COLLOQUIUM MATHEMATICUM

VOL. 108 2007 NO. 2

BIMINIMAL LEGENDRIAN SURFACES IN 5-DIMENSIONAL SASAKIAN SPACE FORMS

BY

TORU SASAHARA (Oita)

Abstract. We classify nonminimal biminimal Legendrian surfaces in 5-dimensional Sasakian space forms.

1. Introduction. The study of Legendrian submanifolds in contact manifolds from the Riemannian geometric point of view was initiated in the 1970's. In particular, the class of minimal Legendrian submanifolds is one of the most interesting objects of study from both the geometric and physical point of views. Needless to say, introducing classes which include such submanifolds is important.

A natural extension of the class of minimal submanifolds is the class of ones with parallel mean curvature vector. During the last three decades, many interesting results on nonminimal submanifolds with parallel mean curvature vector have been obtained by many geometers.

On the other hand, there exist no Legendrian submanifolds with parallel mean curvature vector in Sasakian manifolds apart from the minimal ones (see [11]). Thus, in case the ambient space is a Sasakian manifold, we need to consider some other extensions of minimal Legendrian submanifolds. In this paper, we consider an extension from a variational point of view.

In [7], Loubeau and Montaldo introduced the notion of *biminimal submanifolds*, which are critical points of the bienergy functional with respect to all normal variations. Minimal submanifolds are biminimal, but the converse is not true in general. Recently, in [5] Inoguchi classified nongeodesic biminimal Legendrian curves in 3-dimensional Sasakian space forms. As a next step, it is natural and interesting to classify nonminimal biminimal Legendrian surfaces in 5-dimensional Sasakian space forms. The main result of this paper is the following:

THEOREM 1. *Let M^2 be a nonminimal biminimal Legendrian surface in a 5-dimensional Sasakian space form $N^5(\epsilon)$ of constant ϕ-sectional curva-*

2000 *Mathematics Subject Classification*: Primary 53C42; Secondary 53B25.

Key words and phrases: Legendrian submanifold, biminimal submanifold, Sasakian manifold.

ture ϵ. Then $\epsilon \geq (-11+32\sqrt{2})/41$ and for each point $p \in M^2$ there exists a local coordinate system $\{x, y\}$ on a neighborhood of p such that the metric tensor g and the second fundamental form h take the following forms:

(1) $g = dx^2 + dy^2$,

(2)

$$\begin{aligned} h(\partial_x, \partial_x) &= \frac{\epsilon - 1}{\alpha}\,\phi\partial_x, \\ h(\partial_y, \partial_y) &= \left(\alpha - \frac{\epsilon - 1}{\alpha}\right)\phi\partial_x, \\ h(\partial_x, \partial_y) &= \left(\alpha - \frac{\epsilon - 1}{\alpha}\right)\phi\partial_y, \end{aligned}$$

where $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ and

$$\alpha = \begin{cases} \sqrt{\dfrac{13\epsilon - 9 \pm \sqrt{41\epsilon^2 + 22\epsilon - 47}}{8}} & (\epsilon \neq 1), \\ 1 & (\epsilon = 1). \end{cases}$$

Conversely, suppose that ϵ is a constant satisfying $\epsilon \geq (-11+32\sqrt{2})/41$ and let g be the metric tensor on a simply connected domain $V \subset \mathbb{R}^2$ defined by (1). *Then, up to rigid motions of $N^5(\epsilon)$, there exists a unique Legendrian immersion of (V, g) into $N^5(\epsilon)$ whose second fundamental form is given by* (2). *Moreover such an immersion is nonminimal and biminimal.*

COROLLARY 2. *Let $f : M^2 \to S^5(1) \subset \mathbb{C}^3$ be a nonminimal biminimal Legendrian immersion into the unit 5-sphere. Then the position vector $f(x, y)$ of M^2 in $\mathbb{C}^3$ is given by*

$$f(x,y) = \frac{1}{\sqrt{2}}\,(e^{ix}, ie^{-ix}\sin\sqrt{2}y, ie^{-ix}\cos\sqrt{2}y). \tag{1.1}$$

2. Biminimal submanifolds. Let $f : (M, g) \to (N, \widetilde{g})$ be a smooth map between two Riemannian manifolds. The *bienergy* $E_2(f)$ of f over a compact domain $\Omega \subset M$ is defined by

$$E_2(f) = \int_\Omega \widetilde{g}(\tau(f), \tau(f))\, dv_g, \tag{2.1}$$

where $\tau(f)$ is the tension field of f and dv_g is the volume form of M (see [3]).

E_2 provides a measure for the extent to which f fails to be harmonic. If f is a critical point of (2.1) over every compact domain, then f is called a *biharmonic map* (or *2-harmonic map*). In [6], Jiang proved that f is biharmonic if and only if

$$\mathcal{J}_f(\tau(f)) = 0, \tag{2.2}$$

where the operator $\mathcal{J}_f$ is the *Jacobi operator* defined by

$$
\begin{aligned}
\mathcal{J}_f(V) &:= \overline{\Delta}_f V - \mathcal{R}_f(V), \qquad V \in \Gamma(f^*TN), \\
\overline{\Delta}_f &:= -\sum_{i=1}^{m}(\nabla^f_{e_i}\nabla^f_{e_i} - \nabla^f_{\nabla_{e_i}e_i}), \qquad (2.3)\\
\mathcal{R}_f(V) &:= \sum_{i=1}^{m} R^N(V, df(e_i))df(e_i),
\end{aligned}
$$

where ∇^f is the connection induced by f and R^N is the curvature tensor of N. We call (2.2) the *biharmonic equation.*

If f is an isometric immersion, (2.2) can be rewritten as

$$\overline{\Delta}_f H = \mathcal{R}_f H, \tag{2.4}$$

where H is the mean curvature vector field.

Recently, Loubeau and Montaldo introduced the notion of biminimal immersions.

DEFINITION 3 ([7]). An isometric immersion $f : (M^m, g) \to (N^n, h)$ is called *biminimal* if it is a critical point of the bienergy functional E_2 with respect to all normal variations with compact support. Here, a *normal variation* means a variation f_t of $f = f_0$ such that the variational vector field $V = df_t/dt|_{t=0}$ is normal to M.

f is biminimal if and only if

$$\{\overline{\Delta}_f H\}^\perp = \{\mathcal{R}_f(H)\}^\perp. \tag{2.5}$$

We call (2.5) the *biminimal equation.* Clearly, biharmonic submanifolds are biminimal. There exist many nonbiharmonic biminimal submanifolds (see [7]).

3. Legendrian submanifolds in Sasakian space forms. A $(2n+1)$-dimensional differentiable manifold N^{2n+1} is called a *contact manifold* if there exists a globally defined 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. On a contact manifold there exists a unique global vector field ξ satisfying

$$d\eta(\xi, X) = 0, \qquad \eta(\xi) = 1, \tag{3.1}$$

for all $X \in TN^{2n+1}$.

Moreover it is well known that there exist a tensor field ϕ of type $(1,1)$ and a Riemannian metric g which satisfy

$$
\begin{aligned}
&\phi^2 = -I + \eta \otimes \xi, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \\
&g(\xi, X) = \eta(X), \qquad d\eta(X, Y) = g(X, \phi Y),
\end{aligned} \tag{3.2}
$$

for all $X, Y \in TN^{2n+1}$ (see, for instance, [1]).

The structure (ϕ, ξ, η, g) is called a *contact metric structure* and the manifold N^{2n+1} with a contact metric structure is said to be a *contact metric manifold*. A contact metric manifold is said to be a *Sasakian manifold* if it satisfies $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on N^{2n+1}, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. On Sasakian manifolds, we have

$$(\overline{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{3.3}$$

$$\overline{\nabla}_X \xi = -\phi X, \tag{3.4}$$

for any vector fields X and Y, where $\overline{\nabla}$ is the Levi-Civita connection of N^{2n+1}. In some respects, Sasakian manifolds may be viewed as odd-dimensional analogues of Kähler manifolds.

A tangent plane in T_pN^{2n+1} which is invariant under ϕ is called a *ϕ-section* (see [1]). The sectional curvature of a ϕ-section is called the *ϕ-sectional curvature*. If the ϕ-sectional curvature is constant on N^{2n+1}, then N^{2n+1} is said to be of *constant ϕ-sectional curvature*. Complete and connected Sasakian manifolds of constant ϕ-sectional curvature are called *Sasakian space forms*. Denote Sasakian space forms of constant ϕ-sectional curvature ϵ by $N^{2n+1}(\epsilon)$. The curvature tensor $\overline{R}$ of $N^{2n+1}(\epsilon)$ is given by ([8])

$$\begin{aligned} \overline{R}(X, Y)Z = \frac{\epsilon + 3}{4}\{g(Y, Z)X - g(Z, X)Y\} + \frac{\epsilon - 1}{4}\{&\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi& \\ + g(Z, \phi Y)\phi X - g(Z, \phi X)\phi Y + 2g(X, \phi Y)\phi Z\}&. \end{aligned} \tag{3.5}$$

Sasakian space forms were classified by Tanno (see [10]).

Let M^m be a submanifold in a contact manifold N^{2n+1}. If η restricted to M^m vanishes, then M^m is called an *integral submanifold*, in particular if $m = n$, it is called a *Legendrian submanifold*.

Let $f : M^m \to N^{2n+1}(\epsilon)$ be an isometric immersion. Denote the Levi-Civita connection of $N^{2n+1}(\epsilon)$ (resp. M^m) by $\overline{\nabla}$ (resp. ∇). The formulas of Gauss and Weingarten are given respectively by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \overline{\nabla}_X V &= -A_V X + D_X V, \end{aligned} \tag{3.6}$$

where $X, Y \in TM^m$, $V \in T^\perp M^m$, h, A and D are the second fundamental form, the shape operator and the normal connection. The mean curvature vector H is given by $H = (1/m)\,\mathrm{trace}\, h$. Its length $\|H\|$ is called the *mean curvature function* of M^m. If $H = 0$ at any point of M^m, then M^m is called *minimal*.

For Legendrian submanifolds in Sasakian space forms, we have

$$A_{\phi Y}X = -\phi h(X, Y) = A_{\phi X}Y, \qquad A_\xi = 0 \tag{3.7}$$

(see Lemmas 8.1 and 8.2 in [1]), and moreover a straightforward computation shows that the equations of Gauss, Codazzi and Ricci are equivalent to

$$\langle R(X,Y)Z,W\rangle = \langle [A_{\phi Z}, A_{\phi W}]X, Y\rangle + \langle \overline{R}(X,Y)Z,W\rangle, \tag{3.8}$$

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z), \tag{3.9}$$

where $\overline{\nabla} h$ is defined by $(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

In [5], Inoguchi proved that nongeodesic biminimal Legendrian curves in 3-dimensional Sasakian space forms $N^3(\epsilon)$ are biharmonic, that is, Legendrian helices of curvature $\sqrt{\epsilon - 1}$ (cf. [4]). In the next section, we prove that nonminimal biminimal Legendrian surfaces in 5-dimensional Sasakian space forms $N^5(\epsilon)$ are biharmonic as in the case of curves. In [9], the author classified nonminimal biharmonic Legendrian surfaces in $N^5(\epsilon)$. Applying that result yields Theorem 1 and Corollary 2.

4. Proof of Theorem 1. Let $f : M^2 \to N^5(\epsilon)$ be a biminimal Legendrian surface. We assume that the mean curvature function is nowhere zero. Let $\{e_i\}$ $(i = 1, \ldots, 5)$ be an orthonormal frame along M^2 such that e_1, e_2 are tangent to M^2, $\phi e_1 = e_3$, $\phi e_2 = e_4$, $\xi = e_5$ and $H = (\alpha/2)\phi e_1$, with $\alpha > 0$. It follows from (3.7) that $\langle h(e_1, e_1), \phi e_2\rangle = \langle h(e_1, e_2), \phi e_1\rangle$ and $\langle h(e_2, e_2), \phi e_1\rangle = \langle h(e_1, e_2), \phi e_2\rangle$. Therefore the second fundamental form takes the form

$$\begin{aligned} h(e_1, e_1) &= (\alpha - c)\phi e_1 + b\phi e_2, \\ h(e_1, e_2) &= b\phi e_1 + c\phi e_2, \\ h(e_2, e_2) &= c\phi e_1 - b\phi e_2, \end{aligned} \tag{4.1}$$

for some functions b, c.

We put $\omega_i^j(e_k) = \langle \nabla_{e_k} e_i, e_j\rangle$. By using (3.3) and (4.1) we have

$$(\overline{\nabla}_{e_1} h)(e_2, e_2) = \{e_1 c + 3b\omega_1^2(e_1)\}\phi e_1 - \{e_1 b - 3c\omega_1^2(e_1)\}\phi e_2 + c\xi, \tag{4.2}$$

$$\begin{aligned} (\overline{\nabla}_{e_2} h)(e_1, e_2) = {}& \{e_2 b + (\alpha - 3c)\omega_1^2(e_2)\}\phi e_1 \\ &+ \{e_2 c + 3b\omega_1^2(e_2)\}\phi e_2 + c\xi, \end{aligned} \tag{4.3}$$

$$\begin{aligned} (\overline{\nabla}_{e_1} h)(e_1, e_2) = {}& \{e_1 b + (\alpha - 3c)\omega_1^2(e_1)\}\phi e_1 \\ &+ \{e_1 c + 3b\omega_1^2(e_1)\}\phi e_2 + b\xi, \end{aligned} \tag{4.4}$$

$$\begin{aligned} (\overline{\nabla}_{e_2} h)(e_1, e_1) = {}& \{e_2(\alpha - c) - 3b\omega_1^2(e_2)\}\phi e_1 \\ &+ \{e_2 b + (\alpha - 3c)\omega_1^2(e_2)\}\phi e_2 + b\xi. \end{aligned} \tag{4.5}$$

From (3.9) we get

$$e_1 c + 3b\omega_1^2(e_1) = e_2 b + (\alpha - 3c)\omega_1^2(e_2), \tag{4.6}$$

$$-e_1 b + 3c\omega_1^2(e_1) = e_2 c + 3b\omega_1^2(e_2), \tag{4.7}$$

$$e_2(\alpha - c) - 3b\omega_1^2(e_2) = e_1 b + (\alpha - 3c)\omega_1^2(e_1). \tag{4.8}$$

Combining (4.7) and (4.8) yields

$$e_2\alpha = \alpha\omega_1^2(e_1). \tag{4.9}$$

By the Gauss and Weingarten formulae we have Chen's well-known formula (see p. 273 in [2])

$$\bar{\varDelta}_f H = \operatorname{tr}(\overline{\nabla} A_H) + \varDelta^D H + (\operatorname{tr} A^2_{\phi e_1})H + a(H),$$

where

$$\operatorname{tr}(\overline{\nabla} A_H) = \sum_{i=1}^{2}(A_{D_{e_i}H}e_i + (\nabla_{e_i}A_H)e_i),$$

$$a(H) = \sum_{r=4}^{5}(\operatorname{trace} A_H A_{e_r})e_r,$$

$$\varDelta^D = -\sum_{i=1}^{2}(D_{e_i}D_{e_i} - D_{\nabla_{e_i}e_i})$$

and $\{e_i\}$ is a local orthonormal frame of M^2.

Also, by (3.5) we can easily get

$$\mathcal{R}_f(H) = \frac{5\epsilon + 3}{4}.$$

Thus by comparing the components of ϕe_1, ϕe_2 and ξ of the biminimal equation (2.5), we obtain

$$\begin{aligned} &\varDelta\alpha + \alpha\biggl(\frac{1-5\epsilon}{4} + (\alpha - c)^2 + c^2 + 2b^2 \\ &\qquad\qquad + (\omega_1^2(e_1))^2 + (\omega_1^2(e_2))^2\biggr) = 0, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &2(e_1\alpha)\omega_1^2(e_1) + 2(e_2\alpha)\omega_1^2(e_2) \\ &\qquad\qquad + \alpha\{e_1(\omega_1^2(e_1)) + e_2(\omega_1^2(e_2))\} + \alpha^2 b = 0, \end{aligned} \tag{4.11}$$

$$e_1\alpha + \alpha\omega_1^2(e_2) = 0. \tag{4.12}$$

Using (4.9) and (4.12), we obtain

$$\left[\frac{1}{\alpha}\,e_1, \frac{1}{\alpha}\,e_2\right] = 0. \tag{4.13}$$

Consequently, there exist local coordinates x, y such that

$$e_1 = \alpha\partial_x, \qquad e_2 = \alpha\partial_y. \tag{4.14}$$

It follows from (4.14) that the metric tensor is given by

$$g = \frac{1}{\alpha^2}\,(dx^2 + dy^2). \tag{4.15}$$

Thus we have

$$\omega_1^2(e_1) = \alpha_y, \quad \omega_1^2(e_2) = -\alpha_x, \tag{4.16}$$

where $f_x = \partial_x f$ and $f_y = \partial_y f$.

By substituting (4.14) and (4.16) to (4.11), we get $b = 0$. Therefore it follows from (4.6), (4.7) and (4.10) that

$$\alpha c_x = -(\alpha - 3c)\alpha_x, \tag{4.17}$$

$$3c\alpha_y = \alpha c_y, \tag{4.18}$$

$$-\alpha\alpha_{yy} - \alpha\alpha_{xx} + \frac{1-5\epsilon}{4} + \alpha^2 + 2c^2 - 2\alpha c + (\alpha_x)^2 + (\alpha_y)^2 = 0. \tag{4.19}$$

On the other hand, (3.8) yields

$$\begin{aligned} \alpha c - 2c^2 + \frac{\epsilon+3}{4} &= -(\omega_1^2(e_1))^2 - (\omega_1^2(e_2))^2 + e_2(\omega_1^2(e_1)) - e_1(\omega_1^2(e_2)) \\ &= -(\alpha_y)^2 - (\alpha_x)^2 + \alpha\alpha_{yy} + \alpha\alpha_{xx}. \end{aligned} \tag{4.20}$$

Combining (4.19) and (4.20), we obtain

$$\alpha^2 - 3\alpha c + 4c^2 - \frac{3\epsilon+1}{2} = 0. \tag{4.21}$$

Differentiating (4.21), we have

$$(2\alpha - 3c)\alpha_i + (8c - 3\alpha)c_i = 0, \tag{4.22}$$

where $i = x, y$.

Rewriting (4.17), (4.18) and (4.22), we obtain

$$\begin{pmatrix} \alpha - 3c & \alpha \\ 2\alpha - 3c & 8c - 3\alpha \end{pmatrix} \begin{pmatrix} \alpha_x \\ c_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.23}$$

$$\begin{pmatrix} 3c & -\alpha \\ 2\alpha - 3c & 8c - 3\alpha \end{pmatrix} \begin{pmatrix} \alpha_y \\ c_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.24}$$

The determinants of the systems (4.23) and (4.24) are

$$-5(\alpha - 2c)^2 - 4c^2, \tag{4.25}$$

$$2(\alpha - 3c)^2 + 6c^2, \tag{4.26}$$

respectively. Since $\alpha \neq 0$, from (4.23)–(4.26) we infer that α is constant. Hence $\omega_2^1 = 0$ by (4.16) and we easily see that M^2 satisfies the biharmonic equation (2.4). Theorem 1 and Corollary 2 are obtained from the classification results for nonminimal biharmonic Legendrian surfaces (see p. 300 in [9]). ∎

REFERENCES

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progr. Math. 203, Birkhäuser Boston, Boston, MA, 2002.

[2] B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Sci., Singapore, 1984.

[3] J. Eells and J. H. Sampson, *Variational theory in fibre bundles*, in: Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965), 22–33.

[4] J. Inoguchi, *Submanifolds with harmonic mean curvature vector field in contact 3-manifolds*, Colloq. Math. 100 (2004), 163–179.

[5] —, *Biminimal submanifolds in contact 3-manifolds*, Balkan J. Geom. Appl., to appear.

[6] G. Y. Jiang, *2-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. A 7 (1986), 389–402 (in Chinese).

[7] E. Loubeau and S. Montaldo, *Biminimal immersions in space forms*, preprint, 2004, math.DG/0405320.

[8] K. Ogiue, *On almost contact manifolds admitting axiom of planes or axiom of free mobility*, Kōdai Math. Sem. Rep. 16 (1964), 223–232.

[9] T. Sasahara, *Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors*, Publ. Math. Debrecen 67 (2005), 285–303.

[10] S. Tanno, *Sasakian manifolds with constant ϕ-holomorphic sectional curvature*, Tôhoku Math. J. 21 (1969), 501–507.

[11] K. Yano and M. Kon, *Structures on Manifolds*, Ser. Pure Math. 3, World Sci., Singapore, 1984.

Department of Mathematics
Oita National College of Technology
1666 Maki
Oita, 870-0150, Japan
E-mail: t-sasa@oita-ct.ac.jp

Received 14 February 2006;
revised 17 November 2006 (4722)