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# GRADED BLOCKS OF GROUP ALGEBRAS WITH DIHEDRAL DEFECT GROUPS

BY

DUSKO BOGDANIC (Oxford)

**Abstract.** We investigate gradings on tame blocks of group algebras whose defect groups are dihedral. For this subfamily of tame blocks we classify gradings up to graded Morita equivalence, we transfer gradings via derived equivalences, and we check the existence, positivity and tightness of gradings. We classify gradings by computing the group of outer automorphisms that fix the isomorphism classes of simple modules.

**1. Introduction.** In this paper we study gradings on tame blocks of group algebras of finite groups. Erdmann classified tame blocks of group algebras up to Morita equivalence (cf. [8]). A block of a group algebra over a field of characteristic $p$ is of tame representation type if and only if $p = 2$ and its defect group is a dihedral, semidihedral, or generalized quaternion group. If the defect group of a block is a dihedral (respectively semidihedral, quaternion) group, then we say that the block is of dihedral (respectively semidihedral, quaternion) type. The number of simple modules in a tame block is 1, 2 or 3 (see [8] for more details). Erdmann's classification has been used by Holm [9] to classify tame blocks up to derived equivalence (the case of blocks with dihedral defect groups and three simple modules has been dealt with by Linckelmann in [15]). We will follow Erdmann's and Holm's classification, and use some of the tilting complexes given in [9] and [15] to transfer gradings via derived equivalences in order to prove the existence of non-trivial gradings on an arbitrary dihedral block.

As in the case of Brauer tree algebras (cf. [4]), we classify gradings up to graded Morita equivalence by computing the group of outer automorphisms that fix the isomorphism classes of simple modules. From our computation of these groups we are able to deduce that, in the case of dihedral blocks with two simple modules, for different scalars (which remain undetermined in Erdmann's classification) we get algebras that are not derived equivalent. This is a well known result, first proven in [13], but our proof is more elementary.

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The paper is organized as follows. In the second section we list some preliminary results that will be used throughout this paper. This section contains a classification criterion, and a criterion for tightness and positivity of gradings. In the third section we investigate gradings on dihedral blocks with three simple modules. In the fourth section we investigate gradings on dihedral blocks with two simple modules. The fifth section is devoted to dihedral blocks with one simple module.

**1.1. Notation.** Throughout this text, $k$ will be an algebraically closed field of characteristic 2. All algebras will be finite-dimensional algebras over the field $k$, and all modules will be left modules. The category of finite-dimensional $A$-modules is denoted by $A$-mod and the full subcategory of finite-dimensional projective $A$-modules is denoted by $P_A$. The derived category of bounded complexes over $A$-mod is denoted by $D^b(A)$, and the homotopy category of bounded complexes over $P_A$ will be denoted by $K^b(P_A)$.

**1.1.1.** *Graded modules.* We say that an algebra $A$ is a *graded algebra* if $A$ is the direct sum of subspaces, $A = \bigoplus_{i\in\mathbb{Z}} A_i$, such that $A_iA_j \subset A_{i+j}$ for all $i, j \in \mathbb{Z}$. If $A_i = 0$ for $i < 0$, we say that $A$ is *positively graded*. An $A$-module $M$ is *graded* if it is the direct sum of subspaces, $M = \bigoplus_{i\in\mathbb{Z}} M_i$, such that $A_iM_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. If $M$ is a graded $A$-module, then $N = M\langle i\rangle$ denotes the graded module given by $N_j = M_{i+j}$, $j \in \mathbb{Z}$. An $A$-module homomorphism $f$ between two graded modules $M$ and $N$ is a *homomorphism of graded modules* if $f(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}$. For a graded algebra $A$, we denote by $A$-modgr the category of graded finite-dimensional $A$-modules. We set $\mathrm{Homgr}_A(M, N) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{A\text{-gr}}(M, N\langle i\rangle)$, where $\mathrm{Hom}_{A\text{-gr}}(M, N\langle i\rangle)$ denotes the space of all graded homomorphisms between $M$ and $N\langle i\rangle$ (the space of *homogeneous morphisms of degree* $i$). There is an isomorphism of vector spaces $\mathrm{Hom}_A(M, N) \cong \mathrm{Homgr}_A(M, N)$ that gives us a grading on $\mathrm{Hom}_A(M, N)$ (cf. [16, Corollary 2.4]).

**1.1.2.** *Graded complexes.* Let $X = (X^i, d^i)$ be a complex of $A$-modules. We say that $X$ is a *complex of graded $A$-modules*, or just a *graded complex*, if for each $i \in \mathbb{Z}$, $X^i$ is a graded module and $d^i$ is a homomorphism between graded $A$-modules. If $X$ is a graded complex, then $X\langle j\rangle$ denotes the complex of graded $A$-modules given by $(X\langle j\rangle)^i := X^i\langle j\rangle$ and $d^i_{X\langle j\rangle} := d^i$. Let $X$ and $Y$ be graded complexes. A homomorphism $f = \{f^i\}_{i\in\mathbb{Z}}$ between complexes $X$ and $Y$ is a *homomorphism of graded complexes* if for each $i \in \mathbb{Z}$, $f^i$ is a homomorphism of graded modules. The category of complexes of graded $A$-modules will be denoted by $C_{\mathrm{gr}}(A)$. We set $\mathrm{Homgr}_A(X, Y) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{C_{\mathrm{gr}}(A)}(X, Y\langle i\rangle)$, where $\mathrm{Hom}_{C_{\mathrm{gr}}(A)}(X, Y\langle i\rangle)$

denotes the space of graded homomorphisms between $X$ and $Y\langle i\rangle$ (the space of *homogeneous morphisms of degree* $i$). As for modules, we have an isomorphism of vector spaces $\mathrm{Homgr}_A(X,Y) \cong \mathrm{Hom}_A(X,Y)$ that gives us a grading on $\mathrm{Hom}_A(X,Y)$. From this we get a grading on $\mathrm{Hom}_{K^b(A\text{-mod})}(X,Y)$, since the subspace of zero homotopic maps is homogeneous. We denote this graded space by $\mathrm{Homgr}_{K^b(A\text{-mod})}(X,Y)$.

Unless otherwise stated, for a graded algebra $A$ given by a quiver and relations, we will assume that the projective indecomposable $A$-modules are graded as in Example 2.8 below, i.e. we will assume that their tops are in degree 0 (we recommend [2] as a good introduction to path algebras of quivers). We note here that if we have two different gradings on an indecomposable module (bounded complex), then they differ only by a shift (cf. [3, Lemma 2.5.3]).

## 2. Preliminaries

**2.1. Derived equivalences.** We say that two symmetric algebras $A$ and $B$ are *derived equivalent* if their derived categories of bounded complexes are equivalent. From Rickard's theory we know that $A$ and $B$ are derived equivalent if and only if there exists a tilting complex $T$ of projective $A$-modules such that $\mathrm{End}_{K^b(P_A)}(T) \cong B^{\mathrm{op}}$. For more details on derived categories and derived equivalences we recommend [14].

We remind the reader that derived equivalent algebras share many common properties. Among these is the identity component $\mathrm{Out}^0(A)$ of the group of outer automorphisms (cf. [11, Theorem 17] or [17, Theorem 4.6]).

**2.2. Algebraic groups and a classification criterion.** For a finite-dimensional $k$-algebra $A$, there is a correspondence between gradings on $A$ and homomorphisms of algebraic groups from $\mathbf{G}_m$ to $\mathrm{Aut}(A)$, where $\mathbf{G}_m$ is the multiplicative group $k^*$ of the field $k$. For each grading $A = \bigoplus_{i\in\mathbb{Z}} A_i$ there is a homomorphism of algebraic groups $\pi : \mathbf{G}_m \to \mathrm{Aut}(A)$ where an element $x \in k^*$ acts on $A_i$ by multiplication by $x^i$ (see [17, Section 5]). If $A$ is graded and $\pi$ is the corresponding homomorphism, we will write $(A,\pi)$ to denote that $A$ is graded with grading $\pi$.

Definition 2.1. Let $(A,\pi)$ and $(A,\pi')$ be two gradings on a finite-dimensional $k$-algebra $A$, and let $S_1,\dots,S_r$ be the isomorphism classes of simple $A$-modules. We say that $(A,\pi)$ and $(A,\pi')$ are *graded Morita equivalent* if there exist integers $d_{ij}$, where $1 \le j \le \dim S_i$ and $1 \le i \le r$, such that the graded algebras $(A,\pi')$ and $\mathrm{Endgr}_{(A,\pi)}(\bigoplus_{i,j} P_i\langle d_{ij}\rangle)^{\mathrm{op}}$ are isomorphic, where $P_i$ denotes the projective cover of $S_i$.

Note that two graded algebras $(A, \pi)$ and $(A, \pi')$ are graded Morita equivalent if and only if their categories of graded modules are equivalent (see [17, Section 5].

Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a grading on $A$. If $r \in \mathbb{Z}$, then $A = \bigoplus_{i \in \mathbb{Z}} B_i$, where $B_{ri} := A_i$, $i \in \mathbb{Z}$, and $B_i := 0$ if $r \nmid i$, is a grading on $A$. This procedure of multiplying (or dividing) each degree by the same integer is called *rescaling*.

We now give some background on algebraic groups (more details can be found in [5]). An *algebraic torus* is a linear algebraic group isomorphic to $\mathbf{G}_m^n = \mathbf{G}_m \times \cdots \times \mathbf{G}_m$ ($n$ factors) for some $n \geq 1$. A *maximal torus* in an algebraic group $G$ is a closed subgroup of $G$ which is a torus but is not contained in any larger torus. Tori are contained in $G^0$, the connected component of $G$ that contains the identity element. For a given torus $T$, a *cocharacter* of $T$ is a homomorphism of algebraic groups from $\mathbf{G}_m$ to $T$. A cocharacter of an algebraic group $G$ is a homomorphism of algebraic groups from $\mathbf{G}_m$ to $T$, where $T$ is a maximal torus of $G$. We say that cocharacters $\pi$ and $\pi'$ of $G$ are *conjugate* if there exists $g \in G$ such that $\pi'(x) = g\pi(x)g^{-1}$ for all $x \in \mathbf{G}_m$. We see that a grading on a finite-dimensional algebra $A$ can be seen as a cocharacter $\pi : \mathbf{G}_m \to \mathrm{Aut}(A)$. We will use the same letter $\pi$ to denote the corresponding cocharacter of $\mathrm{Out}(A)$, which is given by composition of $\pi$ and the canonical surjection.

The following proposition tells us how to classify all gradings on $A$ up to graded Morita equivalence.

Proposition 2.2 ([17, Corollary 5.9]). *Two basic graded algebras $(A, \pi)$ and $(A, \pi')$ are graded Morita equivalent if and only if the corresponding cocharacters $\pi : \mathbf{G}_m \to \mathrm{Out}(A)$ and $\pi' : \mathbf{G}_m \to \mathrm{Out}(A)$ are conjugate.*

From this proposition we see that in order to classify gradings on $A$ up to graded Morita equivalence, we need to compute maximal tori in $\mathrm{Out}(A)$. Let $\mathrm{Out}^K(A)$ be the subgroup of $\mathrm{Out}\,(A)$ of those automorphisms fixing the isomorphism classes of simple $A$-modules. Since $\mathrm{Out}^K(A)$ contains $\mathrm{Out}^0(A)$, the connected component of $\mathrm{Out}(A)$ that contains the identity element, we see that maximal tori in $\mathrm{Out}(A)$ are actually contained in $\mathrm{Out}^K(A)$. It follows that it is sufficient to compute maximal tori in $\mathrm{Out}^K(A)$.

Lemma 2.3. *Let $A$ be a basic finite-dimensional algebra such that the maximal tori in $\mathrm{Out}(A)$ are isomorphic to $\mathbf{G}_m$. Up to graded Morita equivalence and rescaling there is a unique grading on $A$.*

*Proof.* We saw at the beginning of this section that gradings on $A$ correspond to cocharacters of $\mathrm{Aut}(A)$. If $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a grading on $A$, then the corresponding cocharacter is given by the action of $x$ on $A_i$ by $x * a_i = x^i a_i$, where $a_i \in A_i$. Let $T$ and $T'$ be two maximal tori in $\mathrm{Out}(A)$. Let $\tau$ be a cocharacter of $\mathrm{Out}(A)$ such that its image is contained in $T'$. Since any two maximal tori in $\mathrm{Out}(A)$ are conjugate, there exists an invertible element $a$ such that $aT'a^{-1} = T$. The cocharacter given by $x \mapsto a\tau(x)a^{-1}$, $x \in \mathbf{G}_m$, is conjugate to $\tau$ and its image is contained in $T$. This cocharacter gives rise to a grading which is graded Morita equivalent to the grading given by $\tau$. It follows that when classifying gradings on $A$ up to graded Morita equivalence it is sufficient to consider cocharacters whose image is in $T$. The only homomorphisms from $\mathbf{G}_m$ to $\mathbf{G}_m \cong T$ are given by maps $x \mapsto x^r$ for $x \in \mathbf{G}_m$ and $r \in \mathbb{Z}$. Let $\pi : \mathbf{G}_m \to \mathrm{Out}(A)$, $x \mapsto x^l$, be the cocharacter that corresponds to the grading $A = \bigoplus_{i\in\mathbb{Z}} A_i$. If we rescale this grading by multiplying by $r \in \mathbb{Z}$, then we get the grading $A = \bigoplus_{i\in\mathbb{Z}} B_i$, where $B_{ri} := A_i$, $i \in \mathbb{Z}$, and $B_i := 0$ if $r \nmid i$. This grading corresponds to the cocharacter $\pi_1 : \mathbf{G}_m \to \mathrm{Out}(A)$, $x \mapsto x^{rl}$. This is easily seen if one thinks of the action of $x \in \mathbf{G}_m$ on $B_{ri}$. If $b_{ri} \in B_{ri}$, then $b_{ri} = a_i$, $a_i \in A_i$. The action of $x$ is given by

$$\pi_1(x)(b_{ri}) = x * b_{ri} = x^{ri} b_{ri} = x^{ri} a_i = (\pi(x))^r(a_i).$$

We see that the grading corresponding to the cocharacter $x \mapsto x^r$, $r \in \mathbb{Z}$, can be obtained by rescaling by $r$ from the grading corresponding to the cocharacter $x \mapsto x$. It follows that there is a unique grading on $A$ up to rescaling (dividing or multiplying each degree by the same integer) and graded Morita equivalence (shifting each projective indecomposable module by an integer). ∎

### 2.3. A criterion for tightness and positivity

PROPOSITION 2.4. *Let $A = \bigoplus_{i\geq 0} A_i$ be a positively graded algebra. Let $e$ and $f$ be homogeneous primitive idempotents such that $Ae \cong Af$. Then $Ae$ and $Af$ are isomorphic as graded $A$-modules.*

*Proof.* The modules $Ae = \bigoplus_{i\geq 0} A_i e$ and $Af = \bigoplus_{i\geq 0} A_i f$ are positively graded. Since $Ae \cong Af$, there exists an invertible element $a$ such that $aea^{-1} = f$. If $a_0$ is the degree 0 component of $a$, then $a_0 e a_0^{-1} = f$. Right multiplication by $a_0$ is an isomorphism between the graded modules $Af$ and $Ae$. ∎

EXAMPLE 2.5. Let $A$ be a positively graded algebra and let $P$ be a projective indecomposable $A$-module. There is a canonical way to grade $P$ as follows. Let $\{f_1, \dots, f_r\}$ be a complete set of primitive orthogonal idempotents. If $e_i$ is the degree 0 component of $f_i$, then by comparing degree 0 components of $f_i^2 = f_i$, we conclude that $e_i$ is a primitive idempotent. Hence, $\{e_1, \dots, e_r\}$ is a complete set of primitive orthogonal idempotents and $A = \bigoplus_{i=1}^r Ae_i$ is a sum of graded modules. The projective indecomposable module $P$ is isomorphic to $Ae_i$ for some $i$. This gives us a grading on $P$, which by the previous proposition does not depend on the choice of the idempotent $e_i$. It follows that every projective $A$-module is graded as a direct sum of graded modules.

DEFINITION 2.6. Let $A$ be a graded algebra. An ideal $I$ of $A$ is called *homogeneous* if it is generated by homogeneous elements.

LEMMA 2.7. *Let $A$ be a graded algebra and let $I$ be a homogeneous ideal of $A$. Then $A/I$ is a graded algebra.*

*Proof.* We define $(A/I)_i := (A_i + I)/I$. ∎

EXAMPLE 2.8. Let $A$ be a finite-dimensional algebra given by the quiver $Q$ and the ideal of relations $I$, i.e. $A = kQ/I$, where $I$ is an admissible ideal of $kQ$. The algebra $kQ$ is generated, as an algebra, by the vertices and arrows of $Q$. In order to grade $kQ$ it is sufficient to define the degrees of the arrows since the vertices of $Q$ will be in degree 0. In order to grade $kQ/I$ it is sufficient to ensure that $I$ is a homogeneous ideal of $kQ$. In other words, if for each relation $\sum_i \lambda_i \alpha_i = 0$ from a generating set of $I$, where the $\lambda$'s are scalars and the $\alpha$'s are paths in $Q$ with the same source and the same target, we have $\deg(\alpha_i) = \deg(\alpha_j)$ for all $i, j$, then $I$ is generated by homogeneous elements.

Let us assume that $A = kQ/I$ is graded in such a way that the arrows and the vertices of $Q$ are homogeneous, and that $I$ is a homogeneous ideal of $kQ$. Let $Ae$ be the projective indecomposable module that corresponds to a vertex $e$ of the quiver $Q$. Then $Ae$ is graded in a natural way as follows. As a vector space

$$Ae = \mathrm{Span}\{\rho \mid \rho \text{ is a path ending at } e\}.$$

If the degree of a path $\rho$ ending at the vertex $e$ is $l$ in $kQ$, then the degree of the 1-dimensional subspace of $Ae$ corresponding to $\rho$ is $l$. This gives us a canonical way to grade projective indecomposable $A$-modules even if the grading on $A$ is not positive.

DEFINITION 2.9 ([6, Section 4]). Let $\mathrm{gr}_{\mathrm{rad}\,A}(A)$ be the graded algebra given by the radical filtration on a $k$-algebra $A$. We say that $A$ is a *tightly graded algebra* if there is an algebra isomorphism

$$A \cong \mathrm{gr}_{\mathrm{rad}\,A}(A).$$

By Proposition 4.4 in [6], $A$ is a tightly graded algebra if and only if there exists a positive grading $A = \bigoplus_{i\geq 0} A_i$ such that $A_0$ is semisimple, and $A$ is generated, as an algebra, by $A_0$ and $A_1$. Such a grading is called *tight*.

We remark here that if $A$ is an algebra given by a quiver $Q$ and an ideal of relations $I$, where the generators of the ideal $I$ are linear combinations of paths of the same length, then $A$ is tightly graded. In this case, if we define that the arrows of $Q$ are in degree 1, then $I$ is a homogeneous ideal of $kQ$.

LEMMA 2.10. *Let $A = \bigoplus_{i\geq 0} A_i$ be a tight grading on a $k$-algebra $A$. If $a$ is an invertible element in $A$, then*

$$A = \bigoplus_{i\geq 0} aA_ia^{-1}$$

*is a tight grading on $A$.*

*Proof.* This is obvious. ■

LEMMA 2.11. *Let $A = \bigoplus_{i\geq 0} A_i$ be a tight grading on $A$. If $A_{\geq i} := \bigoplus_{j\geq i} A_j$, then*

$$\operatorname{rad}^i A = A_{\geq i},$$

*and $A_0$ is a maximal semisimple subalgebra of $A$.*

*Proof.* Since $A$ is an artinian algebra, $A_{\geq 1}$ is a nilpotent ideal. Therefore, $A_{\geq 1} \subset \operatorname{rad} A$. Let $S$ be a maximal semisimple subalgebra of $A$ such that $A = S \oplus \operatorname{rad} A$. Any two maximal semisimple subalgebras of $A$ are conjugate (cf. [7, Theorem 6.2.1]), and hence have the same dimension. Because $A_0$ is a semisimple subalgebra, the dimension argument gives us that $A_0$ is a maximal semisimple subalgebra and that $A_{\geq 1} = \operatorname{rad} A$. It follows easily that $A_{\geq i} = \operatorname{rad}^i A$ for $i \geq 1$. ■

LEMMA 2.12. *Let $A$ be an algebra given by a quiver $Q$ and an ideal of relations $I$. If $A$ is a tightly graded algebra, then there exists a tight grading on $A$ such that for every arrow $\alpha$ of the quiver $Q$, there exists a degree 1 element $t_\alpha$ of the form $\alpha + y_\alpha$, where $y_\alpha \in \operatorname{rad}^2 A$ is a linear combination of paths that have the same source and the same target as $\alpha$.*

*Proof.* Let us assume that $A = \bigoplus_{i\geq 0} A_i$ is a tight grading on $A$. From the previous lemma it follows that $A_0$ is a maximal semisimple subalgebra of $A$. Since any two maximal semisimple subalgebras are conjugate (cf. [7, Theorem 6.2.1]), by Lemma 2.10 we can assume that $A_0 = S$, where $S$ is the maximal semisimple subalgebra given by the linear span of the vertices of $Q$. Let $\alpha$ be an arrow of the quiver $Q$. By the previous lemma, $\operatorname{rad} A = A_1 \oplus A_{\geq 2}$. It follows that the arrow $\alpha$ can be written as a linear combination

$$\alpha = t + y,$$

where $t$ is a homogeneous element of degree 1, and $y \in \operatorname{rad}^2 A$ is a linear combination of homogeneous elements of degree greater than 1. Because vertices are homogeneous, we can multiply this equation from the left by $e_s$, the source vertex of $\alpha$, and from the right by $e_t$, the target vertex of $\alpha$. It follows that

$$\alpha = e_ste_t + e_sye_t.$$

If we define $t_\alpha := e_s t e_t$ and $y_\alpha := -e_s y e_t$, then $t_\alpha = \alpha + y_\alpha$ is a degree 1 element, and $y_\alpha$ is a linear combination of paths that have the same source and the same target as the arrow $\alpha$. ∎

REMARK 2.13. The previous lemma can be used to prove that certain algebras are not tightly graded, as in the following case.

We keep the notation of the previous lemma. Let us assume that one of the generators of $I$, say $v$, is a non-trivial linear combination of paths such that at least two of them are of different lengths. We can assume that $v = \sum_{i=1}^{r} \lambda_i p_i$, where $p_i$ is a path of length $s_i$, i.e. $p_i = \alpha_{i1}\alpha_{i2}\cdots\alpha_{is_i}$, where $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{is_i}$ are arrows of $Q$. If from the structure of $A$ it follows that $p_i = t_{\alpha_{i1}} t_{\alpha_{i2}} \cdots t_{\alpha_{is_i}}$, for all $p_i$, where $t_\alpha$ is as in the previous lemma, then $\deg(t_{\alpha_{i1}} t_{\alpha_{i2}} \cdots t_{\alpha_{is_i}}) = \deg(p_i) = s_i$. Without loss of generality, let us assume that $p_1, \ldots, p_m$ are paths of degree $s$, and that $p_{m+1}, \ldots, p_r$ are paths whose degree is greater than $s$. Then

$$\sum_{i=1}^{m} \lambda_i p_i = -\sum_{j=m+1}^{r} \lambda_j p_j.$$

Since the left-hand side of the above equality is a homogeneous element of degree $s$, and the right-hand side is a sum of homogeneous elements of degrees greater than $s$, we have a contradiction. Hence, $A$ is not tightly graded.

**2.4. The group** $\mathrm{Aut}(k[x]/(x^r))$**.** This group will play an important role in our classification of gradings on dihedral blocks. By Lemma 11.4 in [4], the group $\mathrm{Aut}(k[x]/(x^r))$ is isomorphic to the group $H_r$ from the following definition.

DEFINITION 2.14. We define $H_r$ to be the group

$$(k^* \times \underbrace{k \times \cdots \times k}_{r-1}, *),$$

where the multiplication $*$ is given by

$$\beta * \alpha := \Big(\sum_{i=1}^{l} \alpha_i \Big(\sum_{\substack{k_1+\cdots+k_i=l \\ k_1,\ldots,k_i>0}} \beta_{k_1}\cdots\beta_{k_i}\Big)\Big)_{l=1}^{r} \tag{2.1}$$

where $\beta = (\beta_1, \ldots, \beta_r)$ and $\alpha = (\alpha_1, \ldots, \alpha_r)$.

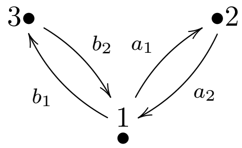
Let $L$ be the subgroup of $H_r$ consisting of the elements of the form $(1, \alpha_2, \ldots, \alpha_r)$ and let $K$ be the subgroup of $H_r$ consisting of the elements of the form $(\alpha_1, 0, \ldots, 0)$. The following proposition is straightforward.

PROPOSITION 2.15. *The group $H_r$ is a semidirect product of $L$ and $K$, where $L \trianglelefteq G$ is unipotent and the subgroup $K \cong \mathbf{G}_m$ is a maximal torus in $H_r$.*

**3. Three simple modules.** For background details about blocks of group algebras and their defect groups, we refer the reader to [1].

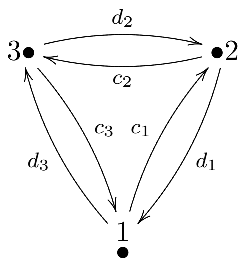
Any block with a dihedral defect group and three isomorphism classes of simple modules is Morita equivalent to some algebra from the following list (cf. [8] or [9]).

(1) For any $r \geq 1$, let $A_r$ be the algebra defined by the quiver and relations



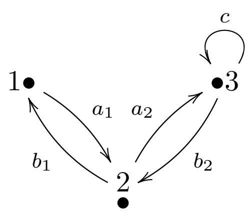

$$a_2a_1 = b_2b_1 = 0,$$
$$(a_1a_2b_1b_2)^r = (b_1b_2a_1a_2)^r.$$

(2) For any $r \geq 1$, let $B_r$ be the algebra defined by the quiver and relations




$$c_1c_2 = c_2c_3 = c_3c_1 = 0,$$
$$d_1d_3 = d_3d_2 = d_2d_1 = 0,$$
$$c_1d_1 = d_3c_3,$$
$$d_1c_1 = (c_2d_2)^r,\ c_3d_3 = (d_2c_2)^r.$$

(3) For any $r \geq 2$, let $C_r$ be the algebra defined by the quiver and relations




$$a_1b_1 = b_2a_2 = a_2c = cb_2 = 0,$$
$$c^r = b_2b_1a_1a_2,$$
$$a_2b_2b_1a_1 = b_1a_1a_2b_2.$$

For $r = 1$ we set $C_1 = A_1$.

**3.1. Classification of gradings.** We start by classifying all gradings up to graded Morita equivalence on $A_r$, $B_r$ and $C_r$. In order to do this we need to compute maximal tori in $\mathrm{Out}(A)$, where $A$ is $A_r$, $B_r$ or $C_r$. Linckelmann proved in [15] that for a fixed integer $r$, the algebras $A_r$, $B_r$ and $C_r$ are derived equivalent, i.e. any two blocks with the same dihedral defect group and with three isomorphism classes of simple modules are derived equivalent. Since $\mathrm{Out}^K(A)$, the group of outer automorphisms that fix the isomorphism classes of simple modules, contains $\mathrm{Out}^0(A)$, and since $\mathrm{Out}^0(A)$ is invariant under derived equivalence (cf. [17, Theorem 4.6] or [11, Theorem 17]), it is sufficient to compute $\mathrm{Out}^K(A)$ for one of these algebras. We will compute $\mathrm{Out}^K(C_r)$. Moreover, we will see that $\mathrm{Out}^K(C_r)$ and $\mathrm{Out}^0(C_r)$ are equal, because $\mathrm{Out}^K(C_r)$ will turn out to be connected.

Let $\varphi$ be an arbitrary automorphism of $C_r$ fixing the isomorphism classes of simple $C_r$-modules. The set $\{e_1, e_2, e_3\}$ of the vertices of the quiver of $C_r$ is a complete set of primitive orthogonal idempotents. Also, the set $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ is a complete set of primitive orthogonal idempotents. From classical ring theory (cf. [12, Theorem 3.10.2]) we know that there exists an invertible element $x$ such that $x^{-1}\varphi(e_i)x = e_{\sigma(i)}$ for all $i$, where $\sigma$ is some permutation. Since $\varphi$ fixes the isomorphism classes of simple modules we can assume that

$$\varphi(e_i) = e_i, \quad i = 1, 2, 3.$$

Since $\varphi(\operatorname{rad} C_r) \subseteq \operatorname{rad} C_r$ and $\varphi(e_i t e_j) = e_i \varphi(t) e_j$, for a given arrow $t$ in the quiver of $C_r$, $\varphi(t)$ is a linear combination of paths whose source is the source of $t$ and whose target is the target of $t$. It follows that

$$\begin{aligned}
\varphi(a_1) &= \alpha_1 a_1 + \beta_1 a_1 a_2 b_2,\\
\varphi(a_2) &= \alpha_2 a_2 + \beta_2 b_1 a_1 a_2,\\
\varphi(b_1) &= \alpha_3 b_1 + \beta_3 a_2 b_2 b_1,\\
\varphi(b_2) &= \alpha_4 b_2 + \beta_4 b_2 b_1 a_1,\\
\varphi(c) &= \sum_{i=1}^{r} \gamma_i c^i,
\end{aligned}$$

where the $\alpha$'s, $\beta$'s and $\gamma$'s are scalars. From $a_1 b_1 = 0$ and $b_2 a_2 = 0$ we conclude that $\alpha_1\beta_3 + \alpha_3\beta_1 = 0$ and $\alpha_4\beta_2 + \alpha_2\beta_4 = 0$. We note here that $\alpha_i \neq 0$ and $\gamma_1 \neq 0$ because $\varphi$ is injective.

We will now compose $\varphi$ with a suitable inner automorphism to get a nice representative of the class of $\varphi$ in $\mathrm{Out}^K(C_r)$ by eliminating $\beta_i$, $i = 1, 2, 3, 4$.

Let $y$ be an arbitrary invertible element in $C_r$. Then $y$ is of the form

$$y = l_1 e_1 + l_2 e_2 + l_3 e_3 + z,$$

where $l_1, l_2, l_3 \in k^*$ and $z \in \operatorname{rad} C_r$ is a linear combination of the remaining paths of strictly positive length. Its inverse $y^{-1}$ is $l_1^{-1}e_1 + l_2^{-1}e_2 + l_3^{-1}e_3 + z_1$, where $z_1 \in \operatorname{rad} C_r$ is easily computed from $yy^{-1} = 1$. Direct computation gives us that

$$ycy^{-1} = c.$$

Let $x := l_1 e_1 + l_2 e_2 + l_3 e_3 + l_4 b_1 a_1 + l_5 a_2 b_2$, where $l_1$, $l_2$ and $l_3$ are invertible, and where we set $l_4 := l_2 \alpha_2^{-1} \beta_2$ and $l_5 := l_2 \alpha_3^{-1} \beta_3$. The inner automorphism given by $x$ has the following action on a set of generators of $C_r$:

$$\begin{aligned}
xa_1x^{-1} &= l_1l_2^{-1}a_1 + l_1l_2^{-2}l_5a_1a_2b_2,\\
xa_2x^{-1} &= l_2l_3^{-1}a_2 + l_4l_3^{-1}b_1a_1a_2,\\
xb_1x^{-1} &= l_2l_1^{-1}b_1 + l_1^{-1}l_5a_2b_2b_1,\\
xb_2x^{-1} &= l_3l_2^{-1}b_2 + l_3l_2^{-2}l_4b_2b_1a_1,\\
xcx^{-1} &= c, \quad xe_ix^{-1} = e_i, \quad i = 1,2,3.
\end{aligned}$$

We denote by $f^x$ the inner automorphism given by this specific $x$, and we define $\varphi_1 := f^x \circ \varphi$. This is an element of $\mathrm{Out}^K(C_r)$ that is a nice class representative. Its action on our set of generators is given by

$$\begin{aligned}
\varphi_1(a_1) &= l_1l_2^{-1}\alpha_1a_1 + (\alpha_1l_1l_2^{-2}l_5 + \beta_1l_1l_2^{-1})a_1a_2b_2,\\
\varphi_1(a_2) &= l_2l_3^{-1}\alpha_2a_2 + (\alpha_2l_4l_3^{-1} + \beta_2l_2l_3^{-1})b_1a_1a_2,\\
\varphi_1(b_1) &= l_2l_1^{-1}\alpha_3b_1 + (\alpha_3l_1^{-1}l_5 + \beta_3l_2l_1^{-1})a_2b_2b_1,\\
\varphi_1(b_2) &= l_3l_2^{-1}\alpha_4b_2 + (\alpha_4l_3l_2^{-2}l_4 + \beta_4l_3l_2^{-1})b_2b_1a_1,\\
\varphi_1(e_i) &= e_i, \quad i = 1,2,3, \quad \varphi_1(c) = c.
\end{aligned}$$

We have chosen $l_4$ and $l_5$ in such a way that, in the above equations, the coefficients of the paths of length 3 are all equal to 0. The automorphism $\phi := f^w \circ \varphi_1$, where $f^w$ is the inner automorphism given by $w$, where $w := l_1^{-1}e_1 + l_2^{-1}e_2 + l_3^{-1}e_3$, represents the same class in $\mathrm{Out}^K(C_r)$ as $\varphi$. It has the following action on a set of algebra generators:

$$\begin{aligned}
\phi(e_i) &= e_i, \quad i = 1,2,3,\\
\phi(a_1) &= \alpha_1a_1, \quad \phi(b_1) = \alpha_3b_1,\\
\phi(a_2) &= \alpha_2a_2, \quad \phi(b_2) = \alpha_4b_2,\\
\phi(c) &= \sum_{i=1}^{r}\gamma_ic^i.
\end{aligned}$$

We see that the $(r+4)$-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma_1, \ldots, \gamma_r)$ completely determines $\phi$, where $\alpha_i$, $i = 1,2,3,4$, and $\gamma_1$ belong to $k^*$ and $\gamma_2, \ldots, \gamma_r \in k$. From the relations of $C_r$ we see that $\alpha_1\alpha_2\alpha_3\alpha_4 = \gamma_1^r$. It follows that an arbitrary element $\phi$ of $\mathrm{Out}^K(C_r)$ is determined by an $(r+3)$-tuple, say $(\alpha_1, \alpha_2, \alpha_3, \gamma_1, \ldots, \gamma_r)$, where $\alpha_4 = (\alpha_1\alpha_2\alpha_3)^{-1}\gamma_1^r$. Composition of homomorphisms induces a group operation on the set of $(r+3)$-tuples, i.e. on the set $k^* \times k^* \times k^* \times (k^* \times \underbrace{k \times \cdots \times k}_{r-1})$. This is componentwise multiplication on the first three coordinates and the operation $*$ of the group $H_r$ from Definition 2.14 on the remaining $r$ coordinates. In other words, we have the group $(k^*)^3 \times H_r$.

Any $(r+3)$-tuple $(\alpha_1, \alpha_2, \alpha_3, \gamma_1, \ldots, \gamma_r)$ gives rise to a representative of an element of $\mathrm{Out}^K(C_r)$, i.e. we have an epimorphism from $(k^*)^3 \times H_r$ onto $\mathrm{Out}^K(C_r)$. The above $(r+3)$-tuple gives us the same class in $\mathrm{Out}^K(C_r)$ as the $(r+3)$-tuple $(l_1 l_2^{-1}\alpha_1, l_2 l_1^{-1}\alpha_2, l_2 l_3^{-1}\alpha_3, \gamma_1, \ldots, \gamma_r)$, where $l_1, l_2$ and $l_3$ are arbitrary scalars from $k^*$. This corresponds to multiplication by an inner automorphism given by an invertible element $l_1e_1 + l_2e_2 + l_3e_3$. If we set $l_1 l_2^{-1} = w$ and $l_2 l_3^{-1} = v$, then $(k^*)^3 \times H_r/R$, where $R$ is the subgroup generated by all $(r+3)$-tuples of the form $(w, w^{-1}, v, 1, 0, \ldots, 0)$, where $v, w \in k^*$, is isomorphic to $\mathrm{Out}^K(C_r)$. This quotient is isomorphic to the direct product of one copy of the multiplicative group $k^*$ and a copy of the group $H_r$. Thus, we see that $\mathrm{Out}^K(C_r)$ is a connected algebraic group, and it follows that it is equal to $\mathrm{Out}^0(C_r)$.

Theorem 3.1. *Let $A$ be one of the algebras $A_r$, $B_r$ or $C_r$. Then*

$$\mathrm{Out}^0(A) \cong k^* \times H_r.$$

*The maximal tori in $\mathrm{Out}^0(A)$ are isomorphic to $\mathbf{G}_m \times \mathbf{G}_m$.*

*Proof.* This follows from the above discussion and the fact that $\mathrm{Out}^0(A)$ is preserved under derived equivalence. ∎

Corollary 3.2. *Let $A$ be one of the algebras $A_r$, $B_r$ or $C_r$. Let $T$ be a maximal torus in $\mathrm{Out}(A)$. Then up to graded Morita equivalence the gradings on $A$ are in one-to-one correspondence with conjugacy classes in $\mathrm{Out}(A)$ of cocharacters of $\mathrm{Out}(A)$ whose image is in $T$. Up to graded Morita equivalence the gradings on $A$ are parameterized by the corresponding subset of $\mathbb{Z}^2$.*

*Proof.* By Proposition 2.2, up to graded Morita equivalence the gradings on $A$ are given by conjugacy classes in $\mathrm{Out}(A)$ of the algebraic group homomorphisms from $\mathbf{G}_m$ to $\mathrm{Out}(A)$. Let $T'$ be another maximal torus in $\mathrm{Out}(A)$ and let $f$ be a cocharacter of $\mathrm{Out}(A)$ such that its image is contained in $T'$. Since any two maximal tori in $\mathrm{Out}(A)$ are conjugate, there exists an invertible element $a$ such that $aT'a^{-1} = T$. The cocharacter given by $x \mapsto af(x)a^{-1}$, $x \in \mathbf{G}_m$, is conjugate to $f$ and its image is contained in $T$. This cocharacter gives rise to a grading which is graded Morita equivalent to the grading given by $f$. It follows that when classifying gradings on $A$ up to graded Morita equivalence it is sufficient to consider cocharacters whose image is in $T$. Algebraic group homomorphisms from $\mathbf{G}_m$ to $T \cong \mathbf{G}_m \times \mathbf{G}_m$ are in one-to-one correspondence with $\mathbb{Z}^2$. ∎

Corollary 3.3. *Up to graded Morita equivalence the gradings on $C_r$, $r \geq 2$, are in one-to-one correspondence with $\mathbb{Z}^2$.*

*Proof.* From the relations of $C_r$ it follows that $\mathrm{Out}(C_r) = \mathrm{Out}^K(C_r)$. Let $T$ be the maximal torus in $\mathrm{Out}(C_r)$ consisting of the $(r+1)$-tuples of the form $(v, d_1, 0, \ldots, 0)$, where $v, d_1 \in k^*$. Let $\pi_1$ and $\pi_2$ be the cocharacters of

$T$ corresponding to the pairs of integers $(m_1, m_2)$ and $(n_1, n_2)$ respectively. If $\pi_1$ and $\pi_2$ are conjugate in $\mathrm{Out}(C_r)$, then from multiplication in $\mathrm{Out}(C_r)$ it follows that $m_1 = n_1$ and $m_2 = n_2$. ∎

REMARK 3.4. There are cases where the group of outer automorphisms of a given algebra $A$ strictly contains the group of outer automorphisms fixing the isomorphism classes of simple modules. In this case it is possible that the normalizer $N_{\mathrm{Out}(A)}(T)$ is not contained in $\mathrm{Out}^0(A)$, where $T$ is a maximal torus in $\mathrm{Out}(A)$.

For example, for the remaining two families $A_r$ and $B_r$, the group of outer automorphisms strictly contains the group of outer automorphisms fixing the isomorphism classes of simple modules. This is because there are outer automorphisms in $\mathrm{Out}(A)$, where $A$ is $A_r$ or $B_r$, that interchange $e_2$ and $e_3$, and fix $e_1$. Also, $\mathrm{Out}^K(A)$ is not necessarily connected, i.e. it is not equal to $\mathrm{Out}^0(A)$. In this case $N_{\mathrm{Out}(A)}(T)$ is not contained in $\mathrm{Out}^0(A)$, and for different pairs of integers we get gradings that are graded Morita equivalent. Thus, $A_r$ and $C_r$ are derived equivalent, but $N_{\mathrm{Out}(A_r)}(T) \not\cong N_{\mathrm{Out}(C_r)}(T')$, where $T$ and $T'$ are maximal tori.

This tells us that derived equivalent algebras, in general, do not have the same number of gradings up to graded Morita equivalence.

**3.2. Transfer of gradings via derived equivalences.** We will use derived equivalences between $A_r$, $B_r$ and $C_r$ to transfer gradings from $A_r$ to $B_r$ and $C_r$. The tilting complexes that we use in this section have been constructed by Linckelmann in [15].

We assume that $A_r$ is graded in such a way that the vertices and the arrows of the quiver of $A_r$ are homogeneous. Moreover, we assume that $\deg(a_1) = \alpha_1$, $\deg(a_2) = \alpha_2$, $\deg(b_1) = \beta_1$, $\deg(b_2) = \beta_2$. Also, we set $\Sigma := \alpha_1 + \alpha_2 + \beta_2 + \beta_2$.

By Example 2.8, the graded radical layers of the projective indecomposable $A_r$-modules with respect to this grading are:

$$\begin{array}{rccccl}
 & & & S_1 & & 0 \\
\alpha_2 & S_2 & & & S_3 & \beta_2 \\
\alpha_1+\alpha_2 & S_1 & & & S_1 & \beta_1+\beta_2 \\
\alpha_1+\alpha_2+\beta_2 & S_3 & & & S_2 & \beta_1+\beta_2+\alpha_2 \\
\Sigma & S_1 & & & S_1 & \Sigma \\
\vdots & & & \vdots & & \vdots \\
(r-1)\Sigma+\alpha_2 & S_2 & & & S_3 & (r-1)\Sigma+\beta_2 \\
r\Sigma-\beta_1-\beta_2 & S_1 & & & S_1 & r\Sigma-\alpha_1-\alpha_2 \\
r\Sigma-\beta_1 & S_3 & & & S_2 & r\Sigma-\alpha_1 \\
 & & & S_1 & & r\Sigma
\end{array}$$

$$\begin{array}{llcll}
S_2 & 0 & \quad & S_3 & 0\\
S_1 & \alpha_1 & & S_1 & \beta_1\\
S_3 & \alpha_1{+}\beta_2 & & S_2 & \beta_1{+}\alpha_2\\
S_1 & \alpha_1{+}\beta_1{+}\beta_2 & & S_1 & \beta_1{+}\alpha_1{+}\alpha_2\\
S_2 & \Sigma & & S_3 & \Sigma\\
\vdots & & & \vdots & \\
S_1 & (r{-}1)\Sigma{+}\alpha_1 & & S_1 & (r{-}1)\Sigma{+}\beta_1\\
S_3 & r\Sigma{-}\alpha_2{-}\beta_1 & & S_2 & r\Sigma{-}\alpha_1{-}\beta_2\\
S_1 & r\Sigma{-}\alpha_2 & & S_1 & r\Sigma{-}\beta_2\\
S_2 & r\Sigma & & S_3 & r\Sigma
\end{array}$$

Here, numbers to the left or right of the composition factors denote their degrees.

Let $T_1$ be the complex given by $T_1 : P_2\langle -\alpha_2\rangle \oplus P_3\langle -\beta_2\rangle \xrightarrow{(\gamma_2,\delta_2)} P_1$, where $P_1$ is in degree 1, and $\gamma_2, \delta_2$ are given by right multiplication by $a_2$ and $b_2$ respectively. Let $T_2$ and $T_3$ be the stalk complexes with $P_2$ and $P_3$ respectively in degree 0. A complex $T$ that tilts from $A_r$ to $B_r$ is given by the direct sum $T := T_1 \oplus T_2 \oplus T_3$.

Viewing $T$ as a graded object and calculating $\mathrm{Homgr}_{K^b(P_{A_r})}(T,T)$ as a graded vector space will give us a grading on $B_r$. It is clear that the following isomorphisms of graded vector spaces hold:

$$\mathrm{Homgr}_{K^b(P_{A_r})}(T_2,T_2) \cong \mathrm{Homgr}_{A_r}(P_2,P_2) \cong \bigoplus_{t=0}^{r} k\langle -t\Sigma\rangle,$$

$$\mathrm{Homgr}_{K^b(P_{A_r})}(T_3,T_3) \cong \mathrm{Homgr}_{A_r}(P_3,P_3) \cong \bigoplus_{t=0}^{r} k\langle -t\Sigma\rangle,$$

$$\mathrm{Homgr}_{K^b(P_{A_r})}(T_2,T_3) \cong \mathrm{Homgr}_{A_r}(P_2,P_3) \cong \bigoplus_{t=0}^{r-1} k\langle -(\beta_1+\alpha_2) - t\Sigma\rangle,$$

$$\mathrm{Homgr}_{K^b(P_{A_r})}(T_3,T_2) \cong \mathrm{Homgr}_{A_r}(P_3,P_2) \cong \bigoplus_{t=0}^{r-1} k\langle -(\beta_2+\alpha_1) - t\Sigma\rangle.$$

It follows that $\deg(d_2) = \alpha_1 + \beta_2$ and $\deg(c_2) = \beta_1 + \alpha_2$ in the quiver of $B_r$. Also, non-zero maps in $\mathrm{Homgr}_{K^b(P_{A_r})}(T_1,T_2)$ and $\mathrm{Homgr}_{K^b(P_{A_r})}(T_1,T_3)$ have to map surjectively $P_2 \oplus P_3$ onto $P_2$ and $P_3$ respectively. We conclude that $\mathrm{Homgr}_{K^b(P_{A_r})}(T_1,T_2) \cong k\langle\alpha_2\rangle$, and $\mathrm{Homgr}_{K^b(P_{A_r})}(T_1,T_3)$ $\cong k\langle\beta_2\rangle$. It follows that $\deg(c_1) = -\alpha_2$ and $\deg(d_3) = -\beta$ in the quiver of $B_r$.

Every non-zero map in $\mathrm{Homgr}_{K^b(P_{A_r})}(T_2, T_1)$ has to map $\operatorname{top} P_2$ onto $\operatorname{soc} P_2$. It follows that $\mathrm{Homgr}_{K^b(P_{A_r})}(T_2, T_1) \cong k\langle -\alpha_2 - r\Sigma\rangle$, and similarly we deduce that $\mathrm{Homgr}_{K^b(P_{A_r})}(T_3, T_1) \cong k\langle -\beta_2 - r\Sigma\rangle$. This implies that $\deg(c_3) = \beta_2 + r\Sigma$ and $\deg(d_1) = \alpha_2 + r\Sigma$.

From the above computation we get a grading on $B_r$. With respect to this grading, the graded quiver of $B_r$ is given by

$$\begin{tikzcd}
3 \arrow[rr, bend left, "\alpha_1+\beta_2"] \arrow[dr, bend right, "\beta_2+r\Sigma"'] & & 2 \arrow[ll, bend left, "\beta_1+\alpha_2"] \arrow[dl, bend left, "\alpha_2+r\Sigma"] \\
& 1 \arrow[ul, "-\beta_2"] \arrow[ur, "-\alpha_2"'] &
\end{tikzcd}$$

If we assume that we started with the tight grading on $A_r$, i.e. if we assume that the arrows of the quiver of $A_r$ are in degree 1, then the resulting graded quiver of $B_r$ is given by

$$\begin{tikzcd}
3 \arrow[rr, bend left, "2"] \arrow[dr, bend right, "4r+1"'] & & 2 \arrow[ll, bend left, "2"] \arrow[dl, bend left, "4r+1"] \\
& 1 \arrow[ul, "-1"] \arrow[ur, "-1"'] &
\end{tikzcd}$$

We remark here that the resulting grading on $B_r$ is not tight. Moreover, it is not a positive grading. This example tells us that when we transfer a tight (respectively positive) grading via derived equivalence, the resulting grading is not necessarily tight (respectively positive). We state this known fact in the following proposition.

Proposition 3.5. *Tightness and positivity of a grading are not preserved, in general, under the transfer of gradings via derived equivalence.*

Let us now assume that the algebra $B_r$ is graded in such a way that the vertices and the arrows of the quiver of $B_r$ are homogeneous. Furthermore, we assume that $\deg(c_1) = \gamma_1$, $\deg(c_2) = \gamma_2$, $\deg(c_3) = \gamma_3$, $\deg(d_1) = \delta_1$, $\deg(d_2) = \delta_2$ and $\deg(d_3) = \delta_3$. We set $\Sigma := \gamma_2 + \delta_2$.

The graded radical layers of the projective indecomposable $B_r$-modules are:

$$
\begin{array}{ccccccc}
 & & & S_1 & & & 0 \\
\delta_1 & S_2 & & & & S_3 & \gamma_3 \\
 & & & S_1 & & & r\Sigma
\end{array}
$$

$$
\begin{array}{cccccc}
 & & S_2 & & & 0 \\
 & & & & S_3 & \delta_2 \\
 & & & & S_2 & \Sigma \\
\gamma_1 & S_1 & & & S_3 & \Sigma+\delta_2 \\
 & & & & S_2 & 2\Sigma \\
 & & & & \vdots & \\
 & & & & S_3 & r\Sigma-\gamma_2 \\
 & & S_2 & & & r\Sigma
\end{array}
\qquad
\begin{array}{cccccc}
 & & S_3 & & & 0 \\
 & & & & S_2 & \gamma_2 \\
 & & & & S_3 & \Sigma \\
\delta_3 & S_1 & & & S_2 & \Sigma+\gamma_2 \\
 & & & & S_3 & 2\Sigma \\
 & & & & \vdots & \\
 & & & & S_2 & r\Sigma-\delta_2 \\
 & & S_3 & & & r\Sigma
\end{array}
$$

We will now transfer this grading from $B_r$ to $C_r$. Let $T_1$ and $T_3$ be the stalk complexes with $P_1$ and $P_3$ respectively in degree 0. Let $T_2$ be the complex

$$T_2 : P_1\langle -\gamma_1\rangle \oplus P_3\langle -\delta_2\rangle \xrightarrow{(\rho_1,\tau_2)} P_2,$$

where $P_2$ is in degree 1, and $\rho_1, \tau_2$ are given by right multiplication by $c_1$ and $d_2$ respectively. Define $T$ to be the direct sum $T := T_1 \oplus T_2 \oplus T_3$. The complex $T$ is a tilting complex for $B_r$ and $\mathrm{End}_{K^b(P_{B_r})}(T) \cong C_r^{\mathrm{op}}$.

As above, we conclude that the space $\mathrm{Homgr}_{K^b(P_{B_r})}(T_3, T_3)$ is isomorphic to $\bigoplus_{t=0}^{r} k\langle -t\Sigma\rangle$, $\mathrm{Homgr}_{K^b(P_{B_r})}(T_1, T_1)$ is isomorphic to $k\langle 0\rangle \oplus k\langle -r\Sigma\rangle$, $\mathrm{Homgr}_{K^b(P_{B_r})}(T_3, T_1) \cong k\langle -\gamma_3\rangle$, and $\mathrm{Homgr}_{K^b(P_{B_r})}(T_1, T_3) \cong k\langle -\delta_3\rangle$. It follows that $\deg(c) = \Sigma$ in the quiver of $C_r$. Since $\ker(\rho_1, \tau_2)$ contains two copies of $S_1$, one copy in degree $\delta_3 + \delta_2$ and one copy in degree $\gamma_1 + r\Sigma$, $\mathrm{Homgr}_{K^b(P_{B_r})}(T_1, T_2) \cong k\langle -(\delta_3+\delta_2)\rangle \oplus k\langle -(\gamma_1+r\Sigma)\rangle$. The same arguments give us that $\mathrm{Homgr}_{K^b(P_{B_r})}(T_3, T_2) \cong k\langle -(\gamma_1 + \gamma_3)\rangle \oplus k\langle -(\delta_2 + r\Sigma)\rangle$. Similarly, there are isomorphisms $\mathrm{Homgr}_{K^b(P_{B_r})}(T_2, T_3) \cong k\langle \delta_2\rangle \oplus k\langle \gamma_1 - \delta_3\rangle$, and $\mathrm{Homgr}_{K^b(P_{B_r})}(T_2, T_1) \cong k\langle \gamma_1\rangle \oplus k\langle \delta_2 - \gamma_3\rangle$.

Using these data and looking at the relations of $C_r$, we find that in the quiver of $C_r$, $\deg(a_1) = \delta_2 + \delta_3$, $\deg(a_2) = -\delta_2$, $\deg(b_1) = -\gamma_1$ and $\deg(b_2) = \gamma_1 + \gamma_3$. With respect to this grading, the graded quiver of $C_r$ is given by

$$
\begin{tikzcd}
1\bullet \arrow[dr, bend right, "\delta_2+\delta_3"'] & & \bullet 3 \arrow[dl, bend left, "\gamma_1+\gamma_3"] \arrow[loop above, "\Sigma"] \\
 & \underset{\bullet}{2} \arrow[ul, bend right, "-\gamma_1"'] \arrow[ur, bend left, "-\delta_2"] &
\end{tikzcd}
$$

The graded radical layers of the projective indecomposable $C_r$-modules are:

$$\begin{array}{ll} S_1 & 0 \\ S_2 & -\gamma_1 \\ S_3 & \gamma_3 \\ S_2 & \gamma_3-\delta_2 \\ S_1 & r\Sigma \end{array} \qquad \begin{array}{rcccl} & & S_2 & & 0 \\ \delta_2+\delta_3 & S_1 & & S_3 & \gamma_1+\gamma_3 \\ \delta_2+\delta_3-\gamma_1 & S_2 & & S_2 & \gamma_1+\gamma_3-\delta_2 \\ \delta_2+\delta_3+\gamma_3 & S_3 & & S_1 & \gamma_1+\gamma_3+\delta_3 \\ & & S_2 & & r\Sigma \end{array}$$

$$\begin{array}{rcccl} & & S_3 & & 0 \\ \Sigma & S_3 & & S_2 & -\delta_2 \\ 2\Sigma & S_3 & & S_1 & \delta_3 \\ & \vdots & & S_2 & \delta_3-\gamma_1 \\ (r-1)\Sigma & S_3 & & & \\ & & S_3 & & r\Sigma \end{array}$$

### 3.3. Positivity and tightness

Proposition 3.6. *Let $A$ be one of the algebras $A_r$, $B_r$ and $C_r$. Then $A$ can be positively graded.*

*Proof.* This follows directly from the relations of these algebras. For $A_r$ we can set that every arrow is in degree 1 and we will get homogeneous relations. For the algebra $B_r$, if $\deg(c_1) = \deg(d_1) = \deg(c_3) = \deg(d_3) = r$ and $\deg(c_2) = \deg(d_2) = 1$, then the relations of $B_r$ are homogeneous. If $\deg(c) = 4$, $\deg(a_1) = \deg(a_2) = \deg(b_1) = \deg(b_2) = r$, then the relations of $C_r$ are homogeneous. ∎

Proposition 3.7. *For every positive integer $r$, $A_r$ is a tightly graded algebra.*

*Proof.* From the proof of the previous proposition, if the vertices of the quiver of $A_r$ are in degree 0, and the arrows are in degree 1, then the ideal of relations of $A_r$ is homogeneous. Therefore, there exists a positive grading on $A$ such that the subalgebra of degree 0 elements is semisimple, and $A$ is generated by the homogeneous elements of degrees 0 and 1. ∎

Proposition 3.8. *The algebra $B_r$ is tightly graded if and only if $r = 1$.*

*Proof.* It is clear that $B_1$ is tightly graded. Let us assume that $B_r$ is tightly graded. By Lemma 2.12, for each arrow $a$ of the quiver of $B_r$, there exists a degree 1 element of the form $a + \sum_i \lambda_i z_i$, where $z_i \in \operatorname{rad}^2 A$ is a path with the same source and the same target as $a$. It follows that $c_1$, $c_3$, $d_1$ and $d_3$ are homogeneous elements of degree 1, since there are no other paths with the same source and the same target. Also, there are degree 1 elements of the form

$$t_{c_2} := c_2 + \sum_{i=1}^{r-1} \lambda_i c_2 (d_2 c_2)^i, \qquad t_{d_2} := d_2 + \sum_{i=1}^{r-1} \mu_i d_2 (c_2 d_2)^i,$$

where the $\lambda$'s and $\mu$'s are scalars. It follows that $(t_{c_2} t_{d_2})^r = (c_2 d_2)^r$ is a homogeneous element of degree $2r$. Since $(c_2 d_2)^r = d_1 c_1$ is a homogeneous element of degree 2, it follows that $r = 1$. ∎

PROPOSITION 3.9. *The algebra $C_r$ is tightly graded if and only if $r = 4$.*

*Proof.* If $r = 4$, then it is obvious that $C_r$ is tightly graded.

Let us assume that $C_r$, $r \geq 2$, is tightly graded. By Lemma 2.12, there are degree 1 elements of the form

$$\begin{aligned} t_{a_1} &:= a_1 + \lambda_1 a_1 a_2 b_2, \qquad t_{b_1} := b_1 + \lambda_3 a_2 b_2 b_1, \\ t_{a_2} &:= a_2 + \lambda_2 b_1 a_1 a_2, \qquad t_{b_2} := b_2 + \lambda_4 b_2 b_1 a_1, \\ t_c &:= c + \sum_{i=2}^{r} \mu_i c^i, \end{aligned}$$

where the $\lambda$'s and $\mu$'s are scalars. It follows that $b_2 b_1 a_1 a_2 = t_{b_2} t_{b_1} t_{a_1} t_{a_2}$ is a homogeneous element of degree 4. At the same time $b_2 b_1 a_1 a_2 = c^r = t_c^r$ is a homogeneous element of degree $r$. It follows that $r = 4$. ∎

We note here that from the previous propositions it follows that the existence of a tight grading is not preserved under derived equivalence, unlike under Morita equivalence (see Proposition 4.4 in [6]).

It is worth noting that for dihedral blocks with three simple modules, in every derived equivalence class there is at least one block that is positively graded and there is at least one block that is tightly graded. The same statement does not hold for all derived equivalence classes of tame blocks.

**4. Two simple modules.** Any block with a dihedral defect group and two isomorphism classes of simple modules is Morita equivalent to some algebra from the following list (cf. [8] or [9]).

(1) For any $r \geq 1$ and $c \in \{0, 1\}$ let $D(2A)^{r,c}$ be the algebra defined by the quiver and relations

$$\begin{tikzcd} 0\bullet \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta"] & \bullet 1 \arrow[l, bend left, "\gamma"] \end{tikzcd} \qquad \begin{aligned} &\gamma\beta = 0,\ \alpha^2 = c(\alpha\beta\gamma)^r, \\ &(\alpha\beta\gamma)^r = (\beta\gamma\alpha)^r. \end{aligned}$$

(2) For any $r \geq 1$ and $c \in \{0, 1\}$ let $D(2B)^{r,c}$ be the algebra defined by the quiver and relations

$$\begin{tikzcd} 0\bullet \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta"] & \bullet 1 \arrow[loop right, "\eta"] \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

$$\begin{aligned} &\beta\eta = \eta\gamma = \gamma\beta = 0, \\ &\alpha\beta\gamma = \beta\gamma\alpha, \\ &\alpha^2 = c(\alpha\beta\gamma),\ \gamma\alpha\beta = \eta^r. \end{aligned}$$

**4.1. Classification of gradings.** In [9], Holm proved that for fixed $r$ and $c$, the algebras $D(2A)^{r,c}$ and $D(2B)^{r,c}$ are derived equivalent. Since the identity component of the group of outer automorphisms is invariant under derived equivalence, it is sufficient to compute this group for $D(2B)^{r,c}$.

As before, for an arbitrary outer automorphism $\varphi$ in $\mathrm{Out}^K(D(2B)^{r,c})$, we will find a suitable automorphism that represents the same element as $\varphi$, but which is easy to work with.

We assume that $\varphi(e_i) = e_i$ for $i = 1, 2$. It follows that

$$\begin{aligned} &\varphi(\alpha) = a_1\alpha + a_2\beta\gamma + a_3\alpha\beta\gamma, \\ &\varphi(\beta) = b_1\beta + b_2\alpha\beta, \qquad \varphi(\gamma) = c_1\gamma + c_2\gamma\alpha, \\ &\varphi(\eta) = \sum_{i=1}^{r} d_i\eta^i, \end{aligned}$$

for some $a_i, b_i, c_i, d_i \in k$. From the relation $\gamma\beta = 0$ we get $b_1c_2 = b_2c_1$. From the relation $\eta^r = \gamma\alpha\beta$ it follows that $d_1^r = a_1b_1c_1$. Since $\varphi(\eta^r) \neq 0$, it follows that $d_1 \neq 0$. Hence, $a_1, b_1$ and $c_1$ are all non-zero. The inner automorphism given by $y$, where $y := l_1e_1 + l_2e_2 + l_3\alpha$ and $l_3 := l_1c_1^{-1}c_2$, when composed with $\varphi$ has the following action on a set of generators:

$$\begin{aligned} &y\varphi(e_i)y^{-1} = e_i, && y\varphi(\eta)y^{-1} = \sum_{i=1}^{r} d_i\eta^i, \\ &y\varphi(\gamma)y^{-1} = c_1l_2l_1^{-1}\gamma, && y\varphi(\beta)y^{-1} = b_1l_1l_2^{-1}\beta, \\ &y\varphi(\alpha)y^{-1} = a_1\alpha + a_2\beta\gamma + a_3\alpha\beta\gamma. \end{aligned}$$

Let $\phi$ be the composition of $y\varphi y^{-1}$ and the inner automorphism given by $l_1^{-1}e_1 + l_2^{-1}e_2$. Then $\phi$ represents the same element in $\mathrm{Out}^K(D(2B)^{r,c})$ as $\varphi$. Its action is given by

$$\begin{aligned} &\phi(e_i) = e_i, \qquad \phi(\eta) = \sum_{i=1}^{r} d_i\eta^i, \\ &\phi(\gamma) = c_1\gamma, \qquad \phi(\beta) = b_1\beta, \\ &\phi(\alpha) = a_1\alpha + a_2\beta\gamma + a_3\alpha\beta\gamma. \end{aligned}$$

It follows that an arbitrary automorphism in $\mathrm{Out}^K(D(2B)^{r,c})$ is completely determined by an $(r+5)$-tuple $(a_1, a_2, a_3, b_1, c_1, d_1, \ldots, d_r)$. By an elementary, but tedious calculation, one can show that it is not possible to eliminate the coefficients $a_2$ and $a_3$ by composing $\phi$ with inner automorphisms.

We have a map from the set of all $(r+5)$-tuples onto $\mathrm{Out}^K(D(2B)^{r,c})$. Composition of morphisms gives us the group multiplication on the set of all $(r+5)$-tuples.

From $d_1^r = a_1b_1c_1$ it follows that one of these four coefficients, say $a_1$, is determined by the remaining three.

If $c=0$, then there are no further restrictions on the coefficients of $\varphi$. In this case, $\varphi$ is determined by the $(r+4)$-tuple $(a_2, a_3, b_1, c_1, d_1, \dots, d_r)$, where $b_1, c_1, d_1 \in k^*$. Multiplication of these $(r+4)$-tuples is given by composition of the corresponding automorphisms, where we replace $a_1$ with $d_1^r(b_1c_1)^{-1}$. If $G$ is the group of all such $(r+4)$-tuples, then multiplication is given by

$$\begin{aligned}(a_2', a_3', b_1', c_1', \mathbf{d}') &* (a_2, a_3, b_1, c_1, \mathbf{d}) \\ &= (d_1^r(b_1c_1)^{-1}a_2' + a_2b_1'c_1', d_1^r(b_1c_1)^{-1}a_3' + a_3(d_1')^r, b_1b_1', c_1c_1', \mathbf{d}'\mathbf{d}),\end{aligned}$$

where $\mathbf{d} = (d_1, \dots, d_r)$ and $\mathbf{d}' = (d_1', \dots, d_r')$, and the product $\mathbf{d}'\mathbf{d}$ is the product of elements of the group $H_r$ from Definition 2.14.

Thus, we have a map from the group $G$ of all $(r+4)$-tuples onto the group $\mathrm{Out}^K(D(2B)^{r,c})$. The kernel of this epimorphism is given by the set of all $(r+4)$-tuples that correspond to inner automorphisms. Let $R$ be the subgroup of $G$ generated by all $(r+4)$-tuples that correspond to inner automorphisms. The $(r+4)$-tuple $(a_2, a_3, b_1, c_1, \mathbf{d})$ represents the same class in the quotient group $M := G/R$ as $(a_2, a_3, l_1l_2^{-1}b_1, l_1^{-1}l_2c_1, \mathbf{d})$, where $l_1, l_2 \in k^*$. In particular, if $l_1l_2^{-1} = c_1$, then the $(r+4)$-tuple $(a_2, a_3, b_1, c_1, \mathbf{d})$ represents the same element as the $(r+4)$-tuple $(a_2, a_3, b_1c_1, 1, \mathbf{d})$. If $v = b_1c_1$, then $M$ can be seen as the group consisting of $(r+3)$-tuples $(a_2, a_3, v, \mathbf{d})$, where multiplication is defined by

$$(a_2', a_3', v', \mathbf{d}') * (a_2, a_3, v, \mathbf{d}) = (d_1^r v^{-1}a_2' + a_2v', d_1^r v^{-1}a_3' + a_3(d_1')^r, vv', \mathbf{d}'\mathbf{d}).$$

PROPOSITION 4.1. *Let $M$ be as above and let $A$ be $D(2B)^{r,0}$ or $D(2A)^{r,0}$. There is an isomorphism of groups*

$$\mathrm{Out}^0(A) \cong M.$$

*The maximal tori in $\mathrm{Out}^0(A)$ are isomorphic to $\mathbf{G}_m \times \mathbf{G}_m$.*

*Proof.* From the above discussion it follows that $\mathrm{Out}^K(D(2B)^{r,c})$ is isomorphic to $M$. Because $\mathrm{Out}^K(D(2B)^{r,c})$ is connected, it is equal to the identity component $\mathrm{Out}^0(D(2B)^{r,c})$. The identity component of the group of outer automorphisms is invariant under derived equivalence. Hence, the first statement of the proposition is true.

The subgroup $L$ of $M$ which is generated by the $(r+3)$-tuples of the form $(a_2, a_3, 1, 1, d_2, \dots, d_r)$ is a normal subgroup of $M$. The subgroup $T$ of $M$ generated by the $(r+3)$-tuples of the form $(0, 0, v, d_1, 0, \dots, 0)$ is isomorphic to the quotient $M/L$. It follows that $M$ is isomorphic to the semidirect product $L \rtimes T$. The group $L$ is unipotent and the group $T$ is semisimple.

Since $T \cong \mathbf{G}_m \times \mathbf{G}_m$, it follows that the maximal tori in $\mathrm{Out}^K(D(2B)^{r,0})$ are isomorphic to $\mathbf{G}_m \times \mathbf{G}_m$. ∎

Corollary 4.2. *Let $A$ be one of the algebras $D(2B)^{r,0}$ or $D(2A)^{r,0}$. Let $T$ be a maximal torus in* $\mathrm{Out}(A)$. *Then up to graded Morita equivalence the gradings on $A$ are in one-to-one correspondence with conjugacy classes in* $\mathrm{Out}(A)$ *of cocharacters of* $\mathrm{Out}(A)$ *whose image is in $T$. Up to graded Morita equivalence the gradings on $A$ are parameterized by the corresponding subset of $\mathbb{Z}^2$.*

*Proof.* The proof is the same as the proof of Corollary 3.2. ∎

Corollary 4.3. *Up to graded Morita equivalence the gradings on $D(2B)^{r,0}$ are in one-to-one correspondence with $\mathbb{Z}^2$.*

*Proof.* It follows from the relations of $D(2B)^{r,0}$ that an arbitrary outer automorphism has to fix the vertices of the quiver of $D(2B)^{r,0}$. Hence, $\mathrm{Out}(D(2B)^{r,0}) = \mathrm{Out}^K(D(2B)^{r,0})$. Let $T$ be the maximal torus consisting of the $(r+4)$-tuples of the form $(0,0,v,d_1,0,\dots,0)$, where $v, d_1 \in k^*$. Let $\pi_1$ and $\pi_2$ be the cocharacters of $T$ corresponding to the pairs of integers $(m_1,m_2)$ and $(n_1,n_2)$ respectively. If $\pi_1$ and $\pi_2$ are conjugate in $\mathrm{Out}(D(2B)^{r,0})$, then from multiplication in $\mathrm{Out}(D(2B)^{r,0})$ it follows that $m_1 = n_1$ and $m_2 = n_2$. ∎

As in the case of three simple modules, the same remarks about the gradings on $D(2A)^{r,0}$ hold, since $\mathrm{Out}^K(D(2A)^{r,0})$ is not a connected group.

If $c = 1$ there is an additional restriction on the coefficients of $\varphi$ coming from the relation $\alpha^2 = \alpha\beta\gamma$. From this relation we have $a_1 = b_1c_1$. This implies that $b_1c_1 = \sqrt{d_1^r}$. It follows that one of these coefficients, say $b_1$, is determined by the remaining two. In this case $\varphi$ is determined by the $(r+3)$-tuple $(a_2,a_3,c_1,d_1,\dots,d_r)$. We have a map from the group $G$ of all $(r+3)$-tuples onto $\mathrm{Out}^K(D(2B)^{r,1})$. Multiplication in $G$ is the same as before, in this case we just set $b_1c_1 = \sqrt{d_1^r}$. The kernel of the above map is the subgroup $R$ generated by all $(r+3)$-tuples corresponding to inner automorphisms. It follows that in the quotient group $G/R$, the $(r+3)$-tuple $(a_2,a_3,c_1,d_1,\dots,d_r)$ represents the same element as the $(r+3)$-tuple $(a_2,a_3,1,d_1,\dots,d_r)$. We find that $G/R$ can be seen as the group consisting of $(r+2)$-tuples $(a_2,a_3,d_1,\dots,d_r)$, with multiplication given by

$$(a_2',a_3',\mathbf{d}') * (a_2,a_3,\mathbf{d}) = (\sqrt{d_1^r}a_2' + a_2\sqrt{(d_1')^r}, \sqrt{d_1^r}a_3' + a_3(d_1')^r, \mathbf{d}'\mathbf{d}).$$

Proposition 4.4. *Let $A$ be one of the algebras $D(2B)^{r,1}$ or $D(2A)^{r,1}$. Let $G$ and $R$ be as above. Then $\mathrm{Out}^0(A) \cong G/R$. The maximal tori in $\mathrm{Out}^0(A)$ are isomorphic to $\mathbf{G}_m$. Up to graded Morita equivalence and rescaling there is a unique grading on the algebra $A$.*

*Proof.* It is obvious that $\mathrm{Out}^K(D(2B)^{r,1})$ is connected, hence it is equal to its identity component $\mathrm{Out}^0(D(2B)^{r,1})$. That $\mathrm{Out}^0(A) \cong G/R$ follows from the above discussion and the fact that the identity component of the group of outer automorphisms is invariant under derived equivalence. It is easily verified that $G/R \cong L \rtimes T$, where $T$ is the subgroup of $G/R$ generated by all $(r+2)$-tuples of the form $(0, 0, d_1, 0, \ldots, 0)$, and $L$ is the subgroup generated by all $(r+2)$-tuples of the form $(a_2, a_3, 1, d_2, \ldots, d_r)$. It follows that the maximal tori are isomorphic to $\mathbf{G}_m$. By Lemma 2.3, there is a unique grading on $A$ up to graded Morita equivalence and rescaling. ∎

An easy corollary of our results is that for different values of the scalar $c$ we get algebras that are not derived equivalent. This statement follows from the fact that $\mathrm{Out}^0(A)$ is invariant under derived equivalence. On the other hand, $\mathrm{Out}^0(D(2B)^{r,0})$ and $\mathrm{Out}^0(D(2B)^{r,1})$ are not isomorphic because they do not have isomorphic maximal tori. Even though this is known (see Corollary 5.3 in [13], or Theorem 1.1 in [10]), we record it in the following corollary.

COROLLARY 4.5. *Let $C^{r,0}$ be one of the algebras $D(2A)^{r,0}$ or $D(2B)^{r,0}$, and let $C^{r,1}$ be one of the algebras $D(2A)^{r,1}$ or $D(2B)^{r,1}$. Then $C^{r,0}$ and $C^{r,1}$ are not derived equivalent.*

**4.2. Transfer of gradings via derived equivalences.** We will use tilting complexes given in [9] to transfer gradings from $D(2A)^{r,c}$ to $D(2B)^{r,c}$. Let us fix an integer $r$ and $c \in \{0, 1\}$, and assume that $D(2A)^{r,c}$ is graded in such a way that the vertices and the arrows of the quiver of $D(2A)^{r,c}$ are homogeneous. We assume that the arrows $\alpha, \beta$ and $\gamma$ of the quiver of $D(2A)^{r,c}$ are in degrees $d_1, d_2$ and $d_3$ respectively. We set $d := d_1 + d_2 + d_3$.

The graded radical layers of the projective indecomposable $D(2A)^{r,c}$-modules are:

$$
\begin{array}{rcccl}
 & & S_0 & & 0 \\
d_1 & S_0 & & S_1 & d_3 \\
d_1+d_3 & S_1 & & S_0 & d_2+d_3 \\
d & S_0 & & S_0 & d \\
\vdots & \vdots & & \vdots & \vdots \\
(r-1)d+d_1 & S_0 & & S_1 & (r-1)d+d_3 \\
rd-d_2 & S_1 & & S_0 & rd-d_1 \\
 & & S_0 & & rd
\end{array}
\qquad\qquad
\begin{array}{cl}
S_1 & 0 \\
S_0 & d_2 \\
S_0 & d_1+d_2 \\
S_1 & d \\
\vdots & \\
S_0 & (r-1)d+d_2 \\
S_0 & (r-1)d-d_3 \\
S_1 & rd
\end{array}
$$

Since the relations are homogeneous we have $(r-2)d_1 + rd_2 + rd_3 = 0$ if $c = 1$. In this case $d_1, d_2$ and $d_3$ cannot all be non-negative (unless they are all equal to zero). If $c = 0$, all relations are trivially homogeneous and we can choose $d_1, d_2$ and $d_3$ arbitrarily. In particular, if $c = 0$, then $D(2A)^{r,c}$ is a tightly graded algebra.

A graded tilting complex $T := T_0 \oplus T_1$ of projective $D(2A)^{r,c}$-modules that tilts from $D(2A)^{r,c}$ to $D(2B)^{r,c}$ is given by the direct sum of the complex $T_1$, which is the stalk complex with $P_1$ in degree 0, and the complex

$$T_0 : 0 \to P_1\langle -d_3\rangle \oplus P_1\langle -(d_1 + d_3)\rangle \xrightarrow{(\gamma,\gamma\alpha)} P_0,$$

where $P_0$ is in degree 1, and where $\gamma$ and $\gamma\alpha$ are given by right multiplication by $\gamma$ and $\gamma\alpha$ respectively. It was shown in [9] that $T$ is a tilting complex for $D(2A)^{r,c}$ and that $\mathrm{End}_{K^b(P_{D(2A)^{r,c}})}(T) \cong (D(2B)^{r,c})^{\mathrm{op}}$. Viewing $T$ as a graded object and calculating $\mathrm{Endgr}_{K^b(P_{D(2A)^{r,c}})}(T)$ as a graded vector space will give us a grading on $D(2B)^{r,c}$.

From $\mathrm{Homgr}_{K^b(P_{D(2A)^{r,c}})}(T_1, T_1) \cong \bigoplus_{t=0}^{r} k\langle -td\rangle$ we have $\deg(\eta) = d$.

To calculate $\mathrm{Homgr}_{K^b(P_{D(2A)^{r,c}})}(T_1, T_0)$ notice that this space is isomorphic to $\mathrm{Homgr}_{D(2A)^{r,c}}(P_1, \ker(\gamma, \gamma\alpha))$. Non-zero maps in the latter space have to map $\mathrm{top}\, P_1$ to $\mathrm{soc}\, P_1\langle -d_3\rangle$, or to $\mathrm{soc}\, P_1\langle -(d_1 + d_3)\rangle$. This gives us that

$$\mathrm{Homgr}_{K^b(P_{D(2A)^{r,c}})}(T_1, T_0) \cong k\langle -(rd + d_3)\rangle \oplus k\langle -(rd + d_1 + d_3)\rangle.$$

Since the only non-zero paths in the quiver of $D(2B)^{r,c}$ that start at vertex 1 and end at vertex 0 are $\gamma$ and $\gamma\alpha$, we have

$$\{\deg(\gamma), \deg(\gamma\alpha)\} = \{rd + d_3, rd + d_1 + d_3\}.$$

To calculate $\mathrm{Homgr}_{K^b(P_{D(2A)^{r,c}})}(T_0, T_1)$ notice that non-zero maps in this space have to map $P_1\langle -d_3\rangle$ or $P_1\langle -(d_1 + d_3)\rangle$ onto $P_1$. It follows that

$$\mathrm{Homgr}_{K^b(P_{D(2A)^{r,c}})}(T_0, T_1) \cong k\langle d_3\rangle \oplus k\langle d_1 + d_3\rangle.$$

Since the only non-zero paths in the quiver of $D(2B)^{r,c}$ that start at vertex 0 and end at vertex 1 are $\beta$ and $\alpha\beta$, we have

$$\{\deg(\beta), \deg(\alpha\beta)\} = \{-d_3, -d_1 - d_3\}.$$

There are two choices for $\deg(\alpha)$. If $\deg(\alpha) = d_1$, then $\deg(\beta) = -d_1 - d_3$ and $\deg(\gamma) = rd + d_3$. This gives us a grading on $D(2B)^{r,c}$. If $\deg(\alpha) = -d_1$, then $\deg(\beta) = -d_3$ and $\deg(\gamma) = rd + d_1 + d_3$. This will not give us a grading on $D(2B)^{r,c}$ if $c = 1$, because the relations are not homogeneous. If $c = 0$, this grading is the same as the previous one via suitable substitution of the integers $d_1, d_2, d_3$, i.e. we get this grading from the former one if we choose $-d_1$, $d_1 + d_2$, $d_1 + d_3$ instead of $d_1$, $d_2$ and $d_3$ respectively for the degrees of the corresponding arrows.

With respect to the resulting grading, the graded quiver of $D(2B)^{r,c}$ is given by

$$\begin{tikzcd} 0\bullet \arrow[loop left, "d_1"] \arrow[r, bend left, "-d_1-d_3"] & \bullet 1 \arrow[loop right, "d"] \arrow[l, bend left, "rd+d_3"] \end{tikzcd}$$

### 4.3. Positivity and tightness

Proposition 4.6. *The algebra $D(2B)^{r,c}$ is positively graded for every $c$ and every $r$. The algebra $D(2B)^{r,c}$ is tightly graded if and only if $c = 0$ and $r = 3$.*

*Proof.* That $D(2B)^{r,c}$ is a positively graded algebra follows easily from its relations. If $\deg(\alpha) = 2r$, $\deg(\beta) = \deg(\gamma) = r$ and $\deg(\eta) = 4$, then the relations are homogeneous.

If $D(2B)^{r,c}$ is tightly graded, then by Lemma 2.12, there are degree 1 elements of the form

$$\begin{aligned} t_\alpha &:= \alpha + a_1\beta\gamma + a_2\alpha\beta\gamma, \\ t_\beta &:= \beta + b_1\alpha\beta, \qquad t_\gamma := \gamma + b_2\gamma\alpha, \\ t_\eta &:= \eta + \sum_{i=2}^{r} d_i\eta^i, \end{aligned}$$

where $a_1, a_2, b_1, b_2, d_1, \ldots, d_r$ are scalars.

It follows that $\alpha^2 = t_\alpha^2$ is a homogeneous element of degree 2, and $\alpha\beta\gamma$ is homogeneous of degree 3. If $c = 1$, then this leads to a contradiction. If $c = 0$, then from $\gamma\alpha\beta = t_\gamma t_\alpha t_\beta$ and $\eta^r = t_\eta^r$, we see that $r = 3$. ∎

Proposition 4.7. *The algebra $D(2A)^{r,0}$ is tightly graded for every $r$. The algebra $D(2A)^{r,1}$ is positively graded if and only if $r \le 2$. The algebras $D(2A)^{1,1}$ and $D(2A)^{2,1}$ are not tightly graded.*

*Proof.* If $r = 0$, then it is obvious that if we put the arrows of the quiver of $D(2A)^{r,0}$ in degree 1, then the relations are homogeneous. Hence, $D(2A)^{r,0}$ is tightly graded.

If $c = 1$ and $r = 1$, then if $\deg(\alpha) = 2$, $\deg(\beta) = 1$ and $\deg(\gamma) = 1$ we get a positive grading on $D(2A)^{1,1}$. If $c = 1$ and $r = 2$, then if $\deg(\alpha) = 2$, $\deg(\beta) = 0$ and $\deg(\gamma) = 0$, we get a positive grading on $D(2A)^{2,1}$.

For $r > 2$, if $\deg(\alpha) = r$, $\deg(\beta) = -(r-2)$ and $\deg(\gamma) = 0$, we get a grading on $D(2A)^{r,1}$. The graded quiver is given by

$$\begin{tikzcd} 0\bullet \arrow[loop left, "r"] \arrow[r, bend left, "2-r"] & \bullet 1 \arrow[l, bend left, "0"] \end{tikzcd}$$

This is not a positive grading. Also, this grading is not graded Morita equivalent to the trivial grading on $D(2A)^{r,1}$. By Proposition 4.4, any other grading on $D(2A)^{r,1}$ can be obtained from this grading by rescaling and graded Morita equivalence. When we rescale a grading such that there are homogeneous elements in both negative and positive degrees, the resulting grading still has the same property. Let $n_0$ and $n_1$ be integers and let $\mathrm{Endgr}_{D(2A)^{r,1}}(P_0\langle n_0\rangle \oplus P_1\langle n_1\rangle)^{\mathrm{op}}$ be a graded algebra that is graded Morita equivalent to the above graded algebra. By Proposition 9.1 in [4], the graded quiver of $\mathrm{Endgr}_{D(2A)^{r,1}}(P_0\langle n_0\rangle \oplus P_1\langle n_1\rangle)^{\mathrm{op}}$ is given by

$$\begin{tikzcd} 0\bullet \arrow[loop left, "r"] \arrow[r, bend left, "(2-r)+n_0-n_1"] & \bullet 1 \arrow[l, bend left, "n_1-n_0"] \end{tikzcd}$$

If $(2-r)+n_0-n_1 \geq 0$, then $n_1-n_0 < 0$. If $n_1-n_0 \geq 0$, then $(2-r)+n_0-n_1 < 0$. It follows that the resulting grading is not positive. Hence, if $r > 2$, then $D(2A)^{r,1}$ is not positively graded.

To prove that $D(2A)^{2,1}$ is not tightly graded we start with the grading on $D(2A)^{2,1}$ given by the graded quiver

$$\begin{tikzcd} 0\bullet \arrow[loop left, "1"] \arrow[r, bend left, "0"] & \bullet 1 \arrow[l, bend left, "0"] \end{tikzcd}$$

This grading is not graded Morita equivalent to the trivial grading on $D(2A)^{2,1}$. As above, it follows easily that any other grading that is graded Morita equivalent to this grading is not positive. Hence, $D(2A)^{2,1}$ is not a tightly graded algebra.

To prove that $D(2A)^{1,1}$ is not tightly graded we again use Lemma 2.12. Assuming that $D(2A)^{1,1}$ is tightly graded, we infer that $\alpha^2$ is a homogeneous element of both degree 2 and degree 3, which is impossible. ∎

**5. One simple module.** Any block with a dihedral defect group and one isomorphism class of simple modules is Morita equivalent to some algebra from the following family (cf. [8] or [9]):

For a given integer $r \geq 1$, let $D := D(1C)^r$ be the algebra defined by the quiver and relations

$$\begin{tikzcd} \bullet \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd} \qquad \alpha^2 = 0 = \beta^2,\ (\alpha\beta)^r = (\beta\alpha)^r.$$

**5.1. Classification of gradings.** The relations of $D$ are homogeneous, regardless of the degrees of $\alpha$ and $\beta$. It follows that for any pair of integers $(a, b)$, we get a grading on $D$ by setting $\deg(\alpha) = a$ and $\deg(\beta) = b$. We denote this graded algebra by $D^{a,b}$. When $a = b = 1$ we get a tight grading on $D$. The graded radical layers of the only projective indecomposable $D^{a,b}$-

module ${}_D D$ are

$$
\begin{array}{rcccl}
 & & S & & {\scriptstyle 0} \\
{\scriptstyle a} & S & & S & {\scriptstyle b} \\
{\scriptstyle a+b} & S & & S & {\scriptstyle a+b} \\
{\scriptstyle 2a+b} & S & & S & {\scriptstyle 2b+a} \\
 & \vdots & & \vdots & \\
{\scriptstyle a+(r-1)(a+b)} & S & & S & {\scriptstyle b+(r-1)(a+b)} \\
 & & S & & {\scriptstyle (a+b)^r}
\end{array}
$$

where $S$ denotes the only simple $D$-module.

For a given integer $d$, the graded algebra $\operatorname{Endgr}_{D^{a,b}}(D\langle d\rangle)^{\mathrm{op}}$ is graded Morita equivalent to $D^{a,b}$ by Definition 2.1. But $\operatorname{Endgr}_{D^{a,b}}(D\langle d\rangle)^{\mathrm{op}} \cong D^{a,b}$ as graded algebras. It follows that the only graded algebra which is graded Morita equivalent to $D^{a,b}$ is $D^{a,b}$ itself. From this we have the following proposition.

Proposition 5.1. *For any pair of integers $(a, b)$ there is a grading $D^{a,b}$ on $D$. For different pairs of integers $(a, b)$ and $(c, d)$, the graded algebras $D^{a,b}$ and $D^{c,d}$ are not graded Morita equivalent.*

It follows from this proposition that the maximal tori in $\operatorname{Out}^K(D)$ are isomorphic to $\mathbf{G}_m^l$, where $l > 1$, for if $l \leq 1$, then we would have a unique grading up to rescaling and graded Morita equivalence on $D$, which is not the case.

If $\varphi$ is an arbitrary automorphism in $\operatorname{Out}^K(D)$, then we can assume that

$$
\begin{aligned}
\varphi(e) &= e, \\
\varphi(\alpha) &= a_1\alpha + a_2\beta + a_3x, \\
\varphi(\beta) &= b_1\alpha + b_2\beta + b_3y,
\end{aligned}
$$

where $a_i, b_i \in k$, and $x, y \in \operatorname{rad}^2 D$. Since $\varphi(\alpha^2) = \varphi(\beta^2) = 0$, we find that $a_1a_2 = 0$ and $b_1b_2 = 0$. From $\varphi((\alpha\beta)^r) \neq 0$ and $\varphi((\beta\alpha)^r) \neq 0$ it follows that either $a_1 \neq 0 \neq b_2$ and $a_2 = b_1 = 0$, or $a_2 \neq 0 \neq b_1$ and $a_1 = b_2 = 0$. The action of $\varphi$ on $\operatorname{rad} D/\operatorname{rad}^2 D$ is given by matrices of the form

$$
\begin{pmatrix} a_1 & 0 \\ 0 & b_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & b_1 \\ a_2 & 0 \end{pmatrix}.
$$

It now follows easily (one can see this directly or by using Remark 3.5 in [17]) that the maximal tori in $\operatorname{Out}^K(D)$ are isomorphic to the product of at most two copies of $\mathbf{G}_m$. Combining this conclusion with the above remarks shows that the maximal tori in $\operatorname{Out}^K(D)$ are isomorphic to $\mathbf{G}_m \times \mathbf{G}_m$.

PROPOSITION 5.2. *The maximal tori in* $\mathrm{Out}^K(D)$ *are isomorphic to* $\mathbf{G}_m^2$. *Up to graded Morita equivalence the gradings on* $D$ *are parameterized by* $\mathbb{Z}^2$ *and are in one-to-one correspondence with algebraic group homomorphisms from* $\mathbf{G}_m$ *to* $\mathbf{G}_m \times \mathbf{G}_m$.

*Proof.* Follows from the above discussion and the previous proposition. ∎

**6. Summary of the results.** In the following table we summarize the results of this paper. The first three columns tell us respectively if there exists a non-trivial, a positive and a tight grading on a given block. The last column gives the isomorphism class of the maximal tori in the group of outer automorphisms of a given block. Derived equivalence classes are separated by horizontal lines.

| Block | Non-trivial | Positive | Tight | Maximal torus |
|---|---|---|---|---|
| $A_r$ | Yes | Yes | Yes | $\mathbf{G}_m \times \mathbf{G}_m$ |
| $B_r$ | Yes | Yes | Only if $r = 1$ | $\mathbf{G}_m \times \mathbf{G}_m$ |
| $C_r$ | Yes | Yes | Only if $r = 4$ | $\mathbf{G}_m \times \mathbf{G}_m$ |
| $D(2A)^{r,0}$ | Yes | Yes | Yes | $\mathbf{G}_m \times \mathbf{G}_m$ |
| $D(2B)^{r,0}$ | Yes | Yes | Only if $r = 3$ | $\mathbf{G}_m \times \mathbf{G}_m$ |
| $D(2A)^{r,1}$ | Yes | Only if $r \leq 2$ | No | $\mathbf{G}_m$ |
| $D(2B)^{r,1}$ | Yes | Yes | No | $\mathbf{G}_m$ |
| $D(1C)^r$ | Yes | Yes | Yes | $\mathbf{G}_m \times \mathbf{G}_m$ |

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Dusko Bogdanic
Mathematical Institute
University of Oxford
24-29 St Giles'
Oxford OX1 3LB, United Kingdom
E-mail: bogdanic@maths.ox.ac.uk

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