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SIMPLE PROOFS OF THE SIEGEL–TATUZAWA AND BRAUER–SIEGEL THEOREMS

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Dedicated to Danièle B.

Abstract. We give a simple proof of the Siegel–Tatuzawa theorem according to which the residues at $s = 1$ of the Dedekind zeta functions of quadratic number fields are effectively not too small, with at most one exceptional quadratic field. We then give a simple proof of the Brauer–Siegel theorem for normal number fields which gives the asymptotics for the logarithm of the product of the class number and the regulator of number fields.

1. Introduction. Let k be a quadratic number field. Let d_k, h_k, w_k and $\varepsilon_k > 1$ denote the absolute value of its discriminant, its class number, its number of complex roots of unity (hence, $w_k = 2$ unless $k = \mathbb{Q}(\sqrt{-1})$ or $k = \mathbb{Q}(\sqrt{-3})$) and its fundamental unit (in the case that k is real). Finally, let κ_k denote the residue at the simple pole $s = 1$ of the Dedekind zeta function $\zeta_k(s)$ of k. The analytic class number formula yields (e.g., see [Lan, Chapter XIII, Theorem 2, p. 259]):

$$h_k = \begin{cases} \dfrac{w_k\sqrt{d_k}}{2\pi}\,\kappa_k & \text{if } k \text{ is imaginary,} \\ \dfrac{\sqrt{d_k}}{2\log\varepsilon_k}\,\kappa_k & \text{if } k \text{ is real.} \end{cases}$$

Since $2 \le w_k \le 6$ and $\varepsilon_k \ge (1+\sqrt{5})/2$, it follows that

$$\kappa_k \gg \frac{1}{d_k^{1/2}}$$

(effectively). Siegel's theorem asserts that for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for any quadratic number field k we have

$$\kappa_k \ge \frac{c_\varepsilon}{d_k^\varepsilon}$$

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(see [Chow], [Dav, Chapter 21], [Est], [Gold], [PinVIII, Theorem 3], [Sie]). Notice that if χ_k denotes the primitive (modulo d_k) quadratic Dirichlet character associated with k, then $\kappa_k = L(1,\chi_k)$. The main drawback of Siegel's theorem is that c_ε is not effective, i.e. for a given $\varepsilon \in (0,1/2)$ we cannot compute any suitable value for c_ε. However, a nice consequence of Siegel's theorem is the asymptotic behavior $\log h_k \sim \frac{1}{2}\log d_k$ for the class number h_k, when k ranges over the imaginary quadratic number fields. In particular, there are only finitely many imaginary quadratic number fields of a given class number, or with one class in each genus.

Because of the ineffectiveness of Siegel's theorem, for solving small class number problems for imaginary quadratic number fields, we must resort to more complicated tools (e.g., see [Wat] for the determination of all the imaginary quadratic number fields of class numbers $h_k \le 100$). Notice that these tools have not yet enabled any one to solve the "one class in each genus" problem for imaginary quadratic number fields. The Siegel–Tatuzawa theorem is a somewhat stronger result according to which for any $\varepsilon > 0$ and any quadratic number field k we have

$$\kappa_k \ge 0.1\,\frac{\varepsilon}{d_k^\varepsilon}$$

with at most one exception (see [Tat]). There are now several different proofs of the Siegel–Tatuzawa theorem (see [Hof], [JL], [PinIV, Theorem 4], [PinVIII, Theorem 6]). The aim of this paper is to give a simpler proof of the following form of the Siegel–Tatuzawa theorem (our proof is not more complicated than that of Siegel's theorem given in [Dav, Chapter 21]):

THEOREM 1. *For any $\varepsilon \in (0,1/2]$ and for any quadratic number field k we have*

$$\kappa_k \ge \frac{\varepsilon}{276(\log d_k)^2 d_k^\varepsilon} \tag{1}$$

with at most one exception.

We have not tried to obtain the best possible right hand term in (1). For example, using [Lou06, Theorem 1] (instead of Proposition 4) and Remark 8 (instead of Proposition 3), we would obtain a better result.

2. Prerequisites. To begin with, we will need a lower bound for κ_K in the case that K is abelian.

LEMMA 2. *Let χ be a non-trivial Dirichlet character modulo $f > 1$. Then, for $|s-a| \le \varrho$ with $0 < a - \varrho < 1 < a$, we have*

$$|L(s,\chi)| \le \frac{af}{2\sqrt{a^2-\varrho^2}}.$$

In particular, for $|s-2| \le 4/3$, we have $|L(s,\chi)| \le f$.

Proof. Set

$$\chi_1(u) = \sum_{1\le n\le u} \chi(n).$$

Then $\chi_1(f) = 0$ and $n \mapsto \chi_1(n)$ is f-periodic. Hence, $|\chi_1(n)| \le f/2$ and

$$L(s,\chi) = s\sum_{n\ge 1}\chi_1(n)\int_n^{n+1}\frac{du}{u^{s+1}},$$

which is valid for $\sigma = \Re(s) > 0$, yield $|L(s,\chi)| \le f|s|/2\sigma$. ∎

PROPOSITION 3. *Let K be an abelian number field. If $26/27 \le \beta < 1$ and $\zeta_K(\beta) \le 0$, then*

$$\kappa_K \ge \frac{1}{4}(1-\beta)d_K^{4(\beta-1)}.$$

Proof. Use [Was, Lemma 11.7] with $f(s) = \zeta_K(s)/\zeta(s) = \prod_{1\ne\chi\in X_K} L(s,\chi)$ and $M = \sup_{|s-2|\le 4/3}|f(s)|$ (where X_K denotes the group of primitive Dirichlet characters associated with K). By Lemma 2 and the conductor-discriminant formula, we obtain $M \le d_K$. ∎

Moreover, we will need upper bounds on $L(1,\chi)$.

PROPOSITION 4. *Let χ be a non-principal character modulo f (hence, $f \ge 3$). Then*

$$|L(1,\chi)| \le 2\log f.$$

Moreover, if $\beta \in (1-\varepsilon, 1) \subseteq [1/\log 3, 1)$ and $L(\beta,\chi) = 0$, then

$$|L(1,\chi)| \le c_\varepsilon(1-\beta)(\log f)f^\varepsilon/\varepsilon,$$

where $c_\varepsilon = 1 + \varepsilon/2$.

Proof. We have

$$L(s,\chi) = \sum_{n=1}^{f-1}\frac{\chi(n)}{n^s} + s\int_f^\infty \chi_1(u)u^{-s-1}\,du \tag{2}$$

$(\chi_1(f) = 0)$. Hence,

$$|L(1,\chi)| \le \sum_{n=1}^{f-1}\frac{1}{n} + \frac{f}{2}\int_f^\infty u^{-2}\,du = \sum_{n=1}^{f-1}\frac{1}{n} + \frac{1}{2} \le 2\log f$$

$(f \ge 3)$. In the same way, if $L(\beta,\chi) = 0$, then

$$|L(1,\chi)| = |L(1,\chi) - L(\beta,\chi)| \le (1-\beta)\sup_{\beta\le s\le 1}|L'(s,\chi)|.$$

Now (see also [Was, Lemma 11.9]), for $1-\varepsilon \le s \le 1$, we have

$$L'(s,\chi) = -\sum_{n=1}^{f-1}\chi(n)\frac{\log n}{n^s} + \int_f^\infty \chi_1(u)(1 - s\log u)u^{-s-1}\,du$$

(differentiate (2)). Hence,

$$|L'(s,\chi)| \leq \sum_{n=2}^{f-1} \frac{\log n}{n^{1-\varepsilon}} + \frac{f}{2} \int_f^\infty (s\log u - 1)u^{-s-1}\,du$$

($s\log u - 1 \geq (1-\varepsilon)\log f - 1 \geq (1-\varepsilon)\log 3 - 1 \geq 0$). Therefore,

$$|L'(s,\chi)| \leq (\log f)\int_1^f \frac{du}{u^{1-\varepsilon}} + \frac{f}{2}\int_f^\infty (s\log u - 1)u^{-s-1}\,du$$

and

$$|L'(s,\chi)| \leq \frac{1}{\varepsilon}(\log f)f^{\varepsilon} + \frac{1}{2}(\log f)f^{1-s} \leq \left(\frac{1}{\varepsilon} + \frac{1}{2}\right)(\log f)f^{\varepsilon}.$$

This proves the proposition. ■

3. The simple proof. We are now in a position to prove Theorem 1. Fix $\varepsilon \in (0, 1/34]$.

1. Assume that $\zeta_k(1-\varepsilon) \leq 0$ for a quadratic number field k. Then

$$\kappa_k \geq \frac{\varepsilon}{4d_k^{4\varepsilon}}, \tag{3}$$

by Proposition 3.

2. Assume that $\zeta_k(1-\varepsilon) > 0$ for some quadratic number field k. Let k_0 be such a quadratic number field of least discriminant d_{k_0}. Since $\zeta_{k_0}(s)$ has a simple pole of positive residue at $s = 1$ (by the analytic class number formula), we have $\lim_{s\to 1^-}\zeta_{k_0}(s) = -\infty$ and there exists $\beta \in (1-\varepsilon, 1)$ such that $\zeta_{k_0}(\beta) = 0$. Let $k \neq k_0$ be any other quadratic number field. We may assume that $d_k \geq d_{k_0}$, otherwise (3) holds true. Let $K = kk_0$. Then K is a biquadratic bicyclic number field. Let k' denote its third quadratic subfield. Then $d_{k'}$ divides $d_k d_{k_0}$, hence $d_{k'} \leq d_k^2$, and $d_K = d_k d_{k_0} d_{k'}$ divides $(d_k d_{k_0})^2$, hence $d_K \leq d_k^4$. Now, $\zeta_K(s)\zeta^2(s) = \zeta_k(s)\zeta_{k_0}(s)\zeta_{k'}(s)$. It follows that $\zeta_K(\beta) = 0$ and

$$\kappa_K \geq \frac{1}{4}(1-\beta)d_K^{4(\beta-1)} \geq \frac{1}{4}(1-\beta)d_K^{-4\varepsilon} \geq \frac{1}{4}(1-\beta)d_k^{-16\varepsilon},$$

by Proposition 3. On the other hand, we also have

$$\begin{aligned}\kappa_K = \kappa_k\kappa_{k_0}\kappa_{k'} &\leq 2c_\varepsilon(1-\beta)(\log d_{k_0})d_{k_0}^{\varepsilon}(\log d_{k'})\kappa_k/\varepsilon \\ &\leq 4c_\varepsilon(1-\beta)(\log d_k)^2 d_k^{\varepsilon}\kappa_k/\varepsilon,\end{aligned}$$

by Proposition 4. Hence,

$$\kappa_k \geq \frac{\varepsilon}{16c_\varepsilon(\log d_k)^2 d_k^{17\varepsilon}} \geq \frac{\varepsilon}{(16+4/17)(\log d_k)^2 d_k^{17\varepsilon}}. \tag{4}$$

Consequently, for any $k \neq k_0$ we have $\kappa_k \geq \min((3),(4)) = (4)$, and the desired result follows by changing ε into $\varepsilon/17$.

4. The Brauer–Siegel theorem. We now want to deduce a simple proof of the Brauer–Siegel theorem for normal number fields (see [Lan, Chapter XVI], and [Nar, Theorem 8.14] for the case of abelian number fields; we also refer the reader to [Lou05] for more explicit and numerically better forms of Theorem 5, as well as for a generalization to the case that K ranges over a family of non-normal number fields of a given degree):

THEOREM 5. *Let K range over an infinite family of normal number fields for which $\varrho_K \to \infty$, ϱ_K being the root discriminant of K. Then $\log(h_K \mathrm{Reg}_K)$ is asymptotic to $\frac{1}{2}\log d_K$ (and this asymptotics is explicit if no K contains a quadratic subfield).*

We need upper bounds for κ_K (generalizing Proposition 4):

PROPOSITION 6 (see [Lou01, Theorem 1]). *Let K be a number field of degree $n > 1$. Then*

$$\kappa_K \le \left(\frac{e\log d_K}{2(n-1)}\right)^{n-1}.$$

Moreover, if $\zeta_K(\beta) = 0$ for some $\beta \in (0,1)$ then

$$\kappa_K \le (1-\beta)\left(\frac{e\log d_K}{2n}\right)^{n}.$$

We also need a lower bound for κ_K (generalizing Proposition 3):

PROPOSITION 7. *There exists $C_n > 0$ with $\log C_n = O(n)$ such that for any number field K of degree $n > 1$ we have*

$$\kappa_K \ge C_n(1-\beta)d_K^{(\beta-1)/2},$$

provided that $\beta \in [3/4, 1)$ and $\zeta_K(\beta) \le 0$.

REMARK 8. We may take

$$C_n = \frac{3}{4}\cdot 2^{-n}e^{-4\pi n},$$

by [Lan, Chapter XVI, Lemma 3, p. 323] (see also [Nar, Lemma 8.15]). For number fields of root discriminants $\varrho_K = d_K^{1/n} \ge 51000$, we may take $C_n = 1/4$ independent of n, by [Lou05]. Indeed, using the lower bound $f_K(\beta) \ge 1$ and the upper bound

$$\frac{2\beta d_K^{(1-\beta)/2}}{2\beta - 1} \le \frac{\frac{3}{2}d_K^{(1-\beta)/2}}{\frac{3}{2}-1} \le 3d_K^{1/8}$$

from [Lou05, (25)], we obtain $\kappa_K \ge C_K(1-\beta)d_K^{(\beta-1)/2}$ with

$$\begin{aligned} C_K &= 1 - 3d_K^{1/8}(A/\varrho_K)^{n/4}J_n = 1 - 3(A^2/\varrho_K)^{n/8}J_n \\ &\ge 1 - \frac{3}{2}(A^2/\varrho_K)^{n/8} \ge 1/4 \end{aligned}$$

for $n \ge 2$ and $\varrho_K \ge 51000$ (for $J_n \le J_2 = 1/2$ and $A = \Gamma^4(1/4)/\pi < 56$).

Proposition 9 (see [Sta, Lemma 3], [Hof, Lemma 2] or [LLO, Lemma 15]). *Set* $c = 2(\sqrt{2}-1)^2 = 0.343\ldots$. *Then* $\zeta_K(s)$ *has at most one simple real zero in the range* $1 - c/\log d_K \le s < 1$. *Moreover* (*see* [Sta, Theorem 3]), *if* K *is a normal number field, then such a simple real zero is a zero of* $\zeta_k(s)$ *for some quadratic subfield* k *of* K.

Using Propositions 6 and 7, Siegel's theorem (a consequence of the Siegel–Tatuzawa theorem) and Proposition 9, we are in a position to prove Theorem 5. To begin with, the analytic class number formula yields (see [Lou05, (7)])

$$\frac{\sqrt{d_K}}{3^n}\,\kappa_K \le h_K \mathrm{Reg}_K \le \frac{4\sqrt{d_K}}{\pi}\,\kappa_K. \tag{5}$$

We also note that $n/\log d_K = 1/\log \varrho_K$ tends to zero as ϱ_K tends to infinity. Now, fix $\varepsilon > 0$ (and notice that for $d_K \ge 4$ we have $1 - c/\log d_K \ge 3/4$).

1. For obtaining an upper bound on κ_K we use Proposition 6 to obtain

$$\kappa_K \le \left(\frac{en\log\varrho_K}{2(n-1)}\right)^{n-1} \le \left(\frac{en}{2(n-1)}\right)^{n-1}\varrho_K^{n\varepsilon} \le 2(e/2)^n d_K^{\varepsilon}, \tag{6}$$

provided that $\varrho_K \ge 1$ is large enough to have $\log\log\varrho_K \le \varepsilon\log\varrho_K$.

2. For obtaining a lower bound on κ_K there are two cases to consider.

(a) If $\zeta_K(1 - c/\log d_K) > 0$, then $\zeta_K(s)$ has a real zero β in the range $1 - c/\log d_K < s < 1$ and, by Proposition 9, K contains a quadratic subfield k for which $\zeta_k(\beta) = \zeta_K(\beta) = 0$. Then $d_K \ge d_k$,

$$\kappa_K \ge C_n(1-\beta)d_K^{(\beta-1)/2} \ge C_n(1-\beta)e^{-c/2}$$

and

$$\kappa_k \le (1-\beta)\left(\frac{e\log d_k}{4}\right)^2 \le (1-\beta)\left(\frac{e\log d_K}{4}\right)^2,$$

by Propositions 7 and 6, and

$$\kappa_K/\kappa_k \ge \frac{16C_n e^{-c/2}}{e^2\log^2 d_K}.$$

Moreover, $\kappa_k \ge c_\varepsilon d_k^{-\varepsilon} \ge c_\varepsilon d_K^{-\varepsilon}$, by Siegel's theorem. Hence,

$$\kappa_K \ge \frac{16C_n c_\varepsilon e^{-c/2}}{e^2(\log d_K)^2 d_K^{\varepsilon}} \ge c'_\varepsilon d_K^{-2\varepsilon}. \tag{7}$$

(b) Otherwise (in particular if K contains no quadratic subfield), we have $\zeta_K(1 - c/\log d_K) \le 0$ and

$$\kappa_K \ge \frac{C_n c e^{-c/2}}{\log d_K} \ge c''_\varepsilon d_K^{-2\varepsilon}, \tag{8}$$

by Proposition 7.

Using (5)–(8) completes the proof.

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