SIMPLE PROOFS OF THE SIEGEL–TATUZA
AND BRAUER–SIEGEL THEOREMS

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Dedicated to Danièle B.

Abstract. We give a simple proof of the Siegel–Tatuzawa theorem according to which
the residues at $s = 1$ of the Dedekind zeta functions of quadratic number fields are
effectively not too small, with at most one exceptional quadratic field. We then give
a simple proof of the Brauer–Siegel theorem for normal number fields which gives the
asymptotics for the logarithm of the product of the class number and the regulator of
number fields.

1. Introduction. Let $k$ be a quadratic number field. Let $d_k$, $h_k$, $w_k$ and $\varepsilon_k > 1$ denote
the absolute value of its discriminant, its class number, its number of complex roots of
unity (hence, $w_k = 2$ unless $k = \mathbb{Q}(\sqrt{-1})$ or $k = \mathbb{Q}(\sqrt{-3})$) and its fundamental
unit (in the case that $k$ is real). Finally, let $\kappa_k$ denote the residue at the simple pole $s = 1$ of
the Dedekind zeta function $\zeta_k(s)$ of $k$. The analytic class number formula yields (e.g., see [Lan,
Chapter XIII, Theorem 2, p. 259]):

\[
h_k = \begin{cases} 
\frac{w_k \sqrt{d_k}}{2\pi} \kappa_k & \text{if } k \text{ is imaginary}, \\
\frac{\sqrt{d_k}}{2 \log \varepsilon_k} \kappa_k & \text{if } k \text{ is real}.
\end{cases}
\]

Since $2 \leq w_k \leq 6$ and $\varepsilon_k \geq (1 + \sqrt{5})/2$, it follows that

\[\kappa_k \gg \frac{1}{d_k^{1/2}}\]

(effectively). Siegel’s theorem asserts that for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that
for any quadratic number field $k$ we have

\[\kappa_k \geq \frac{c_\varepsilon}{d_k^\varepsilon}\]

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(see [Chow], [Dav, Chapter 21], [Est], [Gold], [PinVIII, Theorem 3], [Sie]). Notice that if \( \chi_k \) denotes the primitive (modulo \( d_k \)) quadratic Dirichlet character associated with \( k \), then \( \kappa_k = L(1, \chi_k) \). The main drawback of Siegel’s theorem is that \( c_\varepsilon \) is not effective, i.e. for a given \( \varepsilon \in (0, 1/2) \) we cannot compute any suitable value for \( c_\varepsilon \). However, a nice consequence of Siegel’s theorem is the asymptotic behavior \( \log h_k \sim \frac{1}{2} \log d_k \) for the class number \( h_k \), when \( k \) ranges over the imaginary quadratic number fields. In particular, there are only finitely many imaginary quadratic number fields of a given class number, or with one class in each genus.

Because of the ineffectiveness of Siegel’s theorem, for solving small class number problems for imaginary quadratic number fields, we must resort to more complicated tools (e.g., see [Wat] for the determination of all the imaginary quadratic number fields of class numbers \( h_k \leq 100 \)). Notice that these tools have not yet enabled any one to solve the “one class in each genus” problem for imaginary quadratic number fields. The Siegel–Tatuzawa theorem is a somewhat stronger result according to which for any \( \varepsilon > 0 \) and any quadratic number field \( k \) we have

\[
\kappa_k \geq 0.1 \frac{\varepsilon}{d_k^{1/2}}
\]

with at most one exception (see [Tat]). There are now several different proofs of the Siegel–Tatuzawa theorem (see [Hof], [JL], [PinIV, Theorem 4], [PinVIII, Theorem 6]). The aim of this paper is to give a simpler proof of the following form of the Siegel–Tatuzawa theorem (our proof is not more complicated than that of Siegel’s theorem given in [Dav, Chapter 21]):

**Theorem 1.** For any \( \varepsilon \in (0, 1/2] \) and for any quadratic number field \( k \) we have

\[
\kappa_k \geq \frac{\varepsilon}{276(\log d_k)^2 d_k^{1/2}}
\]

with at most one exception.

We have not tried to obtain the best possible right hand term in (1). For example, using [Lou06, Theorem 1] (instead of Proposition 4) and Remark 8 (instead of Proposition 3), we would obtain a better result.

**2. Prerequisites.** To begin with, we will need a lower bound for \( \kappa_K \) in the case that \( K \) is abelian.

**Lemma 2.** Let \( \chi \) be a non-trivial Dirichlet character modulo \( f > 1 \). Then, for \( |s - a| \leq \varrho \) with \( 0 < a - \varrho < 1 < a \), we have

\[
|L(s, \chi)| \leq \frac{af}{2\sqrt{a^2 - \varrho^2}}.
\]

In particular, for \( |s - 2| \leq 4/3 \), we have \( |L(s, \chi)| \leq f \).
Proof. Set 
\[ \chi_1(u) = \sum_{1 \leq n \leq u} \chi(n). \]
Then \( \chi_1(f) = 0 \) and \( n \mapsto \chi_1(n) \) is \( f \)-periodic. Hence, \( |\chi_1(n)| \leq f/2 \) and 
\[ L(s, \chi) = s \sum_{n \geq 1} \chi_1(n) \int_n^{n+1} \frac{du}{u^{s+1}}, \]
which is valid for \( \sigma = \Re(s) > 0 \), yield \( |L(s, \chi)| \leq f|s|/2\sigma. \)

**Proposition 3.** Let \( K \) be an abelian number field. If \( 26/27 \leq \beta < 1 \) and \( \zeta_K(\beta) \leq 0 \), then
\[ \kappa_K \geq \frac{1}{4} \left( 1 - \beta \right) d_K^{4(\beta-1)}. \]

Proof. Use [Was, Lemma 11.7] with \( f(s) = \zeta_K(s)/\zeta(s) = \prod_{1 \neq \chi \in X_K} L(s, \chi) \) and \( M = \sup_{|s-2| \leq 1/3} |f(s)| \) (where \( X_K \) denotes the group of primitive Dirichlet characters associated with \( K \)). By Lemma 2 and the conductor-discriminant formula, we obtain \( M \leq d_K. \)

Moreover, we will need upper bounds on \( L(1, \chi) \).

**Proposition 4.** Let \( \chi \) be a non-principal character modulo \( f \) (hence, \( f \geq 3 \)). Then
\[ |L(1, \chi)| \leq 2 \log f. \]
Moreover, if \( \beta \in (1 - \varepsilon, 1) \subseteq [1/\log 3, 1) \) and \( L(\beta, \chi) = 0 \), then
\[ |L(1, \chi)| \leq c_\varepsilon (1 - \beta)(\log f)f^{\varepsilon}/\varepsilon, \]
where \( c_\varepsilon = 1 + \varepsilon/2. \)

Proof. We have
\[ L(s, \chi) = \sum_{n=1}^{f-1} \frac{\chi(n)}{n^s} + s \int_f^{\infty} \chi_1(u)u^{-s-1} du \]
\((\chi_1(f) = 0)\). Hence,
\[ |L(1, \chi)| \leq \sum_{n=1}^{f-1} \frac{1}{n} + \frac{f}{2} \int_f^{\infty} u^{-2} du = \sum_{n=1}^{f-1} \frac{1}{n} + \frac{1}{2} \leq 2 \log f \]
\((f \geq 3)\). In the same way, if \( L(\beta, \chi) = 0 \), then
\[ |L(1, \chi)| = |L(1, \chi) - L(\beta, \chi)| \leq (1 - \beta) \sup_{\beta \leq s \leq 1} |L'(s, \chi)|. \]
Now (see also [Was, Lemma 11.9]), for \( 1 - \varepsilon \leq s \leq 1 \), we have
\[ L'(s, \chi) = -\sum_{n=1}^{f-1} \chi(n) \frac{\log n}{n^s} + \int_f^{\infty} \chi_1(u)(1 - s \log u)u^{-s-1} du \]
(differentiate (2)). Hence,

\[
|L'(s, \chi)| \leq \sum_{n=2}^{f-1} \frac{\log n}{n^{1-\varepsilon}} + \frac{f}{2} \int_{1}^{\infty} (s \log u - 1)u^{-s-1} \, du
\]

\[
(s \log u - 1 \geq (1 - \varepsilon) \log f - 1 \geq (1 - \varepsilon) \log 3 - 1 \geq 0).
\]

Therefore,

\[
|L'(s, \chi)| \leq \int_{1}^{f} \frac{du}{u^{1-\varepsilon}} + \frac{f}{2} \int_{1}^{\infty} (s \log u - 1)u^{-s-1} \, du
\]

and

\[
|L'(s, \chi)| \leq \frac{1}{\varepsilon} (\log f)^{f^{\varepsilon}} + \frac{1}{2} (\log f) f^{1-s} \leq \left(\frac{1}{\varepsilon} + \frac{1}{2}\right) (\log f)^{f^{\varepsilon}}.
\]

This proves the proposition. ■

3. The simple proof. We are now in a position to prove Theorem 1.

Fix \( \varepsilon \in (0, 1/34] \).

1. Assume that \( \zeta_k(1 - \varepsilon) \leq 0 \) for a quadratic number field \( k \). Then

\[
\kappa_k \geq \frac{\varepsilon}{4d_k^{16\varepsilon}};
\]

by Proposition 3.

2. Assume that \( \zeta_k(1 - \varepsilon) > 0 \) for some quadratic number field \( k \). Let \( k_0 \) be such a quadratic number field of least discriminant \( d_{k_0} \). Since \( \zeta_{k_0}(s) \) has a simple pole of positive residue at \( s = 1 \) (by the analytic class number formula), we have \( \lim_{s \to 1^-} \zeta_{k_0}(s) = -\infty \) and there exists \( \beta \in (1 - \varepsilon, 1) \) such that \( \zeta_{k_0} (\beta) = 0 \). Let \( k \neq k_0 \) be any other quadratic number field. We may assume that \( d_k \geq d_{k_0} \), otherwise (3) holds true. Let \( K = kk_0 \). Then \( K \) is a biquadratic bicyclic number field. Let \( k' \) denote its third quadratic subfield. Then \( d_k \) divides \( d_k d_{k_0} \), hence \( d_{k'} \leq d_k \), and \( d_K = d_k d_{k_0} d_{k'} \) divides \( (d_k d_{k_0})^2 \), hence \( d_K \leq d_k^2 \). Now, \( \zeta_K(s) = \zeta_k(s) \zeta_{k_0}(s) \zeta_{k'}(s) \). It follows that \( \zeta_K(\beta) = 0 \) and

\[
\kappa_K \geq \frac{1}{4} (1 - \beta)d_K^{A(\beta-1)} \geq \frac{1}{4} (1 - \beta)d_K^{-4\varepsilon} \geq \frac{1}{4} (1 - \beta)d_k^{-16\varepsilon},
\]

by Proposition 3. On the other hand, we also have

\[
\kappa_K = \kappa_k \kappa_{k_0} \kappa_{k'} \leq 2c_{\varepsilon}(1 - \beta)(\log d_{k_0}) d_k^{\varepsilon} (\log d_{k'}) \kappa_k / \varepsilon \\
\leq 4c_{\varepsilon}(1 - \beta)(\log d_k)^2 d_k^{\varepsilon} \kappa_k / \varepsilon,
\]

by Proposition 4. Hence,

\[
\kappa_k \geq \frac{\varepsilon}{16c_{\varepsilon}(\log d_k)^2 d_k^{17\varepsilon}} \geq \frac{\varepsilon}{(16 + 4/17)(\log d_k)^2 d_k^{17\varepsilon}}.
\]

Consequently, for any \( k \neq k_0 \) we have \( \kappa_k \geq \min((3), (4)) = (4) \), and the desired result follows by changing \( \varepsilon \) into \( \varepsilon/17 \).
4. The Brauer–Siegel theorem. We now want to deduce a simple proof of the Brauer–Siegel theorem for normal number fields (see [Lan, Chapter XVI], and [Nar, Theorem 8.14] for the case of abelian number fields; we also refer the reader to [Lou05] for more explicit and numerically better forms of Theorem 5, as well as for a generalization to the case that $K$ ranges over a family of non-normal number fields of a given degree):

**Theorem 5.** Let $K$ range over an infinite family of normal number fields for which $\rho_K \to \infty$, $\rho_K$ being the root discriminant of $K$. Then

$$\log(h_K \text{Reg}_K)$$

is asymptotic to $\frac{1}{2} \log d_K$ (and this asymptotics is explicit if no $K$ contains a quadratic subfield).

We need upper bounds for $\kappa_K$ (generalizing Proposition 4):

**Proposition 6** (see [Lou01, Theorem 1]). Let $K$ be a number field of degree $n > 1$. Then

$$\kappa_K \leq \left( e \frac{\log d_K}{2(n-1)} \right)^{n-1}.$$ 

Moreover, if $\zeta_K(\beta) = 0$ for some $\beta \in (0,1)$ then

$$\kappa_K \leq (1-\beta) \left( e \frac{\log d_K}{2n} \right)^n.$$ 

We also need a lower bound for $\kappa_K$ (generalizing Proposition 3):

**Proposition 7.** There exists $C_n > 0$ with $\log C_n = O(n)$ such that for any number field $K$ of degree $n > 1$ we have

$$\kappa_K \geq C_n (1-\beta) d_K^{(\beta-1)/2},$$

provided that $\beta \in [3/4,1)$ and $\zeta_K(\beta) \leq 0$.

**Remark 8.** We may take

$$C_n = \frac{3}{4} \cdot 2^{-n} e^{-4\pi n},$$

by [Lan, Chapter XVI, Lemma 3, p. 323] (see also [Nar, Lemma 8.15]). For number fields of root discriminants $\rho_K = d_K^{1/n} \geq 51000$, we may take $C_n = 1/4$ independent of $n$, by [Lou05]. Indeed, using the lower bound $f_K(\beta) \geq 1$ and the upper bound

$$\frac{2\beta d_K^{(1-\beta)/2}}{2\beta - 1} \leq \frac{3}{2} d_K^{(1-\beta)/2} \leq 3d_K^{1/8}$$

from [Lou05, (25)], we obtain $\kappa_K \geq C_K (1-\beta) d_K^{(\beta-1)/2}$ with

$$C_K = 1 - 3d_K^{1/8} (A/\rho_K)^{n/4} J_n = 1 - 3(A^2/\rho_K)^{n/8} J_n$$

$$\geq 1 - \frac{3}{2} (A^2/\rho_K)^{n/8} \geq 1/4$$

for $n \geq 2$ and $\rho_K \geq 51000$ (for $J_n \leq J_2 = 1/2$ and $A = \Gamma^4(1/4)/\pi < 56$).
PROPOSITION 9 (see [Sta, Lemma 3], [Hof, Lemma 2] or [LLO, Lemma 15]). Set \( c = 2(\sqrt{2} - 1)^2 = 0.343 \ldots \). Then \( \zeta_K(s) \) has at most one simple real zero in the range \( 1 - c/\log d_K \leq s < 1 \). Moreover (see [Sta, Theorem 3]), if \( K \) is a normal number field, then such a simple real zero is a zero of \( \zeta_k(s) \) for some quadratic subfield \( k \) of \( K \).

Using Propositions 6 and 7, Siegel’s theorem (a consequence of the Siegel–Tatuzawa theorem) and Proposition 9, we are in a position to prove Theorem 5. To begin with, the analytic class number formula yields (see [Lou05, (7)]

\[
\frac{\sqrt{d_K}}{3^n} \kappa_K \leq h_K \text{Reg}_K \leq \frac{4\sqrt{d_K}}{\pi} \kappa_K.
\]

We also note that \( n/\log d_K = 1/\log \varrho_K \) tends to zero as \( \varrho_K \) tends to infinity.

Now, fix \( \varepsilon > 0 \) (and notice that for \( d_K \geq 4 \) we have \( 1 - c/\log d_K \geq 3/4 \)).

1. For obtaining an upper bound on \( \kappa_K \) we use Proposition 6 to obtain

\[
\kappa_K \leq \left( \frac{e n \log \varrho_K}{2(n - 1)} \right)^{n-1} \leq \left( \frac{e n}{2(n - 1)} \right)^{n-1} \varrho_K^{n \varepsilon} \leq 2(e/2)^n d_K^\varepsilon,
\]

provided that \( \varrho_K \geq 1 \) is large enough to have \( \log \log \varrho_K \leq \varepsilon \log \varrho_K \).

2. For obtaining a lower bound on \( \kappa_K \) there are two cases to consider.

(a) If \( \zeta_K(1 - c/\log d_K) > 0 \), then \( \zeta_K(s) \) has a real zero \( \beta \) in the range \( 1 - c/\log d_K < s < 1 \) and, by Proposition 9, \( K \) contains a quadratic subfield \( k \) for which \( \zeta_k(\beta) = \zeta_K(\beta) = 0 \). Then \( d_K \geq d_k \),

\[
\kappa_K \geq C_n (1 - \beta) d_K^{(\beta - 1)/2} \geq C_n (1 - \beta) e^{-c/2}
\]

and

\[
\kappa_k \leq (1 - \beta) \left( \frac{e \log d_k}{4} \right)^2 \leq (1 - \beta) \left( \frac{e \log d_K}{4} \right)^2,
\]

by Propositions 7 and 6, and

\[
\frac{\kappa_K}{\kappa_k} \geq \frac{16 C_n e^{-c/2}}{e^2 \log^2 d_K}.
\]

Moreover, \( \kappa_k \geq c^{-\varepsilon} d_k^{-\varepsilon} \geq c^{-\varepsilon} d_K^{-\varepsilon} \), by Siegel’s theorem. Hence,

\[
\kappa_K \geq \frac{16 C_n c^{-\varepsilon} e^{-c/2}}{e^2 (\log d_K)^2 d_K^{2\varepsilon}} \geq c^{-\varepsilon} d_K^{-2\varepsilon}.
\]

(b) Otherwise (in particular if \( K \) contains no quadratic subfield), we have \( \zeta_K(1 - c/\log d_K) \leq 0 \) and

\[
\kappa_K \geq \frac{C_n c^{-\varepsilon} e^{-c/2}}{\log d_K} \geq c^{-\varepsilon} d_K^{-2\varepsilon},
\]

by Proposition 7.

Using (5)–(8) completes the proof.
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