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HYPERSURFACES WITH ALMOST COMPLEX STRUCTURES IN THE REAL AFFINE SPACE

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Abstract. We study affine hypersurface immersions $f : M \to \mathbb{R}^{2n+1}$, where M is an almost complex n-dimensional manifold. The main purpose is to give a condition for (M, J) to be a special Kähler manifold with respect to the Levi-Civita connection of an affine fundamental form.

1. Introduction. Affine immersions of a manifold M into the complex affine space $\mathbb{C}^{n+1}$ have been studied by many authors. In the case that M is a complex n-dimensional manifold, K. Abe, Dillen, Verstraelen and Vrancken considered an analogue of the real case by choosing a holomorphic transversal vector field (see [1], [4]–[8]). They extended the Blaschke theory to the complex case. Nomizu, Pinkall and Podestà [11] have established the foundations for the geometry of affine Kähler immersions by choosing an anti-holomorphic transversal vector field. An affine Kähler immersion induces on M an affine Kähler connection, that is, a connection ∇ compatible with the complex structure J on M whose curvature tensor satisfies $R(JX, JY) = R(X, Y)$ for all tangent vectors X, Y. Moreover, those authors proved an analogue of the classical theorem of Pick and Berwald. Opozda [13, 14] developed a general theory that includes the above choices of transversal vector fields. Recently, Kurosu [10] studied complex equiaffine immersions of general codimension.

On the other hand, Baues and Cortés [3] proved that any simply connected special Kähler manifold has a canonical realization as a parabolic affine hypersphere. This is an important example of an affine hypersurface with complex structure immersed in the real affine space.

In this paper, we study affine hypersurface immersions $f : M \to \mathbb{R}^{2n+1}$, where M is an almost complex n-dimensional manifold. In particular, we study the case that the affine fundamental form is invariant with respect to the almost complex structure J of M. The main purpose of this paper

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is to give a condition for (M, J) to be a special Kähler manifold. We prove that if the symplectic 2-form ω of the immersion is ∇-parallel, then (M, J) is a special Kähler manifold under some additional conditions. Moreover, we prove that a hypersurface immersion with ∇-parallel almost complex structure J is a hyperquadric in $\mathbb{R}^{2n+1}$.

In Section 2, we recall some basic facts and definitions of affine differential geometry, in particular concerning hypersurface immersions. We prove our main theorems in Section 3. In Section 4, we give some conditions for an affine connection ∇ induced by the immersion to be flat.

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2. Affine hypersurfaces. For the general theory of affine geometry, we refer to Nomizu and Sasaki [12].

Let $\mathbb{R}^{n+1}$ denote the real affine space with the standard flat affine connection D and the standard volume form det. We consider an n-dimensional manifold M together with an immersion $f : M \to \mathbb{R}^{n+1}$. We then call M a *hypersurface* and f a *hypersurface immersion*. Let $\xi \in \Gamma(f^{-1}T\mathbb{R}^{n+1})$ be a transversal vector field on M. Then we have a torsion-free connection ∇ satisfying

$$D_X f_*(Y) = f_*(\nabla_X Y) + g(X, Y)\xi,$$

where g is a symmetric bilinear function on the tangent space $T_m M$, called the *affine fundamental form* relative to the transversal vector field ξ. We call it the *Gauss formula*. Also, for any vector field X, we have the *Weingarten formula*

$$D_X\xi = -f_*(SX) + \tau(X)\xi,$$

where S is a tensor field of type $(1, 1)$, and τ is a 1-form. We call S the *shape operator*, and τ the *transversal connection form* of f with respect to ξ.

The *curvature tensor field* R of ∇ is defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for any vector fields X, Y and Z tangent to M. Since D is a flat connection, we have

$$(D_X D_Y - D_Y D_X - D_{[X,Y]})Z = 0,$$
$$(D_X D_Y - D_Y D_X - D_{[X,Y]})\xi = 0$$

for all vector fields X, Y and Z. By the Gauss and Weingarten formulas, R, ∇, g, S and τ satisfy the following formulas for an arbitrary transversal vector field ξ:

(1) $R(X,Y)Z = g(Y,Z)SX - g(X,Z)SY,$

(2) $(\nabla_X g)(Y,Z) + \tau(X)g(Y,Z) = (\nabla_Y g)(X,Z) + \tau(Y)g(X,Z),$

(3) $(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX,$

(4) $g(X,SY) - g(SX,Y) = d\tau(X,Y).$

Equations (1), (2), (3), and (4) are called the Gauss equation, Codazzi equation for g, Codazzi equation for S and Ricci equation, respectively.

A transversal vector field ξ of f is called an *equiaffine transversal vector field* if $\tau = 0$. If the affine fundamental form g is nondegenerate for some transversal vector field, the hypersurface f is called *nondegenerate*. When f is nondegenerate, there exists a canonical transversal field ξ, called the *affine normal*. The affine normal is uniquely determined up to sign by the following conditions: the metric volume form ν of g is ∇-parallel and coincides with the induced volume form θ, where ν is defined by $\nu(X_1,\dots,X_n) = |\det[g(X_i,X_j)]|^{1/2}$ and θ is defined by $\theta(X_1,\dots,X_n) = \det(f_*(X_1),\dots,f_*(X_n),\xi)$ for tangent vectors X_i $(i = 1,\dots,n)$. Since $\nabla_X\theta = \tau(X)\theta$ for all $X \in T_x(M)$, these conditions lead to $\tau = 0$.

An immersion $f : M \to \mathbb{R}^{n+1}$ with the affine normal ξ is called a *Blaschke immersion*, and the affine fundamental form g associated to the affine normal ξ is called the *Blaschke metric* of f. A Blaschke immersion is called an *affine hypersphere* if the shape operator S satisfies $S = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{R}$. An affine hypersphere is said to be *proper* if $\lambda \neq 0$, and *improper* or *parabolic* if $\lambda = 0$.

For later use, we define the g-conjugate connection $\overline{\nabla}$ on M by the equation

$$Xg(Y,Z) = g(\nabla_X Y,Z) + g(Y,\overline{\nabla}_X Z).$$

We denote by $\widehat{\nabla}$ the Levi-Civita connection of g. When ξ is an equiaffine transversal vector field, the equation $\widehat{\nabla} = \frac{1}{2}(\nabla + \overline{\nabla})$ holds.

3. Affine hypersurfaces with almost complex structure. Let M be a simply connected almost complex manifold with almost complex structure J. Let $f : M \to \mathbb{R}^{2n+1}$ be a nondegenerate hypersurface immersion, ξ a transversal vector field, and g the affine fundamental form. If g satisfies the condition $g(JX,JY) = g(X,Y)$ for any X and Y, then g is said to be *invariant* with respect to J. The 2-form ω is defined by the equation $\omega(X,Y) = g(X,JY)$.

THEOREM 3.1. *Let (M,J) be an almost complex manifold of complex dimension $n \geq 2$ (real dimension $2n$), and $f : M \to \mathbb{R}^{2n+1}$ a nondegenerate hypersurface immersion with a transversal vector field ξ. Suppose that the* *affine fundamental form g is invariant with respect to J. Then $R(X,Y)\omega = 0$ for any $X, Y \in T_mM$ and each $m \in M$ if and only if ∇ is flat.*

Proof. From (1), we have

$$\begin{aligned}(R(X,Y)\omega)(Z,W) &= -\omega(R(X,Y)Z,W) - \omega(Z,R(X,Y)W)\\ &= -g(R(X,Y)Z,JW) - g(Z,JR(X,Y)W)\\ &= -g(Y,Z)g(SX,JW) + g(X,Z)g(SY,JW)\\ &\quad + g(Y,W)g(SX,JZ) - g(X,W)g(SY,JZ)\end{aligned}$$

for any tangent vectors X, Y, Z and W. Taking the contraction with respect to Y and Z, we have

$$(-2n+2)g(SX,JW) + (\operatorname{tr} JS)g(X,W) = 0. \tag{5}$$

Again, taking the contraction, we obtain

$$(4n-2)\operatorname{tr} JS = 0.$$

The conditon $n \geq 2$ implies that $\operatorname{tr} JS = 0$. By (5), we have $g(SX,JW) = 0$ for any $X, W \in T_mM$. This completes the proof. ∎

We now define $d^\nabla J$ by

$$d^\nabla J(X,Y) = (\nabla_X J)Y - (\nabla_Y J)X$$

for any vectors X and Y tangent to M, and we define the *cubic form C* by

$$C(X,Y,Z) = (\nabla_X g)(Y,Z) + \tau(X)g(Y,Z)$$

for any vectors X, Y and Z. According to the Pick–Berwald theorem (see [12]), $f(M)$ is a hyperquadric in $\mathbb{R}^{2n+1}$ if the cubic form C vanishes identically.

LEMMA 3.2. *Let (M,J) be an almost complex manifold, and $f : M \to \mathbb{R}^{2n+1}$ a nondegenerate hypersurface immersion with a transversal vector field ξ. Suppose that the affine fundamental form g is invariant with respect to J. Then*

$$\begin{aligned}&(\nabla_Z\omega)(Y,JX) - (\nabla_X\omega)(Y,JZ)\\ &\qquad = \omega(Y, d^\nabla J(X,Z)) + \tau(Z)g(Y,X) - \tau(X)g(Y,Z)\end{aligned} \tag{6}$$

for any vector fields X, Y and Z.

Proof. By the definition of ω, we have

$$(\nabla_X\omega)(Y,Z) = (\nabla_X g)(Y,JZ) + g(Y,(\nabla_X J)Z), \tag{7}$$

from which

$$(\nabla_X g)(Y,Z) = -(\nabla_X\omega)(Y,JZ) - \omega(Y,(\nabla_X J)Z).$$

Thus, using (2), the Codazzi equation for g, we obtain

$$\begin{aligned} &-(\nabla_X\omega)(Y,JZ)-\omega(Y,(\nabla_X J)Z)+\tau(X)g(Y,Z)\\ &\qquad\qquad = -(\nabla_Z\omega)(Y,JX)-\omega(Y,(\nabla_Z J)X)+\tau(Z)g(Y,X). \end{aligned}$$

Since $d^\nabla J(X,Z) = (\nabla_X J)Z - (\nabla_Z J)X$, we have equation (6). ∎

PROPOSITION 3.3. *Let (M, J) be an almost complex manifold, and $f : M \to \mathbb{R}^{2n+1}$ a nondegenerate hypersurface immersion with a transversal vector field ξ. Suppose that the affine fundamental form g is invariant with respect to J and that $\nabla\omega = 0$. Then $d^\nabla J = 0$ if and only if $\tau = 0$, that is, ξ is an equiaffine transversal vector field.*

For any transversal vector field $\overline{\xi}$, there exists a vector field Z and a nonvanishing function ϕ on M that satisfy $\overline{\xi} = \phi\xi + f_*(Z)$. Let $\overline{g}$ be an affine fundamental form relative to $\overline{\xi}$. Then we have the following equation (see Nomizu and Sasaki [12, Proposition 2.5 in Chapter 2]):

$$\overline{g} = \frac{1}{\phi}\, g.$$

Thus we see that if the affine fundamental form g is invariant with respect to J, then so is $\overline{g}$. Hence we can choose ξ to be an equiaffine transversal vector field.

PROPOSITION 3.4. *Let (M, J) be an almost complex manifold, and $f : M \to \mathbb{R}^{2n+1}$ a nondegenerate hypersurface immersion with an equiaffine transversal vector field ξ. Suppose that the affine fundamental form g is invariant with respect to J. If $\nabla\omega = 0$, then $\widehat{\nabla} J = 0$. Consequently, (M, J, g) is a Kähler manifold.*

Proof. By the definition of the conjugate connection $\overline{\nabla}$, we get

$$\begin{aligned} Zg(X,Y) &= Zg(JX,JY)\\ &= g((\nabla_Z J)X, JY) + g(\nabla_Z X, Y)\\ &\quad + g(JX, (\overline{\nabla}_Z J)Y) + g(X, \overline{\nabla}_Z Y). \end{aligned}$$

So we have

$$g((\nabla_Z J)X, JY) = -g(JX, (\overline{\nabla}_Z J)Y) = -g(X, (\overline{\nabla}_Z J)JY),$$

from which

$$g(X, (\overline{\nabla}_Z J)Y) + g((\nabla_Z J)X, Y) = 0. \tag{8}$$

Since ξ is an equiaffine transversal vector field, the Levi-Civita connection $\widehat{\nabla}$ of g is given by $\widehat{\nabla} = \frac{1}{2}(\nabla + \overline{\nabla})$. Then (8) implies

$$g((\widehat{\nabla}_X J)Y, Z) = \tfrac{1}{2} g((\nabla_X J)Y, Z) - \tfrac{1}{2} g(Y, (\nabla_X J)Z). \tag{9}$$

Suppose that $\nabla\omega = 0$. Then (7) implies

$$0 = (\nabla_X g)(Y, JZ) - (\nabla_X g)(JZ, Y) = -g(Y, (\nabla_X J)Z) + g(Z, (\nabla_X J)Y). \tag{10}$$

From this and (9), we have $\widehat{\nabla} J = 0$. ■

Now we introduce the notion of special Kähler manifold (cf. [2]).

DEFINITION 3.5. Let (M, J, g) be a Kähler manifold with, possibly indefinite, Kähler metric g and Kähler form $\omega := g(\cdot, J\cdot)$. Let ∇ be a torsion-free, flat connection on M. If $\nabla\omega = 0$ and $d^\nabla J = 0$, then (M, J, g, ∇) is called a *special Kähler manifold*.

PROPOSITION 3.6. *Let (M, J) be an almost complex manifold. The almost complex structure J is integrable if and only if $d^\nabla J$ is invariant with respect to J, that is,*

$$d^\nabla J(JX, JY) = d^\nabla J(X, Y).$$

Proof. We denote by N the Nijenhuis tensor with respect to J. Since the induced connection ∇ is torsion-free, we have

$$\begin{aligned} N(X, Y) &= 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\} \\ &= 2\{(\nabla_{JX} J)Y - (\nabla_{JY} J)X - J(\nabla_X J)Y + J(\nabla_Y J)X\} \\ &= 2\{(\nabla_{JX} J)Y - (\nabla_Y J)JX + (\nabla_X J)JY - (\nabla_{JY} J)X\} \\ &= 2\{d^\nabla J(JX, Y) + d^\nabla J(X, JY)\}. \end{aligned}$$

Hence J is integrable if and only if $d^\nabla J(JX, JY) = d^\nabla J(X, Y)$. ■

From Theorem 3.1 and Propositions 3.3, 3.4 and 3.6, we have the next theorem.

THEOREM 3.7. *Let (M, J) be an almost complex manifold of complex dimension $n \geq 2$, and $f : M \to \mathbb{R}^{2n+1}$ a nondegenerate hypersurface immersion with an equiaffine transversal vector field ξ. Suppose that the affine fundamental form g is invariant with respect to J. If $\nabla\omega = 0$, then (M, J, g, ∇) is a special Kähler manifold.*

THEOREM 3.8. *Let (M, J) be an almost complex manifold, and $f : M \to \mathbb{R}^{2n+1}$ a nondegenerate hypersurface immersion with an equiaffine transversal vector field ξ. Suppose that the affine fundamental form g is invariant with respect to J. If $\nabla J = 0$, then $\widehat{\nabla} J = 0$ and the hypersurface $f(M)$ is a hyperquadric in $\mathbb{R}^{2n+1}$.*

Proof. By (8), if $\nabla J = 0$, then $\overline{\nabla} J = 0$. Since ξ is an equiaffine transversal vector field, we have $\widehat{\nabla} = \frac{1}{2}(\nabla + \overline{\nabla})$. Hence $\widehat{\nabla} J = 0$.

From the condition $\nabla J = 0$, we have

$$(\nabla_Z g)(X, JY) + (\nabla_Z g)(JX, Y) = 0.$$

Using the Codazzi equation for g, we get

$$\begin{aligned}(\nabla_Z g)(X, JY) &= -(\nabla_Z g)(JX, Y) = -(\nabla_Y g)(JX, Z)\\ &= (\nabla_Y g)(X, JZ) = (\nabla_X g)(Y, JZ)\\ &= -(\nabla_X g)(JY, Z) = -(\nabla_Z g)(X, JY).\end{aligned}$$

Hence, $f(M)$ is a hyperquadric in $\mathbb{R}^{2n+1}$ because the cubic form C vanishes identically. ∎

4. Conditions for an affine hypersurface to be flat. In this section, we give some conditions for the induced connection ∇ to be flat.

PROPOSITION 4.1. *Let (M, J) be an almost complex manifold of complex dimension $n \geq 2$, and $f : M \to \mathbb{R}^{2n+1}$ a nondegenerate hypersurface immersion with a transversal vector field ξ. Suppose that the affine fundamental form g is invariant with respect to J. The curvature tensor R of the induced connection ∇ satisfies*

$$R(JX, Y) = JR(X, Y)$$

for any X and Y if and only if ∇ is flat. Also, R satisfies

$$R(JX, JY) = R(X, Y)$$

for any X and Y if and only if ∇ is flat.

Proof. We only prove the sufficiency of both curvature assumptions. Under the first assumption, from the Gauss equation, we have

$$\begin{aligned}R(JX, Y)Z - JR(X, Y)Z = g(Y, Z)S(JX) - g(JX, Z)SY\\ -g(Y, Z)JSX + g(X, Z)JSY = 0.\end{aligned} \tag{11}$$

We can choose an orthogonal basis of the tangent space to M of the form $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n\}$. Setting $X = Y = Z = X_i$ in (11), we have $S(JX_i) = 0$ for $i = 1, \ldots, n$. Similarly, setting $X = Y = X_i$ and $Z = JX_i$ in (11), we obtain $SX_i = 0$ for $i = 1, \ldots, n$. Hence $S = 0$, which implies that ∇ is flat.

On the other hand, under the second assumption, we have

$$\begin{aligned}R(JX, JY)Z - R(X, Y)Z = g(JY, Z)S(JX) - g(JX, Z)SJY\\ - g(Y, Z)SX + g(X, Z)SY = 0.\end{aligned} \tag{12}$$

Setting $X = Z = X_i$ and $Y = X_j$ in (12), we have $SX_j = 0$ for $j = 1, \ldots, n$. Similarly, setting $X = Z = X_i$ and $Y = JX_j$, we obtain $SJX_j = 0$ for $j = 1, \ldots, n$. Therefore S vanishes identically, and the connection ∇ is flat. ∎

We define a connection ∇^J by $\nabla^J X = J\nabla(J^{-1}X) = \nabla X - J(\nabla J)X$ (see [2]). The curvature tensor, the Ricci tensor and the Weyl projective curvature tensor with respect to ∇^J will be denoted by R^J, $\mathrm{Ric}^{\mathrm{J}}$ and W^J, respectively.

THEOREM 4.2. *Let (M, J) be an almost complex manifold of complex dimension $n \geq 2$, and $f : M \to \mathbb{R}^{2n+1}$ a nondegenerate hypersurface immersion with a transversal vector field ξ. Suppose that the affine fundamental form g is invariant with respect to J. If ∇^J is projectively flat, then ∇ is flat.*

Proof. By the definition of R^J and the Gauss equation, we obtain

$$R^J(X, Y)Z = -g(Y, JZ)JSX + g(X, JZ)JSY.$$

So we have

$$\begin{aligned}
\mathrm{Ric}^J(Y, Z) &= -\mathrm{tr}(JS)g(Y, JZ) + g(JSY, JZ) \\
&= -\mathrm{tr}(JS)g(Y, JZ) + g(SY, Z), \\
\mathrm{tr}\, R^J(X, Y) &= \mathrm{tr}\{Z \mapsto R^J(X, Y)Z\} = g(Y, SX) - g(X, SY).
\end{aligned}$$

As Ric^J may not be symmetric, the Weyl projective curvature tensor of ∇^J is defined by (see [12])

$$\begin{aligned}
W^J(X, Y)Z = R^J(X, Y)Z &- \frac{1}{2n-1}\{\mathrm{Ric}^J(Y, Z)X - \mathrm{Ric}^J(X, Z)Y\} \\
&- \frac{1}{2n+1}\,\mathrm{tr}\, R^J(X, Y)Z \\
&- \frac{1}{4n^2-1}\{\mathrm{tr}\, R^J(Y, Z)X - \mathrm{tr}\, R^J(X, Z)Y\}.
\end{aligned}$$

Since ∇^J is projectively flat, the Weyl projective curvature tensor W^J vanishes identically. So we have

$$\begin{aligned}
W^J(X, Y)Z = &-g(Y, JZ)JSX + g(X, JZ)JSY \\
&- \frac{1}{2n+1}\{g(Y, SX)Z - g(X, SY)Z\} \\
&- \frac{1}{2n-1}\{-\mathrm{tr}(JS)g(Y, JZ)X + g(SY, Z)X\} \\
&+ \frac{1}{2n-1}\{-\mathrm{tr}(JS)g(X, JZ)Y + g(SX, Z)Y\} \\
&- \frac{1}{4n^2-1}\{g(Z, SY)X - g(Y, SZ)X \\
&- g(Z, SX)Y + g(X, SZ)Y\} = 0.
\end{aligned} \tag{13}$$

If X, Y, Z and JZ are orthogonal with respect to g, then

$$\begin{aligned}
0 = &-\frac{1}{2n-1}\, g(SY, Z)X + \frac{1}{2n-1}\, g(SX, Z)Y \\
&- \frac{1}{2n+1}\{g(Y, SX)Z - g(X, SY)Z\} \\
&- \frac{1}{4n^2-1}\{g(Z, SY)X - g(Y, SZ)X - g(Z, SX)Y + g(X, SZ)Y\}.
\end{aligned}$$

Since X, Y and Z are linearly independent, we have

$$g(SX,Y)=g(SY,X), \tag{14}$$

$$-\frac{2n+2}{4n^2-1}\,g(SY,Z)+\frac{1}{4n^2-1}\,g(Y,SZ)=0, \tag{15}$$

$$\frac{2n+2}{4n^2-1}\,g(SX,Z)-\frac{1}{4n^2-1}\,g(X,SZ)=0. \tag{16}$$

By (13), we obtain

$$g(SY,Z)=g(SZ,Y). \tag{17}$$

Equations (15) and (17) lead to $g(SY,Z)=0$. If X, Y and JY are orthogonal with respect to g, we have

$$W^J(X,Y)Y=-\frac{1}{2n-1}\,g(SY,Y)X=0.$$

So $g(SY,Y)=0$.

Choose an orthogonal basis $\{X_1,\dots,X_n,JX_1,\dots,JX_n\}$ with respect to g. We next prove $g(SX_i,JX_i)=0$ and $g(SJX_i,X_i)=0$ for $i=1,\dots,n$. By the above considerations, there are scalar functions a_i and b_i ($i=1,\dots,n$) such that $SX_i=a_iJX_i$ and $S(JX_i)=b_iX_i$. Then

$$\begin{aligned} W^J(X_j,X_i)JX_i &= -a_jg(X_i,X_i)X_j-\frac{1}{2n-1}\,\mathrm{tr}(JS)g(X_i,X_i)X_j \\ &\quad -\frac{1}{2n-1}\,a_ig(X_i,X_i)X_j-\frac{1}{4n^2-1}\,a_ig(X_i,X_i)X_j \\ &\quad +\frac{1}{4n^2-1}\,b_ig(X_i,X_i)X_j=0, \end{aligned} \tag{18}$$

$$\begin{aligned} W^J(X_j,JX_i)X_i &= a_jg(X_i,X_i)X_j+\frac{1}{2n-1}\,\mathrm{tr}(JS)g(X_i,X_i)X_j \\ &\quad -\frac{1}{2n-1}\,b_ig(X_i,X_i)X_j-\frac{1}{4n^2-1}\,b_ig(X_i,X_i)X_j \\ &\quad +\frac{1}{4n^2-1}\,a_ig(X_i,X_i)X_j=0. \end{aligned} \tag{19}$$

So we have $a_i+b_i=0$. By the definition of trace, we get

$$\begin{aligned} \mathrm{tr}(JS) &= \sum_{k=1}^{n} g(JSX_k,X_k)g(X_k,X_k) \\ &\quad +\sum_{s=1}^{n} g(JS(JX_k),JX_k)g(JX_k,JX_k) \\ &= -\sum_{k=1}^{n} a_k+\sum_{s=1}^{n} b_s=-2\sum_{k=1}^{n} a_k. \end{aligned} \tag{20}$$

Substituting (20) into (18), for any j such that $j\neq i$, we have

$$a_j + \frac{2n+3}{4n^2-1}\,a_i + \frac{2}{2n-1}\sum_{k=1}^{n} a_k = 0. \tag{21}$$

If $i \neq l$ and $j \neq l$, we have

$$a_l + \frac{2n+3}{4n^2-1}\,a_i + \frac{2}{2n-1}\sum_{k=1}^{n} a_k = 0. \tag{22}$$

Hence $a_1 = \cdots = a_n$. By (21), we conclude that $a_i = 0$ $(i = 1, \ldots, n)$. Consequently, $S = 0$, and ∇ is flat. So we have proved the theorem. ∎

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