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SOME RESULTS ON THE KERNELS OF HIGHER DERIVATIONS ON $k[x,y]$ AND $k(x,y)$

BY

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Abstract. Let k be a field and $k[x,y]$ the polynomial ring in two variables over k. Let D be a higher k-derivation on $k[x,y]$ and $\bar{D}$ the extension of D on $k(x,y)$. We prove that if the kernel of D is not equal to k, then the kernel of $\bar{D}$ is equal to the quotient field of the kernel of D.

1. Introduction. Let R be an integral domain with unit and let A be an R-algebra. We recall some definitions on higher derivations. A *higher R-derivation* on A is a set of R-linear endomorphisms $D = \{D_n\}_{n=0}^{\infty}$ of A satisfying the following conditions:

(i) D_0 is the identity map of A.

(ii) For any a, $b \in A$ and for any integer $n \geq 0$,

$$D_n(ab) = \sum_{i+j=n} D_i(a)D_j(b).$$

For a higher R-derivation $D = \{D_n\}_{n=0}^{\infty}$ on A, we define the *kernel* A^D of D by $\{a \in A \mid D_n(a) = 0 \text{ for any } n \geq 1\} = \bigcap_{n\geq 1} \operatorname{Ker} D_n$. It is then clear that A^D is an R-subalgebra of A. A higher R-derivation D is said to be *non-trivial* if $A^D \neq A$.

Derivations and their kernels play an important role and have been studied by many mathematicians (see, e.g., [3] for an excellent account). Recently, several mathematicians have studied the kernels of higher derivations. For example, Kojima and the author [2] proved that the kernel of a non-trivial higher R-derivation D on the polynomial ring $R[x,y]$ in two variables over an HCF-ring R has the form $R[h]$ for some $h \in R[x,y]$ (cf. [2, Theorem 1.1]). When R is a field of characteristic zero and D is an R-derivation, Nowicki and Nagata [4] obtained a similar result (cf. [4, Theorem 2.8]).

In this paper, we study relations between the quotient field of the kernel of a higher k-derivation on $k[x,y]$ and the kernel of $\bar{D}$, the extension of D on $k(x,y)$ (for the precise definition, see Section 2). The main result is the following theorem.

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Theorem 1.1. *Let k be a field and let D be a higher k-derivation on the polynomial ring $A = k[x, y]$ in two variables over k. Let $\overline{D}$ be the extension of D on the quotient field $Q(A)$ of A. If $A^D \neq k$, then $Q(A)^{\overline{D}} = Q(A^D)$.*

By using the proof of Theorem 1.1, we have the following theorem.

Theorem 1.2. *Let k be a field and let D be a non-trivial higher k-derivation on the polynomial ring $A = k[x, y]$. Then there exists $h \in A$ such that $A^D = k[h]$.*

Theorem 1.2 is a special case of [2, Theorem 1.1]. However, the argument as in Section 3 gives an elementary proof of [2, Theorem 1.1] in the case where R is a field.

2. Preliminary results. Let k be a field of characteristic $p \geq 0$ and let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. In this section, we recall some results on higher k-derivations on A and their kernels.

The following lemma is clear from the definition of higher k-derivations.

Lemma 2.1 (cf. [2, Lemma 2.1]). *Let $D = \{D_n\}_{n=0}^{\infty}$ be a set of endomorphisms of A, where we assume that D_0 is the identity map. Then the following conditions are equivalent*:

(1) *D is a higher k-derivation on A.*
(2) *The mapping $\varphi_D : A \to A[[t]]$, where $A[[t]]$ is the formal power series ring in one variable t over A, given by $\varphi_D(a) = \sum_{i \geq 0} D_i(a)t^i$, is a homomorphism of k-algebras.*

For a higher k-derivation D, we call the mapping φ_D as in (2) of Lemma 2.1 the homomorphism *associated to D*.

Let $D = \{D_n\}_{n=0}^{\infty}$ be a higher k-derivation on A and φ_D the morphism associated to D. Let $K = Q(A)$ be the quotient field of A. Then the k-algebra homomorphism $\varphi_D : A \to A[[t]]$ is naturally extended to a k-algebra homomorphism $\Phi : K \to K[[t]]$ by setting

$$\Phi\left(\frac{b}{a}\right) = \frac{\varphi_D(b)}{\varphi_D(a)}$$

for $a, b \in A$ with $a \neq 0$. By Lemma 2.1, the homomorphism Φ defines a higher k-derivation $\overline{D} = \{\overline{D}_n\}_{n=0}^{\infty}$ on K such that $\Phi(\lambda) = \sum_{i \geq 0} \overline{D}_i(\lambda)t^i$ for $\lambda \in K$ and $\overline{D}_i|_A = D_i$ for every $i \geq 0$. We call the higher k-derivation $\overline{D}$ the *extension* of D on K. We set $K^{\overline{D}} := \{\lambda \in K \mid \overline{D}_i(\lambda) = 0 \text{ for any } i \geq 1\}$, which is the kernel of $\overline{D}$. We can easily see that $K^{\overline{D}}$ is a subfield of K and that for $\lambda \in K$, $\lambda \in K^{\overline{D}}$ if and only if $\Phi(\lambda) = \lambda$. The following lemmas are proved in [2].

LEMMA 2.2 (cf. [2, Lemma 2.3]). *With the same notations and assumptions as above, the following assertions hold true*:

(1) $K^{\bar{D}}$ *is algebraically closed in* K.
(2) $K^{\bar{D}} \cap A = A^D$.

LEMMA 2.3 (cf. [2, Lemma 2.4]). *Let* D *be a non-trivial higher* k*-derivation on the polynomial ring* $A = k[x_1, \ldots, x_n]$. *Then* $\mathrm{tr.deg}_k A^D \leq n-1$.

REMARK 2.4. The following examples show that the assumption $A^D \neq k$ is important in Theorem 1.1 and the assertion of Theorem 1.1 does not hold in general in three (or more) variables.

(1) Let D be the higher k-derivation on the polynomial ring $A = k[x, y]$ defined by a k-algebra homomorphism φ_D such that $\varphi_D(x) = x + \sum_{i=1}^{n} xt^i$, $\varphi_D(y) = y + \sum_{i=1}^{n} yt^i$. Then $A^D = k$ and $x/y \in Q(A)^{\bar{D}} \setminus k$. In particular, $Q(A^D) = k \neq Q(A)^{\bar{D}}$.
(2) Let D be the higher k-derivation on the polynomial ring $A = k[x, y, z]$ defined by a k-algebra homomorphism φ_D such that $\varphi_D(x) = x + \sum_{i=1}^{n} xt^i$, $\varphi_D(y) = y + \sum_{i=1}^{n} yt^i$, $\varphi_D(z) = z$. Then $A^D = k[z]$ (so $A^D \neq k$) but $x/y \in Q(A)^{\bar{D}} \setminus k(z)$. In particular, $Q(A^D) = k(z) \neq Q(A)^{\bar{D}}$.

3. Proof of the results

Proof of Theorem 1.1. Let $\varphi_D : A \to A[[t]]$ be the homomorphism associated to D. We note that, for $a \in A$, $a \in A^D$ if and only if $\varphi_D(a) = a$. If D is trivial, then it is clear that $K^{\bar{D}} = K$. Therefore $K^{\bar{D}} = K = Q(A) = Q(A^D)$. From now on, we assume that D is non-trivial. The subsequent argument is almost the same as the proof of [5, Theorem 1.1]. By the condition $A^D \neq k$, we have $\mathrm{tr.deg}_k K^{\bar{D}} \geq 1$. Since $\mathrm{tr.deg}_k K^{\bar{D}} \leq 1$ by Lemma 2.3, we have $\mathrm{tr.deg}_k K^{\bar{D}} = 1$. By Lüroth's theorem, we know that $K^{\bar{D}} = k(h)$ for some $h \in K \setminus k$. Let us set $h = F/G$ for relatively prime elements F, G of A. We may assume that $\deg_y F \geq \deg_y G$ because $k(h) = k(1/h)$. Since $A^D \neq k$, there exists an element $r \in A^D \setminus k$. If $\deg_y r = \deg_x r = 0$, then $r \in k$. This is a contradiction. Thus, we may assume that $\deg_y r > 0$. Let

$$F = f_n y^n + f_{n-1} y^{n-1} + \cdots + f_0, \qquad G = g_m y^m + g_{m-1} y^{m-1} + \cdots + g_0,$$

where $n = \deg_y F$, $m = \deg_y G$ and $f_i, g_j \in k[x]$ for $i = 0, \ldots, n$ and $j = 0, \ldots, m$. Now, we consider the following two cases.

CASE 1: $n = m$ and $\deg_x f_n = \deg_x g_n = l$. Then let

$$f_n = c_l x^l + \cdots + c_0, \qquad g_n = d_l x^l + \cdots + d_0,$$

where $c_i, d_i \in k$ for $i = 0, \ldots, l$. Consider the element $h - c_l/d_l$ in K. It is not equal to zero because $h \notin k$. We have $h - c_l/d_l = H/G$, where H is the polynomial in A equal to $F - (c_l/d_l)G$. Since F and G are relatively prime in A, so are H and G. We also see that either $\deg_y H < \deg_y G$, or they are equal but the coefficients of the highest power of y in H and G are polynomials in $k[x]$ of different degrees. Then we replace h with $1/(h - c_l/d_l)$ and we are in the following second case.

CASE $n > m$, or $n = m$ but $\deg_x f_n \neq \deg_x g_n$. Since $r \in A^D \subseteq K^{\bar{D}} = k(h)$, we can write

$$r = \frac{\sum_{i=0}^{t} a_i h^i}{\sum_{i=0}^{s} b_i h^i} = \frac{\sum_{i=0}^{t} a_i (F/G)^i}{\sum_{i=0}^{s} b_i (F/G)^i} = \frac{\sum_{i=0}^{t} a_i G^{t-i} F^i}{\sum_{i=0}^{s} b_i G^{s-i} F^i} G^{s-t}$$

for $a_i, b_i \in k$ and $a_t, b_s \neq 0$. In this case, we show that

$$\deg_y r = (t - s)(\deg_y F - \deg_y G) = (t - s)(n - m). \tag{3.1}$$

It is clear that $\deg_y G^{s-t} = -(t - s)m$. So, it is sufficient to prove that $\deg_y(\sum_{i=0}^{t} a_i G^{t-i} F^i) = tn$ and $\deg_y(\sum_{i=0}^{s} b_i G^{s-i} F^i) = sn$. If $n > m$, then each term of the form $G^{t-i}F^i$ has a different degree with respect to y. Since the highest degree (equal to nt) terms are contained in $G^0 F^t$ and $a_t \neq 0$, the equality $\deg_y(\sum_{i=0}^{t} a_i G^{t-i} F^i) = tn$ holds true. If $n = m$ and $\deg_x f_n \neq \deg_x g_n$, then it is clear that $\deg_y(\sum_{i=0}^{t} a_i G^{t-i} F^i) \leq tn$. Suppose that the inequality is strict. Then the coefficient polynomial of y^{nt} in $\sum_{i=0}^{t} a_i G^{t-i} F^i$ is equal to 0. Therefore $\sum_{i=0}^{t} a_i g_n^{t-i} f_n^i = 0$. Since $\deg_x f_n \neq \deg_x g_n$, all polynomials of the form $g_n^{t-i} f_n^i$ have different degrees with respect to x. Since at least one of the elements $a_0, \ldots, a_t$ is non-zero, it follows that the above sum cannot be equal to 0. This is a contradiction. Thus, the equality (3.1) is proved. Because $\deg_y r > 0$, we have $n > m$ and $t > s$.

The equality

$$rG^{t-s}\Big(\sum_{i=0}^{s} b_i G^{s-i} F^i\Big) = \sum_{i=0}^{t} a_i G^{t-i} F^i$$

in A implies that the polynomial $a_t F^t + \sum_{i=0}^{t-1} a_i G^{t-i} F^i$ is divisible by G and hence F^t is divisible by G. Since F and G are relatively prime, we have $G \in k$ and $h \in A$. This completes the proof. ■

Proof of Theorem 1.2. The assertion is clear if $A^D = k$. Let $\bar{D}$ be the extension of D on the quotient field K of A. As seen from the proof of Theorem 1.1, if $A^D \neq k$, then there exists an $h \in A$ such that $K^{\bar{D}} = k(h)$. Here we note the following claim stated in [1, Lemma 2.1], which holds in any characteristic. For the reader's convenience, we reproduce the proof.

Claim 3.1. *Let k be a field and let $R = k[x_1, \ldots, x_n]$. If $f \in R$, then $k(f) \cap R = k[f]$.*

Proof. The "$\supseteq$" part is clear. We prove the "$\subseteq$" part. Assume that $u = u(x_1, \ldots, x_n) \in k(f) \cap R$. Then we can write $u = p(f)/q(f)$ for relatively prime elements $p(t)$, $q(t)$ of $k[t]$, where $k[t]$ is the polynomial ring in one variable. There exist $\alpha(t), \beta(t) \in k[t]$ such that $1 = \alpha(t)p(t) + \beta(t)q(t)$. Hence, we have

$$\begin{aligned}1 &= \alpha(f)p(f) + \beta(f)q(f) = \alpha(f)u(x_1, \ldots, x_n)q(f) + \beta(f)q(f)\\ &= (\alpha(f)u(x_1, \ldots, x_n) + \beta(f))q(f)\end{aligned}$$

in R. This implies that the polynomial $q(f)$ is invertible in R. Thus, $q(f) \in k$, i.e., $u = p(f)/q(f) \in k[f]$. ∎

By Lemma 2.2(2) and Claim 3.1, $k[h] = k(h) \cap A = K^{\overline{D}} \cap A = A^D$. ∎

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