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BAND COMBINATORICS OF DOMESTIC STRING ALGEBRAS

BY

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Abstract. We prove that the multiplicity of a simple module as a composition factor in a composition series for a primitive band module over a domestic string algebra is at most two.

1. Introduction. There is a range of results on string and band combinatorics of finite-dimensional algebras. First of all, they played a pivotal role in the classification of finite-dimensional modules over string algebras in Butler and Ringel [1]. They were also used by Geiß [2] to describe certain components in the Auslander–Reiten quiver of a string algebra, and by Ringel [9] to construct new types of generic modules over non-domestic string algebras.

Band combinatorics for string algebras has been developed by Ringel [7, 8] (see also [10]). For instance, he gave a purely combinatorial characterization of domesticity of string algebras. Namely, he proved that a string algebra A is domestic iff for every arrow α, there is at most one band of A with α as a first letter. Another result of Ringel [8, Sect. 11, Cor. 1] says that two primitive cycles of a domestic string algebra with a vertex in a common socle are equivalent.

These results have recently acquired further applications. For instance, Schröer [11] uses the band combinatorics of string algebras to calculate radical series of the category of finite-dimensional modules. Furthermore, some recent progress in the classification of indecomposable pure-injective modules over string algebras (see [3, 4]) relies on string and band combinatorics, and this combinatorics is essential in Puninski's proof [5] of the existence of superdecomposable pure-injective modules over non-domestic string algebras.

The aim of this paper is to add one new result to this list, which is a crucial ingredient in the forthcoming proof of the finiteness of the Krull–Gabriel dimension of 1-domestic string algebras (see [6]).

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By Ringel [7, Sect. 11, Cor. 2], the number (up to equivalence) of primitive cycles over a domestic string algebra A does not exceed $|Q_0|$, the number of vertices of the underlying quiver of A. We prove (in Corollary 5.7) that the length of any primitive cycle over A does not exceed $2|Q_0|$. This is a consequence of the theorem (see Theorem 5.6) that every primitive cycle C over A (i.e. a primitive walk in the quiver of A) has at most double selfintersections. From the point of view of representation theory it can be reformulated as saying that the multiplicity of any simple module as a composition factor in a composition series for a primitive band module is at most two.

2. Strings. Let $Q = (Q_0, Q_1)$ be a *quiver*, that is, a finite oriented graph with vertices Q_0 and arrows Q_1, where both multiple arrows and loops are admitted. If $\alpha \in Q_1$ is an arrow, then $s(\alpha)$ will denote the source vertex of α, and $e(\alpha)$ the end vertex of α.

We say that the composition $\alpha\beta$ of arrows α and β is defined if $s(\alpha) = e(\beta)$, that is, if we can go through α after going through β:

$$\circ \xleftarrow{\ \alpha\ } \circ \xleftarrow{\ \beta\ } \circ$$

A *path* in Q is a word $u_1 \dots u_n$ consisting of direct arrows u_i such that all compositions $u_i u_{i+1}$, $1 \leq i \leq n-1$, are defined. For instance, every arrow is a path of length 1.

We will distinguish a finite number of (forbidden) paths in Q of length ≥ 2, which will be called *relations*. The set of relations should be large enough to guarantee that the number of "allowed" paths in Q, i.e., of paths not containing any relation as a subpath, is finite. Now, if $\Bbbk$ is a field, then the *path algebra* $\Bbbk Q$ has the set of all "allowed" paths in Q (including paths of length zero, i.e., vertices) as a $\Bbbk$-basis. In other words, if u is a path, then $u = 0$ in $\Bbbk Q$ if u belongs to the ideal generated by the relations.

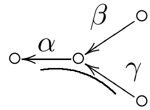
A path algebra $\Bbbk Q$ is a *string algebra* if:

1) For any vertex $S \in Q_0$ there are at most two arrows starting at S, and at most two arrows ending at S.
   Here is a typical configuration in Q without loops.

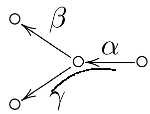
$$\begin{tikzcd} \circ \arrow[dr, "\alpha"'] & & \circ \arrow[dl, "\beta"'] \\ & \circ \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\ \circ & & \circ \end{tikzcd}$$

2) If α, β, γ are arrows such that $s(\alpha) = e(\beta) = e(\gamma)$, that is, if the compositions $\alpha\beta$ and $\alpha\gamma$ are defined, then either $\alpha\beta$ or $\alpha\gamma$ is a relation in Q. In the following diagram the relation $\alpha\gamma = 0$ is shown by a

solid curve:

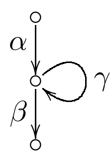


3) If α, β, γ are arrows such that $e(\alpha) = s(\beta) = s(\gamma)$, that is, if the compositions $\beta\alpha$ and $\gamma\alpha$ are defined, then either $\beta\alpha$ or $\gamma\alpha$ is a relation in Q.



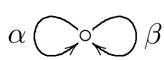
In the following we will not refer to the algebraic structure of a string algebra so often. So the reader may look at a string algebra as just its underlying quiver.

Note that, in the above definition, each loop at a vertex S counts both as an entering and as an outgoing arrow. Thus the following is a typical configuration in Q with one loop:



In particular, there exist at most two loops at each vertex S.

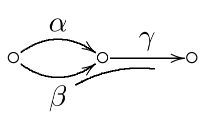
For instance, the quiver



with the relations $\alpha\beta = \beta\alpha = \alpha^2 = \beta^3 = 0$ is the quiver of the (finite-dimensional string) Gelfand–Ponomarev algebra $G_{2,3}$.

Let α be an arrow. We define the *source* of α^{-1}, $s(\alpha^{-1})$, as the vertex $e(\alpha)$; and the *end* of α^{-1}, $e(\alpha^{-1})$, as the vertex $s(\alpha)$. If u and v are direct or inverse arrows, then we say that the composition uv is defined if $s(u) = e(v)$.

For instance, let A be the string algebra



with the relation $\gamma\beta = 0$. Then the composition $\alpha\beta^{-1}$ is defined, but $\alpha\gamma^{-1}$ is not.

In what follows we consider words in the alphabet $Q_1 \cup Q_1^{-1}$, that is, the letters of this alphabet are (direct) arrows α and inverse arrows α^{-1}. We say that a word $u = u_1 \ldots u_n$ in this alphabet is a *string* if:

1) u is a path, i.e., each composition $u_i u_{i+1}$, $i = 1, \ldots, n-1$, is defined.
2) Neither $\alpha\alpha^{-1}$ nor $\alpha^{-1}\alpha$ is a subword of u (here and throughout, v is a *subword* of u if $u = u_1 v u_2$ for some, maybe empty, words u_1, u_2).
3) If $v = v_1 \ldots v_k$ is a relation in Q, then neither v nor $v^{-1} = v_k^{-1} \ldots v_1^{-1}$ is a subword of u.

For instance, $\alpha\beta^{-2}\alpha\beta^{-1}$ is a string over $G_{2,3}$, but $\alpha\beta\alpha^{-1}$ is not a string (since it contains the relation $\alpha\beta$ as a subword). Usually strings are interpreted as "generalized paths".

Clearly, if u is a string, then u^{-1} is also a string. Furthermore, u is not a cyclic permutation of u^{-1}, in particular $u \neq u^{-1}$.

3. Bands. Let A be a (finite-dimensional) string algebra. A string C over A is a *cycle* if:

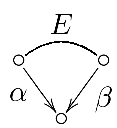
1) C contains a direct arrow and an inverse arrow.
2) The composition C^2 is a string (in particular, it is defined).

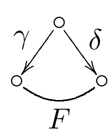
For instance, $\alpha\beta^{-1}$ and $\alpha\beta^{-2}$ are cycles over $G_{2,3}$, but β is not. Usually a cycle is interpreted as a closed path, i.e., a path starting and ending at the same vertex (or just a closed curve on Q, without specifying this vertex). For instance, in the example above, $\alpha\beta^{-1}$ is just the figure "eight" which traverses (the unique) vertex twice.

Note that, if C is a cycle, then every cyclic permutation of C is a cycle, and also C^k is a cycle for every k. Furthermore, C^{-1} is also a cycle.

We say that cycles C and D are *c-equivalent* (cyclically equivalent), written $C \sim_c D$, if C is a cyclic permutation of D. Clearly $\sim_c$ is an equivalence relation. We will distinguish between $\sim_c$ and the broader equivalence relation $\sim$, where $C \sim D$ if D can be obtained from C by a cyclic permutation and (maybe) taking the inverse. Thus $C \sim D$ iff $C \sim_c D$ or $C \sim_c D^{-1}$. Furthermore, since C and C^{-1} are not $\sim_c$-equivalent, each $\sim$-equivalence class splits in exactly two (equipotent) $\sim_c$-equivalence classes.

Clearly every cycle C is c-equivalent to a cycle $\alpha E\beta^{-1}$, where α and β are distinct arrows with a common end; and C is c-equivalent to a cycle $\gamma^{-1}F\delta$, where γ and δ are distinct arrows with a common source.





Note that, if $C = \alpha E\beta^{-1}$ is a string, where α, β are distinct arrows with a common end, then C^2 is also a string, hence C is a cycle. Similarly, if $D = \gamma^{-1}F\delta$ is a string, where γ, δ are distinct arrows with a common source, then D is a cycle.

A cycle C is called *primitive* if C is not a power of a proper subword, that is, $C = u^k$ for some word u implies $k = 1$. Clearly any cyclic permutation of a primitive cycle is a primitive cycle. A primitive cycle of the form $\alpha E \beta^{-1}$ will be called a *band*. For instance, $\alpha\beta^{-1}\alpha\beta^{-2}$ is a band over $G_{2,3}$.

If $B = \alpha E \beta^{-1}$ is a band, then β and α are distinct arrows with a common end, hence β is uniquely determined by α and vice versa. Furthermore, $B^{-1} = \beta E^{-1} \alpha^{-1}$ is also a band.

4. The domestic case. We will not reproduce the general definition of a finite-dimensional domestic (tame) algebra, but for string algebras there is a very useful equivalent combinatorial condition.

Fact 4.1 ([7, Sect. 11, Prop. 2]). *Let A be a finite-dimensional string algebra. Then A is domestic if and only if for every arrow α there is at most one band over A with the first letter α.*

For instance, the Gelfand–Ponomarev algebra $G_{2,3}$ is not domestic, since it has two distinct bands $\alpha\beta^{-1}$ and $\alpha\beta^{-2}$ with the first letter α.

Let A_2 be the Kronecker algebra, that is, the path algebra of the following quiver without relations:

$$\begin{tikzcd} \circ \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & \circ \end{tikzcd}$$

Then $C = \alpha\beta^{-1}$ and $C^{-1} = \beta\alpha^{-1}$ is a complete list of bands of A_2, hence A_2 is domestic (even 1-domestic).

A primitive cycle B is said to be a *coband* if $B = \gamma^{-1} E \delta$ (hence γ and δ are different arrows with a common source). The theory of cobands can be developed parallel to the theory of bands. For instance, a string algebra A is domestic iff, for every arrow γ, there is at most one coband B over A starting with γ^{-1}. A more elaborate explanation of this phenomenon is the following. Applying the duality $\mathrm{Hom}(-, \Bbbk)$ (between the categories of left and right A-modules), we just invert arrows in the quiver of A, hence bands go to cobands and vice versa. Furthermore, the definition of string algebra implies that A is a string algebra iff A^{op} is.

The following lemma implies that (over a domestic string algebra) there are no proper inclusions between bands and cobands.

Lemma 4.2. *Let C be a primitive cycle of a domestic string algebra. Then no proper substring of C is a band, and no proper substring of C is a coband.*

Proof. By [4, L. 3.3] (or see a similar proof below) no proper substring of C is a band.

Suppose that $B=\gamma^{-1}E\delta$ is a coband which is a (proper) substring of C. Then B contains a substring $\tau^{-1}\theta$ somewhere between γ^{-1} and δ. Thus $B=B_1\tau^{-1}\theta B_2$, hence $C=C_1\tau^{-1}\theta C_2$. Then $C\sim_c C'=\theta C_2C_1\tau^{-1}$ and $B\sim_c B'=\theta B_2B_1\tau^{-1}$.

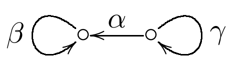
Since A is domestic, both C' and B' are powers of a unique band $D=\theta F\tau^{-1}$. But B' is primitive, hence $D=B'$, and similarly $D=C'$. Comparing the lengths, we obtain $|C|=|B|$, a contradiction. ∎

5. Main results

FACT 5.1 (see [4, L. 3.4]). *Let C be a primitive cycle of a domestic string algebra. Then no arrow occurs in C twice as a direct arrow, and no arrow occurs in C twice as an inverse arrow.*

If we count direct and inverse occurrences of an arrow α together, then α can appear in a primitive cycle twice, as the following example shows.

EXAMPLE 5.2. Let A be the string algebra



with the relations $\beta^2=\gamma^2=\beta\alpha\gamma=0$. It is easily seen that $B=\alpha\gamma\alpha^{-1}\beta^{-1}$ and $B^{-1}=\beta\alpha\gamma^{-1}\alpha^{-1}$ is a complete list of bands of A (that is, A is 1-domestic), and B contains α and α^{-1}.

In the following proposition we show how to "improve" certain primitive cycles of a domestic string algebra. Recall that, by Fact 4.1, any band of a domestic string algebra is uniquely determined by its first letter.

PROPOSITION 5.3. *Let C be a primitive cycle of a domestic string algebra A. Suppose that C contains α or α^{-1}, and C contains β or β^{-1}, where α and β are distinct arrows with a common end. Then there is a band $B=\alpha U\beta^{-1}$ over A such that C is equivalent to B.*

Proof. Taking the inverse if necessary, we may assume that α occurs in C as a direct arrow. Furthermore, applying a cyclic permutation, we can put α at the beginning of C, hence $C=\alpha E\beta^{-1}F$, or $C=\alpha E\beta F$.

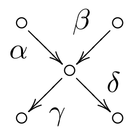
Suppose that $C=\alpha E\beta^{-1}F$. If F is empty, then we can take $B=C$. Otherwise $F\neq\emptyset$. The cycle $\alpha E\beta^{-1}$ is a power of a unique band $B=\alpha U\beta^{-1}$. Since F is not empty, B is a proper substring of C, which contradicts Lemma 4.2.

It remains to consider the case $C=\alpha E\beta F$.

CASE 1: *E and F are non-empty.* If the last letter of F is an inverse arrow, it must be β^{-1}, hence $C=\alpha V\beta^{-1}$ is a band. Then we can take $B=C$.

Thus we may assume that $F = F'\gamma$, where the composition $\gamma\alpha$ is defined. If the last letter of E is an inverse arrow, it must be α^{-1}. Then $E = E'\alpha^{-1}$, hence $C = \alpha E'\alpha^{-1}\beta F$. A cyclic permutation of C is the band $B' = \beta F\alpha E'\alpha^{-1}$. Then $B = B'^{-1} = \alpha E'^{-1}\alpha^{-1}F^{-1}\beta^{-1}$ is a band, and $C \sim B$.

Thus we may assume that $E = E'\delta$, where the composition $\delta\beta$ is defined. Then $C = \alpha E'\delta\beta F'\gamma$.



From $\gamma\alpha \neq 0$ and $\delta\beta \neq 0$ we conclude that $\gamma \neq \delta$ (but we have not excluded the possibility, say, that $\gamma = \alpha$, i.e., α is a loop).

Since γ and δ have a common source, $D = \alpha E'\delta\gamma^{-1}F'^{-1}\beta^{-1}$ is a string, hence D is a cycle. Since A is domestic, $D = B^k$ for a unique band $B = \alpha U\beta^{-1}$. Because $C^2 = \alpha E'\delta\beta F'\gamma\alpha E'\delta\beta F'\gamma$ is a string,

$$CD = \alpha E'\delta\beta F'\gamma\alpha E'\delta\gamma^{-1}F'^{-1}\beta^{-1}$$

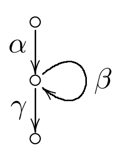
is a string. Since CD starts with α and ends with β^{-1}, CD is a cycle, hence $CD = B^l$ for some l. Then $D = B^k$ yields $C = B^{l-k}$.

Comparing the last letters of C and B^{l-k}, we obtain $\gamma = \beta^{-1}$, a contradiction.

Case 2: *F is empty.* Then $C = \alpha E\beta$, hence β is a loop. Since C contains an inverse arrow, E is not empty.

If E ends with an inverse arrow, it must be α^{-1}. Then $C = \alpha E'\alpha^{-1}\beta$ is c-equivalent to the cycle $B' = \beta\alpha E'\alpha^{-1}$. Since C is primitive, B' is a band, and $C \sim B'^{-1} = \alpha E'^{-1}\alpha^{-1}\beta^{-1}$.

Thus we may assume that $E = E'\gamma$, hence $C = \alpha E'\gamma\beta$. Since $\beta\alpha$ is defined, we must have $\beta^2 = 0$. Then $\gamma\beta \neq 0$ implies $\gamma \neq \beta$.



Thus $D = \alpha E'\gamma\beta^{-1}$ is a cycle. Considering the string CD, we obtain a contradiction ($\beta = \beta^{-1}$) as in Case 1.

Case 3: *E is empty, but F is non-empty.* Then $C = \alpha\beta F$, hence α is a loop. If F ends with an inverse arrow, it must be β^{-1}, hence $C = \alpha\beta F'\beta^{-1}$ is a band.

Otherwise $F = F'\gamma$ and $C = \alpha\beta F'\gamma$. Note that $\alpha\beta \neq 0$ implies $\alpha^2 = 0$, and then from $\gamma\alpha \neq 0$ we conclude that $\gamma \neq \alpha$.

Thus, considering the cycles $D = \beta F'\gamma\alpha^{-1}$ and $\beta F'\gamma\alpha D$, we obtain $\alpha = \alpha^{-1}$, a contradiction. ∎

By Fact 5.1, a primitive cycle of a domestic string algebra may contain at most four (direct or inverse) occurrences of a pair of arrows with a common end. In the following proposition we improve this result.

PROPOSITION 5.4. *Let C be a primitive cycle of a domestic string algebra A. If α and β are distinct arrows with a common end, then C contains at most three occurrences of α and β* (*as direct or inverse arrows*).

Proof. By Fact 5.1, α may occur in C at most once as a direct arrow, and at most once as an inverse arrow; and the same is true for β. Hence the total number of occurrences of α and β in C does not exceed 4. Thus, looking for a contradiction, we may assume that this number is equal to 4, that is, C contains α, α^{-1} and β, β^{-1}.

By Proposition 5.3, there is a band $B = \alpha U\beta^{-1}$ such that C is equivalent to B. Thus we may assume that $C = B$, hence B contains α^{-1} and β.

Suppose that β occurs in B before α^{-1}, i.e. $B = \alpha E\beta F\alpha^{-1}G\beta^{-1}$. Then $\beta F\alpha^{-1}$ is a cycle, hence a power of B^{-1}. But the length of $\beta F\alpha^{-1}$ is less than the length of B, a contradiction (to Lemma 4.2).

Thus we may assume that α^{-1} occurs in B before β: $B = \alpha E\alpha^{-1}F\beta G\beta^{-1}$ (in particular, E and G are non-empty). If $B' = \alpha E\alpha^{-1}F\beta G^{-1}\beta^{-1}$ is a string, it is a power of B. From $|B| = |B'|$ we conclude that $B = B'$, hence $G = G^{-1}$, a contradiction.

Thus B' is not a string. It follows that F is non-empty and the last letter of F is a direct arrow δ (such that $F\beta G^{-1}$ contains a relation of A as a subword).

Similarly, considering the word $\alpha E^{-1}\alpha^{-1}F\beta G\beta^{-1}$, we conclude that the first letter of F is an inverse arrow γ^{-1} (such that $E^{-1}\alpha^{-1}F$ contains the inverse of a relation of A).

Thus $F = \gamma^{-1}F'\delta$ and $B = \alpha E\alpha^{-1}\gamma^{-1}F'\delta\beta G\beta^{-1}$. If S is the common end of α and β, then S is the common source of γ and δ. Since the compositions $\gamma\alpha$ and $\delta\beta$ are defined, we conclude that $\gamma \neq \delta$.

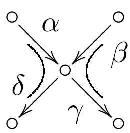
Then $\gamma^{-1}F'\delta$ is a cycle, hence a power of a unique coband $\gamma^{-1}H\delta$. The length of this coband is less than the length of B, a contradiction (to Lemma 4.2). ∎

Given a (primitive) cycle $C = u_1 \ldots u_n$ and a vertex S, one can count how many times C goes through S. Thus we assign to C the word $u(C) = S_1 \ldots S_n$, where S_i is the source vertex of the letter u_i (i.e., the vertex between u_i and u_{i+1}, or between u_n and u_1). Now, the *number of occurrences of S in C* is just the number of occurrences of S in $u(C)$. This is obviously the same as the multiplicity of the simple module corresponding to S as a factor in a composition series for a primitive band module corresponding to C (see [10, S. 1.11] or [1, p. 161] for a definition of band modules). Clearly this number is an invariant of the $\sim$-equivalence class of C. For instance, the unique vertex S occurs three times in the band $\alpha\beta^{-2}$ over $G_{2,3}$.

PROPOSITION 5.5. *Let $B = \alpha U \beta^{-1}$ be a band of a domestic string algebra, and let S be the common end of α and β. Then S cannot occur three times in B.*

Proof. Otherwise $B = \alpha E.F.G\beta^{-1}$, where the dots denote two (internal) occurrences of S. In particular, F is not empty.

CASE 1: *There is no loop in S.* Then E, G are non-empty, and $|F| \geq 2$. Thus we need four (direct or inverse) arrows to surround the two dotted occurrences of S in B. By Fact 5.1, α and β^{-1} cannot occur in B a second time (except for α at the beginning of B, and β^{-1} at the end of B). Furthermore, by Proposition 5.4, α^{-1} and β cannot occur in B simultaneously. Thus there are at least three positions around the two dots to be occupied by arrows starting at S. Therefore (by Fact 5.1 again) there must be two different arrows γ and δ with a common source at S occurring in B as direct or inverse arrows.



We may assume that $\delta\alpha = 0$ and $\gamma\beta = 0$. Furthermore (by a dual to Proposition 5.4) the total number of occurrences of γ and δ in B (as direct or inverse arrows) does not exceed 3. Thus either α^{-1} or β occurs in B around the dots. By symmetry we may assume that α^{-1} occurs in B (hence β does not occur).

Clearly α^{-1} must occur on the left of a dot. Thus there are two cases to consider: (a) $B = \alpha E\alpha^{-1}.F.G\beta^{-1}$ and (b) $B = \alpha E.F\alpha^{-1}.G\beta^{-1}$.

SUBCASE (a): $B = \alpha E\alpha^{-1}.F.G\beta^{-1}$. Since β does not occur in B, the first letter of F must be an inverse arrow. From $\delta\alpha = 0$ it follows that $F = \gamma^{-1}F'$, hence $B = \alpha E\alpha^{-1}.\gamma^{-1}F'.G\beta^{-1}$, where F' is non-empty. To surround the second dot, we need either $\gamma.\delta^{-1}$ or $\delta.\gamma^{-1}$. But γ^{-1} cannot occur twice in B (Lemma 5.1), hence $F' = F''\gamma$, $G = \delta^{-1}G'$, and $B = \alpha E\alpha^{-1}.\gamma^{-1}F''\gamma.\delta^{-1}G'\beta^{-1}$.

But then $B' = \alpha E\alpha^{-1}.\beta G'^{-1}\delta.\gamma^{-1}F''\gamma.\delta^{-1}G'\beta^{-1}$ is clearly a cycle, hence a power of the (unique) band B. Note that there is no α in E, G' and F''. Comparing letters at the beginning of B' and B, we obtain $\beta = \gamma^{-1}$, a contradiction.

Subcase (b): $B = \alpha E.F\alpha^{-1}.G\beta^{-1}$. If the first letter of G is a direct arrow, then $B' = \alpha F^{-1}.E^{-1}\alpha^{-1}.G\beta^{-1}$ is a cycle, hence B' is a power of B. Since $|B'| = |B|$, we have $B' = B$, hence $EF = (EF)^{-1}$, a contradiction. Thus the first letter of G is an inverse arrow. Since $\delta\alpha = 0$, it must be γ^{-1}.

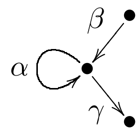
Thus $G = \gamma^{-1}G'$, hence $B = \alpha E.F\alpha^{-1}.\gamma^{-1}G'\beta^{-1}$. The first dot in B is surrounded by either $\gamma.\delta^{-1}$ or $\delta.\gamma^{-1}$. Since γ^{-1} cannot occur in B twice, it must be $\gamma.\delta^{-1}$.

Thus $B = \alpha E'\gamma.\delta^{-1}F'\alpha^{-1}.\gamma^{-1}G'\beta^{-1}$. Then

$$B' = \alpha E'\gamma.\delta^{-1}F'\alpha^{-1}.\beta G'^{-1}\gamma.\delta^{-1}F'\alpha^{-1}.\gamma^{-1}G'\beta^{-1}$$

is a cycle beginning with α, hence B' is a power of B. Comparing B' and B, we obtain $\beta = \gamma^{-1}$, a contradiction.

Case 2: *There is exactly one loop at S.* Since α and β are the only arrows ending at S, one of them must be a loop. By symmetry we may assume that α is a loop, hence β is not a loop. Then there exists at most one arrow $\gamma \neq \alpha$ starting at S.



We have $B = \alpha E.F.G\beta^{-1}$. Since β is not a loop, G is not empty. By Fact 5.1, there are no additional occurrences of α and β^{-1} in B. Furthermore (see Proposition 5.4), α^{-1} and β can occur in B at most once, and cannot occur simultaneously.

Suppose first that $|F| = 1$, i.e. F is a loop. Then $F = \alpha^{-1}$, hence $B = \alpha E.\alpha^{-1}.G\beta^{-1}$. It follows that β does not occur in B. If E starts with an inverse arrow, then $B' = \alpha E^{-1}.\alpha^{-1}.G\beta^{-1}$ is a cycle. Comparing lengths of B and B' we obtain $B' = B$. But this leads to $E = E^{-1}$, a contradiction. Otherwise the first letter of E is a direct arrow, i.e., β, a contradiction again.

Thus we may assume that $|F| \geq 2$ in $B = \alpha E.F.G\beta^{-1}$. Now the first and the last letters of F and the first letter of G surround S. We have only three candidates to fill in this gap: γ, γ^{-1}, and either α^{-1} or β (at most one of those). Thus all three letters must occur once in these positions. Also E should be empty, i.e., $B = \alpha.F.G\beta^{-1}$.

Since γ cannot occur to the right of a dot, the last letter of F must be γ: $B = \alpha.F'\gamma.G\beta^{-1}$. Then G cannot start with γ^{-1}, hence γ^{-1} should be the first letter of F'. Thus $B = \alpha.\gamma^{-1}F''\gamma.G\beta^{-1}$, where the first letter of G is either β or α^{-1}. If it is α^{-1}, i.e., $B = \alpha.\gamma^{-1}F''\gamma.\alpha^{-1}G'\beta^{-1}$, we obtain a contradiction (to the domesticity) by considering the cycle $\alpha.\gamma^{-1}F''^{-1}\gamma.\alpha^{-1}G'\beta^{-1}$.

Thus $B = \alpha.\gamma^{-1}F''\gamma.\beta G'\beta^{-1}$. Then $B' = \alpha.\gamma^{-1}F''^{-1}\gamma.\alpha^{-1}.\beta G'\beta^{-1}$ is a cycle. By domesticity, B' must be a power of B. Comparing B' and B, we derive $F'' = F''^{-1}$, a contradiction (since F'' cannot be empty).

Case 3: *There are two loops at S.* Then α and β must be loops, hence α and β are the only arrows starting or ending at S. If $B = \alpha U \beta^{-1}$, then U is non-empty (otherwise there is just one internal occurrence of S). Since β^{-1} cannot occur twice (and α cannot occur twice), the first letter of U must be β: $B = \alpha\beta U'\beta^{-1}$. Now U' is not empty, and the first letter of U' must be α^{-1}. But then the total number of occurrences of α and β in B is at least 4, a contradiction (to Proposition 5.4). ∎

Now we are in a position to prove the main result of the paper.

Theorem 5.6. *Let C be a primitive cycle of a domestic string algebra A. Then no vertex occurs in C three times. Therefore the multiplicity of a simple module as a composition factor in a composition series of a primitive band module over A is at most* 2.

Proof. Suppose that S occurs in C three times. If one of these occurrences is (up to a cyclic permutation of C) of the form $\beta^{-1}.\alpha$, then C is c-equivalent to a band $B = \alpha E \beta^{-1}$. Thus we may apply Proposition 5.5.

If one of the occurrences of S in C is of the form $\gamma.\delta^{-1}$, then C is c-equivalent to a coband $D = \delta^{-1}F\gamma$, hence we may apply a dual of Proposition 5.5.

Otherwise S occurs in C (up to a cyclic permutation of C again) either of the form $\gamma.\alpha$ or $\alpha^{-1}.\gamma^{-1}$. Since α appears in C at most once, $\gamma.\alpha$ can occur just once, and the same is true for $\alpha^{-1}.\gamma^{-1}$. So using $\gamma.\alpha$ and $\alpha^{-1}.\gamma^{-1}$ we can surround the only two occurrences of S.

Thus we need one more path $\delta.\beta$ or $\beta^{-1}.\delta^{-1}$ to surround the third copy of S. In particular, $\beta \neq \alpha$ ends at S.

Then C contains α or α^{-1}, and β or β^{-1}. By Proposition 5.3, C is equivalent to a band $B = \alpha U \beta^{-1}$. Now we can apply Proposition 5.5. ∎

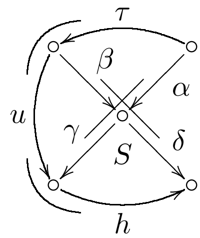
Corollary 5.7. *Let C be a primitive cycle of a domestic string algebra. Then $|C| \leq 2|Q_0|$.*

Proof. By Theorem 5.6, each vertex $S \in Q_0$ may occur in C at most twice. Thus the length of C does not exceed twice the number of vertices. ∎

Note that the estimate given by Corollary 5.7 is sharp. Indeed, let A be the string algebra from Example 5.2. Then A is domestic, has two vertices, and the length of the unique band $B = \alpha\gamma\alpha^{-1}\beta^{-1}$ over A is 4.

One may wonder if it is possible to "dualize" Theorem 5.6 by proving that three different bands of a domestic string algebra have no vertices in common. The following example shows that this is not the case.

Let A be the string algebra



with the relations $\gamma\beta\tau = h\gamma\beta = 0$ and $u\tau = hu = \delta\beta = \gamma\alpha = 0$. It is not difficult to check that A is 3-domestic with the following bands: $C = \alpha\tau^{-1}\beta^{-1}$, $D = h\gamma\delta^{-1}$ and $E = u\beta^{-1}\gamma^{-1}$, which have vertex S in common.

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