0 * D^\alpha\varphi.$$

Hence $g_0 * D^\alpha\varphi = g_\alpha * \varphi$ and consequently

$$D^\alpha g_0 = \frac{g_0 * D^\alpha\varphi}{\varphi} = \frac{g_\alpha * \varphi}{\varphi} = g_\alpha.$$

6.2. p.v.$\left(\frac{1}{x}\right)$. Now we turn our attention to the distribution p.v.$\left(\frac{1}{x}\right)$. We show how it can be introduced as an element of $\mathcal{K}$. Since we use two different kinds of quotients, we need to indicate which one we mean. We adopt the following convention. If the variables are not explicitly used (for instance $\frac{f}{\varphi}$), then it is a convolution quotient. On the other hand, if the variables are explicitly used (for instance $\frac{f(x)}{\varphi(x)}$), then it is a quotient with respect to pointwise multiplication. In order to use this notation consistently, we need to name the functions we use:

$$L(x) = \ln|x| \quad \text{and} \quad M(x) = x.$$

Let $\theta : \mathbb{R} \to [0,1]$ be a smooth function with compact support such that $\theta(-x) = \theta(x)$ for all $x \in \mathbb{R}$ and $\theta(x) = 1$ for all $x \in [-1,1]$. If $\varphi \in \mathcal{G}$ (for instance, $\varphi(x) = e^{-x^2}$), then

$$\begin{aligned} \varphi' * L &= \varphi' * ((1-\theta)L) + \varphi' * (\theta L) \\ &= \varphi * ((1-\theta)L' - \theta' L) + \varphi' * (\theta L). \end{aligned} \tag{6.1}$$

Consequently,

$$\frac{\varphi' * L}{\varphi} = (1-\theta)L' - \theta' L + \frac{\varphi' * (\theta L)}{\varphi}.$$

Since $(1-\theta)L', \theta' L, \theta L \in L^2(\mathbb{R})$, we have $\frac{\varphi' * L}{\varphi} \in \mathcal{K}$. We will show that, for every $\psi \in \mathcal{S}(\mathbb{R})$,

$$(M\psi) \cdot \frac{\varphi' * L}{\varphi} = \psi. \tag{6.2}$$

Let $\theta_n(x) = \theta(nx)$, $n \in \mathbb{N}$. Using (6.1) and (4.3) we obtain

$$\begin{aligned} (M\psi) \cdot \frac{\varphi' * L}{\varphi} &= (M\psi) \cdot \left\{ \frac{\varphi * [(1-\theta_n)L' - \theta_n' L]}{\varphi} + \frac{\varphi' * (\theta_n L)}{\varphi} \right\} \\ &= \frac{\varphi * [\psi(1-\theta_n) - M\psi\theta_n' L]}{\varphi} + \frac{\varphi' * (M\psi\theta_n L)}{\varphi} + \frac{\varphi * [(\psi + M\psi')\theta_n L]}{\varphi} \end{aligned}$$

for all $n \in \mathbb{N}$. Now (6.2) can be obtained by letting $n \to \infty$, as $M\psi\theta_n' L \to 0$, $M\psi\theta_n L \to 0$, $(\psi + M\psi')\theta_n L \to 0$ and $\psi(1-\theta_n) \to \psi$ in $L^2(\mathbb{R})$.

Let $F = \frac{\varphi' * L}{\varphi}$. From (6.2) we obtain $-\frac{1}{2\pi i}\widehat{\psi}' * \widehat{F} = \widehat{\psi}$. Now consider the function $\omega = \widehat{F} + 2\pi i H$, where H is the Heaviside function. Then $\widehat{\psi}' * \omega = 0$, and hence $\widehat{\psi} * \omega = c_\psi$, where c_ψ is a complex number. Since $\psi \in \mathcal{S}(\mathbb{R})$ is arbitrary, we must have $\omega = c$, where c is a complex number. Thus we have $\widehat{F} = -2\pi i H + c$. Since $\widehat{F}$ is an odd function, it follows that $\widehat{F} = -\pi i\sigma$, where σ is the sign function. Consequently,

$$F = \text{p.v.}\left(\frac{1}{x}\right) = \frac{\varphi' * L}{\varphi}.$$

6.3. *Hilbert transform.* Let

$$Q(x) = \frac{x}{x^2+1}.$$

It can be shown (for example, see [4, 399]) that, for any $\psi \in L^2(\mathbb{R})$, we have $Q * \psi \in L^2(\mathbb{R})$ and $\mathcal{F}(Q * \psi) = \mathcal{F}(Q)\mathcal{F}(\psi)$. If we define

$$P(x) = \begin{cases} \dfrac{1}{x(x^2+1)} & \text{if } |x| \geq 1, \\ -\dfrac{x}{x^2+1} & \text{if } |x| < 1, \end{cases}$$

then we have

$$P(x) + Q(x) = \begin{cases} \dfrac{1}{x} & \text{if } |x| \geq 1, \\ 0 & \text{if } |x| < 1, \end{cases}$$

Note that $P \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Now

$$\begin{aligned} \varphi' * L &= \varphi * \big[(1-\theta)L' - \theta' L\big] + \varphi' * (\theta L) \\ &= \varphi * \big[(1-\theta)(P+Q) - \theta' L\big] + \varphi' * (\theta L) \end{aligned}$$

and

$$\text{p.v.}\left(\frac{1}{x}\right) = \frac{\varphi' * L}{\varphi} = (1-\theta)P + (1-\theta)Q - \theta' L + \frac{\varphi' * (\theta L)}{\varphi}. \tag{6.3}$$

Consequently, for every $\psi \in L^2(\mathbb{R})$, $\text{p.v.}\left(\frac{1}{x}\right) * \psi \in \mathcal{K}$ and

$$\mathcal{F}\left(\text{p.v.}\left(\frac{1}{x}\right) * \psi\right) = -i\pi\sigma\widehat{\psi}.$$

Since $\sigma\widehat{\psi} \in L^2(\mathbb{R})$, we must have $\text{p.v.}\left(\frac{1}{x}\right) * \psi \in L^2(\mathbb{R})$. On the other hand, (6.3) implies $\text{p.v.}\left(\frac{1}{x}\right) * \text{p.v.}\left(\frac{1}{x}\right) \in \mathcal{K}$ and so

$$\mathcal{F}\left(\text{p.v.}\left(\frac{1}{x}\right) * \text{p.v.}\left(\frac{1}{x}\right)\right) = -\pi^2.$$

Hence

$$\text{p.v.}\left(\frac{1}{x}\right) * \text{p.v.}\left(\frac{1}{x}\right) = -\pi^2\delta,$$

where $\delta = \frac{\varphi}{\varphi}$. Therefore

$$\text{p.v.}\left(\frac{1}{x}\right) * \text{p.v.}\left(\frac{1}{x}\right) * \psi = -\pi^2\psi$$

for any $\psi \in L^2(\mathbb{R})$. This shows that the mapping $\psi \mapsto \frac{1}{\pi}\text{p.v.}\left(\frac{1}{x}\right) * \psi$ is an injection from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ and its inverse is given by the mapping

$\psi \mapsto -\frac{1}{\pi}\mathrm{p.v.}\left(\frac{1}{x}\right) * \psi$. The transformation $\psi \mapsto \frac{1}{\pi}\mathrm{p.v.}\left(\frac{1}{x}\right) * \psi$ is known as the Hilbert transform.

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