

RINGS WHOSE PROPER FACTORS ARE RIGHT PERFECT

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Abstract. We show that practically all the properties of almost perfect rings, proved by Bazzoni and Salce [Colloq. Math. 95 (2003)] for commutative rings, also hold in the non-commutative setting.

1. Introduction. We say that a ring R is *right almost perfect* if R/I is a right perfect ring for every proper non-zero two-sided ideal I of R . Similarly we define *left almost perfect* rings. Commutative almost perfect rings were defined by Bazzoni and Salce in [3]. The class of commutative almost perfect rings has several interesting properties and characterizations, and has been studied by Bazzoni, Fuchs, Sang Bum Lee, Salce, Zanardo and others. We will show that most of the properties of these rings presented in [4] still hold in the non-commutative setting: non-prime right almost perfect rings are right perfect, over the prime rings the torsion modules in a suitable torsion theory are semiartinian, etc. Our main results are Theorems 3.1 and 4.6 and Proposition 5.1.

We also introduce the notion of (non-commutative) *h-local* ring. This is also an extension to non-commutative rings of the corresponding notion studied in the commutative setting first by Jaffard [9, Theorem 6], and then by Matlis [11] and others.

Bazzoni and Salce also proved that a commutative integral domain R is almost perfect if and only if it is *h-local* and every localization of R at a maximal ideal is almost perfect. As localization at maximal ideals is typical of the commutative setting, this property clearly does not have an obvious non-commutative counterpart. The next step now is to see whether other properties of commutative almost perfect rings, for instance those in [3], also hold in the non-commutative case. Some examples are given.

In this paper, all rings R are associative rings with identity $1 \neq 0$, and $J(R)$ denotes the Jacobson radical of the ring R .

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2. Bazzoni and Salce's results. We will very frequently use the various characterizations of right perfect rings that appear in the famous result due to Bass and called *Theorem P* [1, Theorem 28.4]. We state it here for later reference. Recall that a (not necessarily commutative) ring R is said to be: (1) *semilocal* if $R/J(R)$ is semisimple artinian; (2) a *right max ring* if every non-zero right R -module has a maximal submodule; and, dually, (3) *right semiartinian* if every non-zero right R -module has a simple submodule (equivalently, if every right R -module is an essential extension of its socle).

THEOREM 2.1 (Bass's Theorem P). *The following conditions are equivalent for a ring R :*

- (1) R is right perfect.
- (2) R is semilocal and its Jacobson radical $J(R)$ is right T -nilpotent.
- (3) R is semilocal and right max.
- (4) Every flat right R -module is projective.
- (5) R satisfies the descending chain condition on principal left ideals.
- (6) R is left semiartinian and contains no infinite orthogonal set of idempotents.

We now briefly recall the main results on commutative almost perfect rings obtained by Bazzoni and Salce in [4]. They defined in [3] *almost perfect rings* as those commutative rings R for which R/I is a perfect ring for every non-zero proper ideal I of R . Recall that a commutative integral domain R is *h -local* [11] if (1) R/I is semilocal for every non-zero proper ideal I of R , and (2) R/P is local for every non-zero prime ideal P of R . In the commutative case, a ring R is perfect if and only if it is semilocal and every localization of R at a maximal ideal is a perfect ring [4, Theorem 1.1]. The following are the four main results that appear in the paper [4] by Bazzoni and Salce.

PROPOSITION 2.2 ([4, Proposition 1.3]). *If R is an almost perfect commutative ring and R is not an integral domain, then R is perfect.*

THEOREM 2.3 ([8, Theorem 4.4.1]). *The following conditions are equivalent for a commutative integral domain R :*

- (1) Every non-zero torsion module contains a simple module.
- (2) Every torsion R -module is semiartinian.
- (3) For every non-zero ideal I of R , R/I contains a simple module.
- (4) For every non-zero proper R -submodule A of Q , Q/A contains a simple module.
- (5) Q/R is semiartinian.

THEOREM 2.4 ([4, Theorem 2.3]). *The following conditions are equivalent for a commutative integral domain R :*

- (1) R is almost perfect.
- (2) R is h -local and every localization of R at a maximal ideal is almost perfect.
- (3) R is h -local and satisfies one of the equivalent conditions of Theorem 2.3.

COROLLARY 2.5. *The following conditions are equivalent for a commutative local integral domain R :*

- (1) R is almost perfect.
- (2) Q/R is semiartinian.
- (3) Every non-zero torsion module is semiartinian.

3. Non-commutative almost perfect rings. We will now see how the previous results modify when the ring R is not commutative. As we have already said in the Introduction, we call a ring R *right almost perfect* if R/I is a right perfect ring for every proper two-sided ideal $I \neq 0$ of R .

Let us see some immediate examples of right almost perfect rings.

- (1) Right perfect rings are right almost perfect [1, Corollary 28.7].
- (2) Simple rings are trivially right and left almost perfect.

(3) For any ring R , the intersection of all non-zero two-sided ideals is either 0 or the least non-zero two-sided ideal of R . Assume that this second case holds, that is, that R has a least non-zero two-sided ideal, I say. Then the ring R is right almost perfect if and only if R/I is right perfect. For instance, assume that the ring R has exactly three ideals, necessarily 0, R and I . Then R is right almost perfect if and only if the simple ring R/I is right perfect. As simple rings are right perfect if and only if they are simple artinian, it follows that a ring R with exactly three ideals 0, R and I is right almost perfect if and only if R/I is simple artinian. In particular, such a ring R is right almost perfect if and only if it is left almost perfect.

An interesting example of these rings with three ideals is given by the nearly simple chain rings considered by Dubrovin [6, 7] and Puninski [13, 14]. A *nearly simple* chain ring R is a non-commutative right and left chain ring (i.e., the right ideals and the left ideals are linearly ordered under inclusion), with exactly three two-sided ideals, necessarily the ideals 0, R and the maximal ideal $J(R)$. As chain rings R are local, $R/J(R)$ is a division ring, hence a perfect ring. It follows that nearly simple chain rings are right and left almost perfect. There are both examples of nearly simple chain rings that are (non-commutative) integral domains, and examples of nearly simple chain rings that are prime rings but not integral domains.

For another example of right and left almost perfect ring R with exactly three ideals, see the example in Remark 5.2(2).

(4) There are left almost perfect rings that are not right almost perfect. For instance, let k be a field and let k_ω be the k -algebra of all matrices with entries in k , countably many rows and columns, and in which each row has only finitely many non-zero entries. Let N be the set of all strictly lower triangular matrices in k_ω with only finitely many non-zero entries and let R be the subalgebra $k + N$ of k_ω . If we denote by $E_{i,j}$ the matrix units, where i and j are positive integers, one has $E_{i,j} \in N$ if and only if $i > j$. The Jacobson radical of R is N . It is known that R is left perfect but not right perfect [2, Example (5), p. 476]. In particular, R is left almost perfect. In order to show that R is not right almost perfect, consider the principal two-sided ideal I of R generated by $E_{2,1}$. It is easily seen that I is the vector space generated by all products $E_{i,j}E_{2,1}E_{k,\ell}$, with $i \geq j$ and $k \geq \ell$, that is, I is the vector space generated by all $E_{i,1}$ with $i \geq 2$. Then $I \subseteq J(R)$, so that $J(R/I) = J(R)/I = N/I$. In the sequence $\dots, E_{5,4}, E_{4,3}, E_{3,2}$, all the elements are in N and the products $E_{n,n-1}E_{n-1,n-2} \dots E_{4,3}E_{3,2} = E_{n,2}$ are not in I . This proves that N/I is not right T -nilpotent, and so R/I is not right perfect. Thus R is left almost perfect, but not right almost perfect.

(5) There is no relation between being an almost perfect ring and being a semiperfect ring. For instance, the ring \mathbb{Z} of integers is almost perfect but not semiperfect, and a commutative valuation domain of Krull dimension ≥ 2 is semiperfect but not almost perfect.

Our first result is the non-commutative analogue of Proposition 2.2. It shows that, for non-prime rings, the notions of right almost perfect ring and right perfect ring coincide.

THEOREM 3.1. *If a ring R is right almost perfect and not a prime ring, then R is right perfect.*

Proof. Let R be a right almost perfect ring that is not a prime ring. We will distinguish two cases.

First case: R has a non-zero nilpotent two-sided ideal.

In this case, R has a two-sided ideal $K \neq 0$ with $K^2 = 0$. In particular, K is nilpotent, hence $K \subseteq J(R)$. Thus $J(R) \neq 0$, so that $R/J(R)$ is right perfect, hence semisimple artinian (Theorem 2.1). It follows that R is semilocal, and so it has no infinite orthogonal set of idempotents. In order to conclude, by Theorem 2.1(6), it suffices to show that every non-zero left R -module contains a simple submodule. Let ${}_R M \neq 0$ be a left R -module. If $KM = 0$, then M is a left R/K -module. But R/K is right perfect, so that M has a simple R/K -submodule, which is also a simple R -submodule. If $KM \neq 0$, then KM is a non-zero left R/K -module. But R/K is right perfect, so that KM has a simple R/K -submodule, which is also a simple R -submodule. Thus M has a simple R -submodule. This shows that R is right perfect in this first case.

Second case: R has no non-zero nilpotent two-sided ideals.

Since R is not a prime ring, R has two non-zero two-sided ideals I and J with $IJ = 0$. Then $(I \cap J)^2 \subseteq IJ = 0$, so that $I \cap J = 0$ because R has no non-zero nilpotent two-sided ideals. We will now show that R contains no infinite orthogonal set of idempotents. Assume the contrary, and let E be an infinite orthogonal set of distinct idempotents of R . Then $E_I := \{e + I \mid e \in E\}$ is an orthogonal set of idempotents of R/I , which is right perfect, so that E_I must be a finite set. It follows that there is a partition of E into finitely many subsets E_1, \dots, E_n with the property that, for every $e, f \in E$, $e - f \in I$ if and only if e and f belong to the same block E_i of the partition. Since E is infinite, one of the blocks, E_t say, is an infinite set. Thus E_t is an infinite orthogonal set of distinct idempotents of R . The set $E_{t,J} := \{e + J \mid e \in E_t\}$ is an orthogonal set of idempotents of R/J , which is right perfect, so that $E_{t,J}$ must be a finite set. It follows that there is a partition of E_t into finitely many subsets E'_1, \dots, E'_m with the property that, for every $e, f \in E_t$, $e - f \in J$ if and only if e and f belong to the same block E'_j of this partition of E_t . As E_t is infinite, one of these blocks, E'_ℓ say, is infinite. But for every $e, f \in E'_\ell$, we have $e - f \in J$ because e and f belong to the same block E'_ℓ , and $e - f \in I$ because both e and f belong to E_t . Thus $e - f \in I \cap J = 0$, i.e., $e = f$ for every $e, f \in E'_\ell$. This shows that E'_ℓ has exactly one element, which contradicts what we have previously proved. Thus R contains no infinite orthogonal set of idempotents. In order to conclude, by Theorem 2.1(6), it suffices to show that every non-zero left R -module contains a simple submodule. Let ${}_R M \neq 0$ be a left R -module. If $IM = 0$, then M is a left R/I -module. But R/I is right perfect, so that M has a simple R/I -submodule, hence a simple R -submodule. If $IM \neq 0$, then IM is a non-zero left R/J -module, because $(JI)^2 = J(IJ)J = 0$ implies $JI = 0$ (R has no non-zero nilpotent two-sided ideals). But R/J is right perfect, so that IM has a simple R/J -submodule, hence a simple R -submodule. Thus M has a simple R -submodule. This proves that R is right perfect in this second case also. ■

Puninski's example of a nearly simple chain domain R [13] immediately shows that the conditions of Theorem 2.3 do not hold for the right almost perfect Ore domain R if we consider on R the natural torsion theory in which the torsion modules are those in which every element is annihilated by a non-zero element of R and Q is the classical ring of fractions of R , which is a division ring. In order to recall the definition of the ring R considered by Puninski, let G be the group of affine linear functions on \mathbb{Q} ,

$$G = \{\alpha_{a,b}: \mathbb{Q} \rightarrow \mathbb{Q} \mid a, b \in \mathbb{Q}, a > 0\},$$

where $\alpha_{a,b}(t) = at + b$ and the group operation on G is the composition of functions. Fix a positive irrational number ϵ in the field of real numbers.

It is possible to define a right order on G with generalized positive cone $P := \{\alpha_{a,b} \in G \mid \epsilon \leq \alpha_{a,b}(\epsilon)\}$ [5]. Let k be a division ring, $k[P]$ the semigroup ring, and consider $M := \sum_{\alpha \in P^+} \alpha k[P] \subseteq k[P]$, where $P^+ = \{\alpha_{a,b} \in P \mid \epsilon < \alpha_{a,b}(\epsilon)\}$. Then M is a maximal ideal in $k[P]$. The subset $k[P] \setminus M$ is a right and left Ore set in $k[P]$. Let R denote the localization of $k[P]$ with respect to $k[P] \setminus M$. The ring R is a nearly simple chain domain, hence it is a right and left almost perfect ring. In particular, R is a local ring and is an Ore domain. The valuation group of R is G . This group is not a discrete group. It follows that Q/R , where Q is the division ring of fractions of R , is a torsion R -module with zero socle, so that the conditions of Theorem 2.3 do not hold for the almost perfect ring R with the torsion theory natural for an Ore domain. In the next section, we will see that we need a different torsion theory, which is more natural for prime rings.

4. Prime rings. Let us pass to the structure of right perfect rings. We begin with an elementary lemma.

LEMMA 4.1. *Every prime right perfect ring is a simple artinian ring.*

Proof. If R is a prime right perfect ring, then R satisfies the descending chain condition on principal left ideals by Theorem 2.1(5), so that R is semisimple artinian by [10, Theorem 10.24]. Since R is prime, it must be a simple ring. ■

COROLLARY 4.2. *A non-zero two-sided ideal of a right almost perfect ring R is a maximal ideal if and only if it is a prime ideal, if and only if it is a right primitive ideal.*

Proof. Every maximal ideal is prime. If $I \neq 0$ is a prime ideal, then R/I is a prime right perfect ring, so that R/I is simple artinian by the previous lemma. Thus I is the right annihilator of the unique simple right R/I -module. In particular, I is a right primitive ideal. Finally, let $I \neq 0$ be a right primitive ideal. Then R/I is a right perfect ring with a faithful simple right module. In particular, $J(R/I) = 0$. But right perfect rings are semisimple artinian modulo their Jacobson radical, so that R/I is a semisimple artinian ring. Since it has a faithful simple right module, it follows that R/I is simple artinian. Thus I is maximal in R . ■

We say that a ring R is h -local if: (1) for every non-zero proper two-sided ideal I of R , the factor ring R/I is semilocal; and (2) every non-zero prime two-sided ideal of R is contained in a unique maximal two-sided ideal of R .

Clearly, local rings are h -local. From Theorem 2.1 and Lemma 4.1, we immediately deduce:

COROLLARY 4.3. *Every right almost perfect ring is h -local.*

Any prime ring R has a natural topology with respect to which it is a topological ring that is right and left linearly topological [16, p. 144]. This natural topology on R is defined as follows. Let \mathcal{B} be the set of all non-zero two-sided ideals of R . The topology on the prime ring R has \mathcal{B} as a basis of neighborhoods of 0. This is not a Hausdorff topology in general. For instance, if R is a nearly simple prime chain ring, the closure of zero in this topology is the maximal ideal of R . Moreover, the left ideals of a prime ring R that are open in this linear topology do not form a left Gabriel topology in general, but only a *divisible left Oka family* [15]. More precisely, recall that if I is a left ideal of a ring R , the *core* of I , denoted $\text{core}(I)$, is the largest two-sided ideal of R contained in I . It coincides with the annihilator in R of the left R -module ${}_R R/I$. If R is a prime ring, the class \mathcal{P} of all left R -modules whose elements are annihilated by an element of \mathcal{B} is closed under submodules, homomorphic images and direct sums. That is, \mathcal{P} is a hereditary pretorsion class [16, Proposition VI.4.2].

The hereditary torsion class generated by \mathcal{P} is the class \mathcal{T} consisting of all left R -modules ${}_R M$ such that each non-zero homomorphic image of ${}_R M$ has a non-zero submodule in \mathcal{P} [16, Proposition VI.2.5]. The left Gabriel topology corresponding to the hereditary torsion class \mathcal{T} is $J(\mathcal{B}) = \{I \leq {}_R R \mid \text{for every } I' \leq {}_R R \text{ with } I \subset I' \subset {}_R R, \text{ there exists } a \in R \setminus I' \text{ such that } \text{core}(I' : a) \neq 0\}$ [16, Proposition VI.5.4]. *In the rest of the paper, whenever we say a torsion module over a prime ring we mean a module in \mathcal{T} .* The module ${}_R R$ is torsion-free in this torsion theory, otherwise it would contain a non-zero cyclic submodule Rr that is torsion, that is, $Rr \in \mathcal{T}$. Thus Rr would contain a non-zero submodule in \mathcal{P} . Therefore Rr would contain a non-zero element annihilated by a non-zero two-sided ideal of R , which is impossible because R is prime.

LEMMA 4.4. *In a prime ring R , every left ideal with a non-zero core is essential in ${}_R R$.*

Proof. Let I and J be left ideals of R with $\text{core}(I) \neq 0$ and $J \neq 0$. We must prove that $I \cap J \neq 0$. Now $\text{core}(I)J \neq 0$ because R is prime, and $\text{core}(I)J \subseteq \text{core}(I) \cap J \subseteq I \cap J$. This proves that I is essential in ${}_R R$. ■

Thus the left Gabriel topology $J(\mathcal{B})$ is contained in the Goldie topology of R [16, Section VI.6.2].

Proposition 4.5 gives the non-commutative analogue of Theorem 2.3.

PROPOSITION 4.5. *The following conditions are equivalent for a prime ring R :*

- (1) *Every non-zero torsion left R -module contains a simple submodule.*
- (2) *Every torsion left R -module is semiartinian.*

- (3) R/I is a left semiartinian ring for every non-zero proper two-sided ideal I of R .
- (4) For every proper left ideal L with a non-zero core, the cyclic left R -module R/L contains a simple submodule.

Proof. (1) \Rightarrow (2) follows immediately from the fact that every homomorphic image of a torsion module is a torsion module.

(2) \Rightarrow (3). Assume that (2) holds. Let $I \neq 0$ be a proper ideal of R . We must prove that every non-zero left R/I -module has a simple submodule. Let M be a non-zero left R/I -module. Then M is a non-zero left R -module in which every element is annihilated by I , which is in \mathcal{B} . Therefore $M \in \mathcal{P} \subseteq \mathcal{T}$, that is, M is a non-zero torsion left R -module. By (2), M is a semiartinian R -module, hence a semiartinian R/I -module. This proves that R/I is left semiartinian.

(3) \Rightarrow (4). Suppose that (3) holds. Let L be a proper left ideal with a non-zero core. By (3), $R/\text{core}(L)$ is a left semiartinian ring. Therefore R/L is a semiartinian left $R/\text{core}(L)$ -module. Thus R/L is a semiartinian left R -module, that is, it contains a simple R -submodule.

(4) \Rightarrow (1). Let ${}_R M$ be a non-zero module in \mathcal{T} . Then ${}_R M$ contains a non-zero submodule ${}_R N$ in \mathcal{P} . Let x be a non-zero element in ${}_R N$. There exists an ideal $I \in \mathcal{B}$ with $Ix = 0$. If L is the annihilator of x in R , then L is a proper left ideal of L containing I . In particular, L has a non-zero core. By (4), the cyclic left R -module $R/L \cong Rx$ contains a simple submodule. Thus ${}_R M$ contains a simple submodule. This proves that (1) holds. ■

We are ready to state and prove the non-commutative analogue of Theorem 2.4.

THEOREM 4.6. *The following statements are equivalent for a prime ring R :*

- (1) *The ring R is right almost perfect.*
- (2) *The ring R is h -local and satisfies one of the equivalent conditions of Proposition 4.5.*

Proof. (1) \Rightarrow (2). We have already shown in Corollary 4.3 that (1) implies that R is h -local. It suffices to show that condition (3) of Proposition 4.5 holds. Let I be a non-zero proper two-sided ideal of R . Then R/I is right perfect, so that R/I is left semiartinian by Theorem 2.1(6).

(2) \Rightarrow (1). Let R be an h -local ring and suppose that every torsion left R -module is semiartinian. In particular, every left R -module with a non-zero annihilator is semiartinian. We want to show that R is right almost perfect. Let I be a proper non-zero two-sided ideal of R . Then every non-zero left R/I -module is a left R -module annihilated by I , hence is semiartinian. This proves that R/I is left semiartinian. Moreover, the ring R/I is semilocal because R is h -local. Semilocal rings have no infinite orthog-

onal set of idempotents. By Theorem 2.1(6), it follows that R/I is right perfect. ■

The following result is the non-commutative analogue of [4, Corollary 2.4].

COROLLARY 4.7. *Let R be a local ring. Then R right almost perfect if and only if it satisfies one of the equivalent conditions of Proposition 4.5.*

We conclude this section with a further characterization of right almost perfect rings. Recall that the *radical* $\text{rad}(M_R)$ of a right module M_R is the intersection of all maximal submodules of M_R . Thus $\text{rad}(M_R) = M_R$ if and only if M_R has no maximal submodule. For instance, this trivially holds for the zero module.

PROPOSITION 4.8. *A ring R is a right almost perfect ring if and only if it satisfies the following two conditions:*

- (1) R is h -local;
- (2) every non-zero right R -module M_R with $\text{rad}(M_R) = M_R$ is faithful.

Proof. Let R be a right almost perfect ring. We have already seen in Corollary 4.3 that R is h -local. Let M_R be a non-zero right R -module with $\text{rad}(M_R) = M_R$. Let I be the annihilator of M_R in R , so that I is a two-sided ideal of R . If $I \neq 0$, then M is a right module over R/I , which is right perfect, hence right max (Theorem 2.1(3)). Thus M has a maximal submodule as an R/I -module, which implies $\text{rad}(M_R) \neq M_R$, a contradiction. This proves that $I = 0$, i.e., M_R is faithful.

Conversely, suppose that (1) and (2) hold. Let I be a non-zero proper two-sided ideal of R . By (1), the factor ring R/I is semilocal. By Theorem 2.1(3), in order to conclude it suffices to show that R/I is right max. Let $M_{R/I}$ be a non-zero right R/I -module. Then M , viewed as a right R -module, is not faithful, so that $\text{rad}(M_R) \neq M_R$ by (2). It follows that $M_{R/I}$ has a maximal submodule. This proves that R/I is right max. ■

5. Left noetherian rings and final remarks. For left noetherian prime rings, the situation is much better than in the case of arbitrary prime rings. Recall that a left noetherian ring is right perfect if and only if it is left artinian. (To see this, let R be a left noetherian right perfect ring. Then R right perfect implies R semilocal and $J(R)$ right T -nilpotent. In particular, $J(R)$ is nil, hence nilpotent by Levitzki's Theorem [1, Theorem 15.22]. By Hopkins's Theorem [1, Theorem 15.20], R is left artinian.)

Let R be a left noetherian prime ring. Let \mathcal{B} be the set of all non-zero two-sided ideals of R . By [16, Proposition 6.10], the set \mathcal{B} now is a basis for a left Gabriel topology, so that, when R is a left noetherian prime ring, the left Gabriel topology $J(\mathcal{B})$ turns out to be the linear topology with \mathcal{B} as a basis of neighborhoods of zero, and $\mathcal{P} = \mathcal{T}$.

Over a left noetherian ring, Proposition 4.5 and Theorem 4.6 have the following improvement.

PROPOSITION 5.1. *The following conditions are equivalent for a left noetherian prime ring R :*

- (1) *Every non-zero torsion left R -module contains a simple module.*
- (2) *Every torsion left R -module is semiartinian.*
- (3) *For every non-zero proper two-sided ideal I of R , the ring R/I is left semiartinian.*
- (4) *For every non-zero proper two-sided ideal I of R , the ring R/I is left artinian.*
- (5) *R is a right almost perfect ring.*

Proof. The equivalence of (1)–(3) has been proved in Proposition 4.5.

(3) \Rightarrow (4). Since R is left noetherian, the left Gabriel topology corresponding to semiartinian modules consists of the left ideals A of R with R/A of finite length [16, Corollary VIII.2.2]. If I is two-sided and R/I is left semiartinian, then $1 + I$ is annihilated by I , hence I belongs to the left Gabriel topology corresponding to semiartinian modules, and so R/I is a left R -module of finite length. Thus R/I is a left artinian ring.

(4) \Rightarrow (5) follows immediately from the fact that every left artinian ring is right perfect.

(5) \Rightarrow (1). If ${}_R M$ is a non-zero torsion module over the right almost perfect ring R , then ${}_R M$ has a non-zero element x , and Rx is not faithful because it is annihilated by the two-sided ideal that annihilates x . Therefore Rx , hence ${}_R M$, contains a simple module (Theorem 2.1(6)). ■

REMARKS 5.2. (1) One could think that over an h -local ring R , every torsion module ${}_R M$ decomposes into its primary components, one for each maximal ideal, like for commutative h -local rings, but this is not true, not even in the best of the cases we are dealing with. In our example, the ring R will be right and left artinian, hence right and left noetherian, right and left perfect, and semilocal. In particular, it is h -local. Let k be a field and $R := \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ the ring of all lower triangular 2×2 matrices, so that R has all the properties mentioned in the previous sentence. This ring R has exactly two maximal two-sided ideals. The cyclic projective module ${}_R M := R \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}$ has its endomorphism ring isomorphic to k , hence ${}_R M$ is an indecomposable R -module and is the extension of two non-isomorphic simple modules. Therefore ${}_R M$ does not decompose into two primary components corresponding to the two simple left R -modules.

(2) Let R be an arbitrary ring. There are two mutually exclusive cases according to whether or not the ideal 0 is maximal. Also, a ring R can

be semilocal or not. Correspondingly, right almost perfect rings belong to exactly one of the following three classes:

First class: simple rings. It corresponds to the case in which 0 is a maximal ideal. We already know that simple rings are right and left almost perfect.

Second class: corresponds to the case in which 0 is not a maximal ideal and R is semilocal. A ring R belongs to this class if and only if it is a semilocal non-simple ring, the factor rings R/P are simple artinian for all non-zero prime ideals P of R and every non-zero non-faithful left R -module is semiartinian. Notice that these conditions are either two-sided conditions on the ring or conditions on *left* R -modules, and they characterize semilocal non-simple right almost perfect rings. For instance, nearly simple chain rings belong to this class.

Third class: corresponds to the case in which 0 is not a maximal ideal and R is not semilocal. A ring R belongs to this class if and only if it is a non-simple ring, $J(R) = 0$, every non-zero element of R belongs to only finitely many maximal ideals of R , the factor rings R/P are simple artinian for all non-zero prime ideals P of R and every non-zero non-faithful left R -module is semiartinian.

Here is an example of a ring which belongs to this third class. It is due to Faith and Michler-Villamayor [12, Remark 4.5]. It is a von Neumann regular ring that is a right V -ring but not a left V -ring. Recall that right V -rings are right max rings [12, Theorem 2.1].

Let k be a field and let V_k be an infinite-dimensional vector space over k . Consider the endomorphism ring $E := \text{End}(V_k)$ and its two-sided ideal S consisting of endomorphisms of finite rank. It is easily seen that the k -subalgebra $R := k + S$ of E has just three two-sided ideals: 0 , R and S . In particular, R is prime. Trivially, the ring R is right and left almost perfect. The Jacobson radical $J(R)$ of R is zero. In fact, $J(R)$ is a proper two-sided ideal, so it can only be 0 or S . But the element $1 - E_{11} \in 1 + S$ is not invertible. Thus $J(R) = 0$. In particular, R is not semilocal, otherwise $R/J(R) \cong R$ would be semisimple artinian, which is not. Finally, R not semilocal implies that R is not right perfect and is not left perfect.

(3) In order to continue the investigation of right almost perfect rings, it is now necessary to introduce an analogue of the field of fractions Q of an integral domain for an arbitrary (right almost perfect) prime ring. The natural candidate for Q is the ring of quotients $\varinjlim \text{Hom}({}_R I, {}_R R)$, ${}_R I \in J(\mathcal{B})$, of the prime ring R with respect to the Gabriel topology $J(\mathcal{B})$.

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