ON PARTITIONS IN CYLINDERS OVER CONTINUA
AND A QUESTION OF KRASINKIEWICZ

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Abstract. We show that a metrizable continuum $X$ is locally connected if and only if every partition in the cylinder over $X$ between the bottom and the top of the cylinder contains a connected partition between these sets.

J. Krasinkiewicz asked whether for every metrizable continuum $X$ there exists a partition $L$ between the top and the bottom of the cylinder $X \times I$ such that $L$ is a hereditarily indecomposable continuum. We answer this question in the negative. We also present a construction of such partitions for any continuum $X$ which, for every $\epsilon > 0$, admits a confluent $\epsilon$-mapping onto a locally connected continuum.

1. Introduction. Our terminology follows [Ku]. All spaces are meant to be metrizable unless otherwise stated. All mappings are continuous. A closed subset $L \subset X$ is a partition in $X$ between the sets $A, B \subset X$ if there exist open disjoint subsets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $X \setminus L = U \cup V$.

**Theorem 1.1 ([Ku, Ch. VIII, §57, I, Theorem 9, and III, Theorem 1]).** If $X$ is a locally connected continuum, then any partition in the cylinder over $X$ between the bottom and the top of the cylinder contains a connected partition between these sets.

We shall show that the converse is also true, i.e., we will prove the following theorem.

**Theorem 1.2.** If $X$ is a non-locally connected metrizable continuum, then there exists a partition $L$ in $X \times I$ between the top and the bottom of the cylinder such that $L$ does not contain any connected partition between these sets.

**Corollary 1.3.** A metrizable continuum $X$ is locally connected if and only if every partition in the cylinder over $X$ between the bottom and the top of the cylinder contains a connected partition between these sets.

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Let us recall that a continuum is indecomposable if it is not the union of two proper subcontinua. A continuum $X$ is hereditarily indecomposable if any subcontinuum of $X$ is indecomposable. Bing [B] proved that for any continuum $X$ and any disjoint closed subsets $A, B$ of $X$ there is a partition $L$ between $A$ and $B$ such that every component of $L$ is hereditarily indecomposable. Bing’s theorem combined with Theorem 1.1 yields the following corollary.

**Corollary 1.4.** If $X$ is a locally connected continuum, then there is a partition $L$ in $X \times I$ between the top and the bottom of the cylinder such that $L$ is a hereditarily indecomposable continuum.

Let us recall that a closed set $F \subset X$ disjoint from $A, B \subset X$ cuts $X$ between the sets $A$ and $B$ if $F$ intersects any continuum $K \subset X$ such that $K \cap A \neq \emptyset \neq K \cap B$. Every partition between $A$ and $B$ in $X$ cuts $X$ between $A$ and $B$, and the converse is true for locally connected continua.

For any continuum $X$, J. Krasinkiewicz [Kr] constructed an “arc of hereditarily indecomposable continua” which cut $X \times I$ between the top and the bottom. More precisely, he constructed a continuum $Y \subset X \times I$ with a monotone surjection $s : Y \to I$ such that every fiber $s^{-1}(t), \; t \in (0, 1)$, is a hereditarily indecomposable continuum which cuts $X \times I$ between $X \times \{0\}$ and $X \times \{1\}$. He posed the following problem.

**Problem 1.5 ([Kr, Problem 6.1]).** Does there exist, for any continuum $X$, a partition $L$ in $X \times I$ between the top and the bottom of the cylinder such that $L$ is a hereditarily indecomposable continuum?

Let us denote by $\mathcal{K}$ the class of all continua $X$ satisfying the condition described in Problem 1.5. In Section 4 we will give an example of a continuum which does not belong to $\mathcal{K}$ and answers negatively the question of Krasinkiewicz (see Example 4.2). This also implies a negative answer to the first question of Problem 6.2 in [Kr] (see Section 5 for more details).

Let us recall that a surjective mapping $f : X \to Y$ between compacta is confluent if for any continuum $K \subset Y$ and any component $C$ of the set $f^{-1}(K)$ we have $f(C) = K$. The confluent mappings between compacta, defined by J. J. Charatonik in [JCh], form a class of mappings including the classes of open mappings and of monotone mappings.

Let us recall that a continuum $X$ is confluent $\mathcal{LC}$-like if, for every $\epsilon > 0$, $X$ admits a confluent $\epsilon$-mapping onto a locally connected continuum. The class of confluent $\mathcal{LC}$-like continua was defined and investigated by L. G. Oversteegen and J. R. Prajs in [O-P]. Non-locally connected examples of confluent $\mathcal{LC}$-like continua include Knaster type continua (i.e., the inverse
limits of arcs with open bonding mappings), solenoids, and fans that are cones over compact zero-dimensional sets.

Slightly modifying an argument by Bing [13], we shall prove the following theorem.

**Theorem 1.6.** Any confluently \( L^C \)-like continuum \( X \) belongs to \( K \). Moreover, there exists a partition \( L \) in \( X \times I \) between the top and the bottom of the cylinder such that

(i) \( L \) is a hereditarily indecomposable continuum,

(ii) \( (X \times I) \setminus L = U \cup V \) where \( U, V \) are disjoint open connected subsets in \( X \times I \).

**2. Proof of Theorem 1.2.** Suppose that \( X \) is a continuum which is not locally connected at \( x \in X \). Then \( x \) has a closed neighbourhood \( Z \) such that

(1) \( x \notin \text{int}_Z C \), where \( C \) is a component of \( Z \) and \( x \in Z \).

Let \( E \) be the space of components of \( Z \) equipped with the quotient topology and let \( q : Z \to E \) be the quotient map. Since \( Z \) is a metrizable compact space we have

(2) \( E \) is a metrizable compact zero-dimensional space.

By (1), (2) and the sequential continuity of \( q \) we can find a sequence \( x_n \) of points of \( \text{int}_X Z \) converging to \( x \) and a sequence \( C_n \ni x_n \) of pairwise different components of \( Z \) such that the sequence \( q(C_n) \) converges to \( q(C) \) in \( E \). Since the sets \( S' = \{q(C_{2i}) \mid i = 1, 2, \ldots\} \) and \( S'' = \{q(C_{2i-1}) \mid i = 1, 2, \ldots\} \) are closed subsets of the subspace \( E \setminus \{q(C)\} \), there exists a pair of disjoint, clopen subsets \( E' \) and \( E'' \) of \( E \setminus \{q(C)\} \) such that \( E \setminus \{q(C)\} = E' \cup E'' \), \( S' \subset E' \) and \( S'' \subset E'' \) by (2). Let us define \( Z' = q^{-1}(E') \), \( Z'' = q^{-1}(E'') \).

Obviously,

(3) \( Z' \) and \( Z'' \) are pairwise disjoint open subsets of \( Z \) and \( Z \setminus C = Z' \cup Z'' \).

Observe that

(4) \( x \) is a limit point of both \( \text{int}_X Z' \) and \( \text{int}_X Z'' \).

By (2), the set \( E'' \) is the union of clopen subsets of \( E \) and hence

(5) \( Z'' = q^{-1}(E'') \) is the union of clopen subsets \( W_j, j \in J \), of \( Z \). Let \( Y \) be the partition in \( X \times I \) between \( X \times \{0\} \) and \( X \times \{1\} \) defined by

(6) \( Y = \text{bd}_{X \times I} B \) where \( B = (X \times [0, 1/4]) \cup ((Z' \cup C) \times [1/4, 1/2]) \cup (Z \times [1/2, 3/4]) \).

Since the set \( B \) is compact, we have \( Y \subset B \).
By (5) we have $Z'' \times [1/2, 3/4] = \bigcup_{j \in J} W_j \times [1/2, 3/4]$. One can easily check that the sets $W_j \times [1/2, 3/4]$ are clopen subsets of $B$ and hence

(7) the set $A = Z'' \times [1/2, 3/4]$ is the union of clopen subsets of $B$.

Let us assume that $Y$ contains a connected set $L$ that is a partition in $X \times I$ between $X \times \{0\}$ and $X \times \{1\}$. Then by (7) we have

(8) $L \subset B \setminus A \subset (X \times [0, 1/4]) \cup ((Z' \cup C) \times [1/4, 3/4]) = D$.

Let $(X \times I) \setminus L = U \cup V$, where $U$ and $V$ are non-empty open subsets of $X \times I$ such that $X \times \{0\} \subset U$ and $X \times \{1\} \subset V$. Observe that by (6) any point of $G = \text{int}_X Z' \times [0, 3/4] \cup X \times [0, 1/4]$ can be connected with $X \times \{0\}$ by a “vertical” interval contained in $(X \times I) \setminus L$. Similarly, by (8) any point of $(X \times I) \setminus D$ can be connected with $X \times \{1\}$ by a vertical interval contained in $(X \times I) \setminus L$. Therefore we have

(9) $G \subset U$ and $(X \times I) \setminus D \subset V$.

The definition of $Y$ implies that the point $z = (x, 5/8)$ does not belong to $L$. By (3), (4) and (9), $z \in \text{cl}_{X \times I} U \cap \text{cl}_{X \times I} V$, a contradiction.

Remark 2.1. The arguments in the proof of Theorem 1.2 extend to the case of non-locally connected perfectly normal continua. Indeed, in this case the space $E$ defined as in the proof of Theorem 1.2 is a perfectly normal (hence Fréchet) zero-dimensional compact space. One can easily check the remaining details of the proof.

3. Proof of Theorem 1.6

Lemma 3.1 ([L-R, Corollaries 4.3 and 5.2]). If $Y$ is a locally connected continuum, then for any mapping $f : X \to Y$ the following conditions are equivalent:

(i) $f$ is confluent,

(ii) $f$ is the composition $g \circ k$ of an open mapping $g$ and a monotone mapping $k$.

Corollary 3.2. If $f_i : X_i \to Y_i$, for $i = 1, 2$, are confluent mappings of continua onto locally connected continua, then the mapping $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is also confluent.

In the proof of Theorem 1.6 we shall use the following well-known lemma, the proof of which we recall for the reader’s convenience.

Lemma 3.3. If $f : X \to Y$ is a confluent mapping between continua and $K, L$ are continua in $Y$ such that $L \subset K$ and the set $f^{-1}(L)$ is connected, then the set $f^{-1}(K)$ is also connected.
Proof. Since $f$ is confluent, any component of $f^{-1}(K)$ intersects the continuum $f^{-1}(L) \subset f^{-1}(K)$ and hence the set $f^{-1}(K)$ is connected.

We shall prove Theorem 1.6 slightly modifying a proof of Bing from [B]. Let us recall that an embedding $f : I \to X$ is $\epsilon$-crooked if there exist $0 < a < b < 1$ with $d(f(0), f(b)) < \epsilon$ and $d(f(1), f(a)) < \epsilon$. In our reasoning we will need the following lemma proved in [B].

**Lemma 3.4.** For any disjoint continua $K_1, K_2$ in the Hilbert cube $E$ and any $\epsilon > 0$ there exist disjoint closed subsets $F_1, F_2$ of $E$ such that $K_i \subset F_i$ for $i = 1, 2$ and any embedding $f : I \to X \setminus (F_1 \cup F_2)$ is $\epsilon$-crooked.

We will say that a sequence $(A_i, B_i)$, $i = 0, 1, 2, \ldots$, of pairs of subcontinua of a space $Y$ is a Bing sequence if the following conditions are satisfied:

1. $A_i \cap B_i = \emptyset$, $\text{int} A_{i+1} \supset A_i$ and $\text{int} B_{i+1} \supset B_i$ for $i = 0, 1, \ldots$,
2. $Y \setminus (A_i \cup B_i)$ is contained in an open subset $W_i$ of a locally connected continuum $Z \supset Y$ such that any embedding $f : I \to W_i$ is $(1/i)$-crooked for $i = 1, 2, \ldots$.

Let us recall that a continuum $X$ is unicoherent if for any continua $A, B$ in $X$ such that $X = A \cup B$ the set $A \cap B$ is a continuum. Any contractible continuum is unicoherent (cf. [Ku, §57, II, Theorem 2]).

**Lemma 3.5.** If $X$ is a continuum such that there exists a Bing sequence in $X \times I$ with $A_0 = X \times \{0\}$ and $B_0 = X \times \{1\}$, then $X$ belongs to $\mathcal{K}$. Moreover, there exists a partition $L$ in $X \times I$ between the top and the bottom of the cylinder such that

(i) $L$ is a hereditarily indecomposable continuum,
(ii) $(X \times I) \setminus L = U \cup V$ where $U, V$ are disjoint open connected subsets in $X \times I$.

**Proof.** Let $(A_i, B_i)$, $i = 0, 1, \ldots$, be a Bing sequence in $X \times I$ such that $A_0 = X \times \{0\}$ and $B_0 = X \times \{1\}$ and let $W_i$ and $Z$ be as in the definition of a Bing sequence. Let $U', V'$ be subsets of $X \times I$ defined by

3. $U' = \bigcup_{i=0}^{\infty} A_i$, $V' = \bigcup_{i=0}^{\infty} B_i$
and let

4. $K = (X \times I) \setminus (U' \cup V') = \bigcap_{i=0}^{\infty} [(X \times I) \setminus (A_i \cup B_i)]$.
By the definition and (1),

5. $U'$ and $V'$ are open connected disjoint subsets of $X \times I$
and $K$ is a closed subset of $X \times I$. Slightly modifying a reasoning in [B], we shall show that

6. for any pair of intersecting continua $M_1, M_2$ in $K$, either $M_1 \subset M_2$ or $M_2 \subset M_1$. 

Aiming at a contradiction, assume that there are continua $M_1, M_2$ in $K$ such that $M_1 \cap M_2 \neq \emptyset$ and $M_1 \setminus M_2 \neq \emptyset \neq M_2 \setminus M_1$. Let $x_1 \in M_2 \setminus M_1$, $x_2 \in M_1 \setminus M_2$ and let $k$ be such that $1/k$ is less than the distance between $x_j$ and $M_j$ for $j = 1, 2$. By (2) and (4) we have $M_1, M_2 \subset W_k$. Let $V_i \supset M_i$ be an open subset of $W_k$ such that

(7) the distance between $x_i$ and $V_i$ is greater than $1/k$

for $i = 1, 2$, and let

(8) $U_i \subset V_i \subset W_k$

be a component of $V_i$ such that $M_i \subset U_i$. By condition (2) and the Mazurkiewicz-Moore Theorem, $U_i$ is an open, arcwise connected subset of $Z$ for $i = 1, 2$. One can easily check that there is an embedding $f : I \to U_1 \cup U_2$ such that $f(0) = x_1$, $f(1) = x_2$, $f([0,1/2]) \subset U_2$ and $f([1/2,1]) \subset U_1$.

By (7) and (8), $f$ is not $(1/k)$-crooked, a contradiction with (2) and (8). This finishes the proof of (6).

From (6) it follows that $K$ does not contain any non-trivial arc and hence $K$ is a boundary subset of the cylinder $X \times I$. Thus

(9) $X \times I = \text{cl} U' \cup \text{cl} V'$

by (4). The cone over $X$ is unicoherent by Theorem 2 in [Ku, §57, II], and hence the set

(10) $L = \text{cl} U' \cap \text{cl} V'$

is a connected partition in $X \times I$ contained in $K$ by (4), (5) and (9). It follows that

(11) $L$ is a hereditarily indecomposable continuum

by (6). Moreover, from (9) and (10) we have

(12) $(X \times I) \setminus L = U \cup V$,

where

(13) $U = \text{cl} U' \setminus L = (X \times I) \setminus \text{cl} V'$, $V = \text{cl} V' \setminus L = (X \times I) \setminus \text{cl} U'$.

Since $U' \subset U \subset \text{cl} U'$ and $V' \subset V \subset \text{cl} V'$ by (5) and (13), the sets $U$ and $V$ are open connected disjoint subsets of $X \times I$ such that $A_0 = X \times \{0\} \subset U$ and $B_0 = X \times \{1\} \subset V$, by (3), (5) and (13). This completes the proof of Lemma 3.5 by (11) and (12).

**Lemma 3.6.** If $Z_0, Z_1, \ldots, Z_n$, $n \geq 2$, are non-empty pairwise disjoint subcontinua of a locally connected continuum $Y$, then there is a pair $K_0, K_1$ of disjoint continua in $Y$ such that $Z_i \subset K_i$ for $i = 0, 1$ and $\bigcup_{i=0}^{n} Z_i \subset K_0 \cup K_1$.

**Proof.** We proceed by induction on $n$. For $n = 1$ we can put $K_i = Z_i$ for $i = 0, 1$. Let us suppose that the statement of Lemma 3.6 is valid
for $n = 1, \ldots, k$, $k \geq 1$, and let $Z_0, Z_1, \ldots, Z_{k+1}$ be non-empty pairwise disjoint subcontinua of a locally connected continuum $Y$. Since $Y$ is arcwise connected, there is an arc $K$ in $Y$ intersecting both $Z_0$ and $Z_{k+1}$. One can easily check that there is a subarc $K'$ of $K$ intersecting $Z_{k+1}$ and such that $K' \cap Z_i \neq \emptyset$ for some $l \in \{0, 1, \ldots, k\}$ and $K \cap Z_i = \emptyset$ for $i \in \{0, 1, \ldots, k\} \setminus \{l\}$. Let us define

$$Z_i' = Z_i \cup K' \cup Z_{k+1} \text{ and } Z_{i}'' = Z_i \text{ for } i \in \{0, 1, \ldots, k\} \setminus \{l\}.$$  

One can easily check that $Z_0', Z_1', \ldots, Z_k'$ are non-empty pairwise disjoint subcontinua of $Y$. By the inductive hypothesis there is a pair $K_0, K_1$ of disjoint continua such that $Z_i' \subset K_i$ for $i = 0, 1$ and $\bigcup_{i=0}^{k} Z_i' \subset K_0 \cup K_1$. It follows that $Z_i \subset K_i$ for $i = 0, 1$ and $\bigcup_{i=0}^{k+1} Z_i \subset K_0 \cup K_1$ by (14), which finishes the proof of the inductive step and of Lemma 3.6.

**Lemma 3.7.** If $Z$ is a locally connected continuum, then for any pair $C_0, C_1$ of closed disjoint subsets of $Z$ and any continua $K_0, K_1$ in $Z$ such that $C_0 \cup C_1 \subset K_0 \cup K_1$ and $D_i \subset \text{int } K_i$ for $i = 0, 1$, there is a pair of disjoint continua $K_0, K_1$ in $Z$ such that $C_0 \cup C_1 \subset K_0 \cup K_1$ and $D_i \subset \text{int } K_i$ for $i = 0, 1$.

**Proof.** Since the space $Z$ is locally connected, there exists a finite cover $\mathcal{A}$ of $Z$ consisting of continua with diameter less than $\epsilon/3$, where $\epsilon$ is less than the distance between $C_0$ and $C_1$. Let us define $A_i = \bigcup\{A \in \mathcal{A} \mid A \cap C_i \neq \emptyset\}$ for $i = 0, 1$. One can easily check that

$$A_0 \text{ and } A_1 \text{ are disjoint closed sets having finitely many components such that}$$

$$C_i \subset \text{int } A_i \text{ for } i = 0, 1.$$  

Let $Z_i$ be a component of $A_i$ containing $D_i$ for $i = 0, 1$, and let $Z_2, Z_3, \ldots, Z_k$ be the remaining components of $A_0 \cup A_1$. By (15) and (16) we have

$$D_i \subset \text{int } Z_i \text{ for } i = 0, 1.$$  

By Lemma 3.6 there is a pair $K_0, K_1$ of disjoint continua in $Z$ such that

$$Z_i \subset K_i \text{ for } i = 0, 1 \text{ and } A_0 \cup A_1 = \bigcup_{i=0}^{k} Z_i \subset K_0 \cup K_1.$$  

By (16)–(18) we have $D_i \subset \text{int } Z_i \subset \text{int } K_i$ for $i = 0, 1$ and $C_0 \cup C_1 \subset K_0 \cup K_1$. This finishes the proof of Lemma 3.7.

**Proof of Theorem 1.6.** Let $X$ be any confluent $\text{LC}$-like continuum. Without loss of generality we can assume that $X \times I$ equipped with the product metric is a subset of the Hilbert cube $E$. By Lemma 3.5, to prove Theorem 1.6 it suffices to define a Bing sequence $(A_i, B_i)$ in $X \times I$ with $A_0 = X \times \{0\}$, $B_0 = X \times \{1\}$. Let us suppose that we have defined elements $(A_0, B_0), \ldots, (A_k, B_k)$ satisfying the conditions from the definition of a Bing
sequence. By Lemma 3.4 there is a pair of disjoint closed subsets $A, B$ in $E$ such that

(19) $A_k \subset A$, $B_k \subset B$

and

(20) any function $f : I \to U = E \setminus (A \cup B)$ is $(1/(k + 1))$-crooked.

Let $\epsilon > 0$ be less than the distance between $A$ and $B$ and let $g : X \to Y$ be a confluent $\epsilon$-mapping onto a locally connected continuum. By Corollary 3.2,

(21) $g \times \text{Id}_I$ is also a confluent $\epsilon$-mapping.

One can easily check that the sets

$C_0 = (g \times \text{Id}_I)(A \cap (X \times I))$, $C_1 = (g \times \text{Id}_I)(B \cap (X \times I))$, $D_0 = (g \times \text{Id}_I)(A_k)$, $D_1 = (g \times \text{Id}_I)(B_k)$ satisfy the assumption of Lemma 3.7 with $Z = Y \times I$, hence there is a pair of disjoint continua $K_0, K_1$ in $Y \times I$ with $C_0 \cup C_1 \subset K_0 \cup K_1$ and

(22) $Y \times \{i\} \subset D_i \subset \text{int} K_i$ for $i = 0, 1$.

By Lemma 3.3, (21) and (22) the sets

$A_{k+1} = (g \times \text{Id}_I)^{-1}(K_0) \supset (g \times \text{Id}_I)^{-1}(Y \times \{0\}) = X \times \{0\}$

and

$B_{k+1} = (g \times \text{Id}_I)^{-1}(K_1) \supset (g \times \text{Id}_I)^{-1}(Y \times \{1\}) = X \times \{1\}$

are connected and hence, by (19) and (20), $A_{k+1}, B_{k+1}$ are disjoint continua in $X \times I$ satisfying conditions (1) and (2) in the definition of a Bing sequence. This finishes the proof of Theorem 1.6.

**Remark 3.8.** From the proof of Theorem 1.6 it follows that if $X$ is a continuum such that $X \times I$ satisfies the condition described in Lemma 3.7 with $D_i = X \times \{i\}$ for $i = 0, 1$, then $X$ belongs to $\mathcal{K}$. We do not know, however, whether such continua form a class larger than the class of confluent $\mathbb{L}/\mathbb{C}$-like continua.

**Remark 3.9.** Observe that if

(23) $U, V \subset X \times I$ are disjoint open sets

with

(24) $\text{cl} U \cup \text{cl} V = X \times I$,

then the set

(25) $N = \text{cl} U \cap \text{cl} V$

is a partition in $X \times I$ between $U$ and $V$. Indeed, by (23)–(25), $(X \times I) \setminus N = U' \cup V'$, where the sets

(26) $U' = \text{cl} U \setminus N = (X \times I) \setminus \text{cl} V \supset U$, $V' = \text{cl} V \setminus N = (X \times I) \setminus \text{cl} U \supset V$

are disjoint open subsets of $X \times I$ with
(27)  \( \text{cl} U' = U' \cup N = \text{cl} U, \text{cl} V' = V' \cup N = \text{cl} V. \)

By (25)–(27) and the openness of \( U' \) and \( V' \),

(28)  \( \text{bd} U' = \text{cl} U' \setminus U' = \text{cl} U \cap \text{cl} V = N, \) and similarly, \( \text{bd} V' = N. \)

If moreover \( U \) and \( V \) are connected, then the sets \( U' \), \( V' \) are also connected by (26), (27), and the set \( N \subset (X \times I) \setminus (U \cap V) \) is a continuum by (24), (25) and the unicoherence of the cone over \( X \).

It follows that the condition (ii) in Theorem 1.6 can be replaced by

(ii')  \( (X \times I) \setminus L = U \cup V \) where \( U, V \) are disjoint open connected subsets in \( X \times I \) such that \( \text{bd} U = \text{bd} V = L. \)

4. An example of a continuum not belonging to \( K \). Let us recall that a space \( X \) has the property of Kelley if for each \( x \in X \), for each sequence \( x_n \) converging to \( x \) in \( X \) and for any continuum \( C \ni x \) in \( X \) there exists a sequence \( C_n \) of continua in \( X \) converging to \( C \) with respect to the Hausdorff metric and such that \( x_n \in C_n \) for \( i = 1, 2, \ldots. \)

Lemma 4.1 ([W, Theorem 3.1], [Ke] and [Ch-P, Theorem 2.2]). Each hereditarily indecomposable continuum and each \( \mathbb{L}\mathbb{C} \)-like continuum has the property of Kelley.

Example 4.2. Let \( X \subset \mathbb{R}^2 \) be a continuum defined by \( X = A \cup \bigcup_{i=1}^{\infty} A_i \), where \( A \subset \mathbb{R}^2 \) is a segment with endpoints \( (0,0), (0,2) \) and \( A_i, \ i = 1, 2, \ldots, \) is a segment with endpoints \( (0,0), (1/i,1) \). We shall show that there is no partition \( L \) in \( X \times I \) between the top and the bottom of the cylinder such that

(1)  \( L \) is a hereditarily indecomposable continuum

and hence \( X \) does not belong to \( K \).

On the contrary, let us assume that such a partition \( L \) exists. Let \( W, Z \) be disjoint open subsets in \( X \times I \) such that \( (X \times I) \setminus L = W \cup Z \). By the same argument as in Remark 3.9, we can prove that the set

(2)  \( N = \text{cl} W \cap \text{cl} Z \subset L \)

is a partition in \( X \times I \) between the top and the bottom of the cylinder satisfying the following condition:

(3)  \( (X \times I) \setminus N = U \cup V \) for some open disjoint subsets \( U, V \) of \( X \times I \) such that \( \text{bd} U = \text{bd} V = N. \)

Let \( J \) be the segment with endpoints \( (0,1/2), (0,3/2). \) By Theorem 1.1 there is a continuum \( K \) in \( N \cap (J \times I) \) such that

(4)  \( p(K) = J, \) where \( p \) stands for the projection of \( X \times I \) onto the first factor.
Let \( x = (0, 3/4) \in J \) and let \( y \in K \) be such that \( p(y) = x \). By the definition of \( p \) we have \( y = (x, z) \) for some \( z \in I \setminus \{0, 1\} \).

We shall show that

(5) there is a sequence \( y_n \) in \( N \setminus (A \times I) \) converging to \( y \).

It suffices to prove that every closed neighbourhood \( G \) of \( y \) intersects \( N \setminus (A \times I) \). We can consider only neighbourhoods \( G \) of \( y \) in \( X \times I \) of the form \( G = (B \cup \bigcup_{i=k}^{\infty} B_i) \times S \), where \( B \ni x \) is a segment in \( A \) with endpoints \((0, a), (0, b)\), \( 1 > b > 3/4 > a > 0 \), \( B_i \) is a segment in \( A_i \) with endpoints \(((1/i) \cdot a, a), ((1/i) \cdot b, b)\) for \( i = k, k+1, \ldots \), and \( S \) is a segment in \( I \).

Assume on the contrary that for some \( G \) as described above, we have \( G \cap N \subset A \times I \). By (3) and the connectivity of the sets \( B_i \times S \) we have

(6) \( B_i \times S \subset U \) or \( B_i \times S \subset V \) for \( i = k, k+1, \ldots \).

From (3) it follows that

(7) for any neighbourhood \( M \) of \( y \) in \( G \) the sets \( M \cap U \), \( M \cap V \) are non-empty open subsets of \( G \).

Since the set \( G \setminus (\bigcup_{i=k}^{\infty} B_i \times S) = B \times S \not\ni y \) is a boundary set in \( G \), any neighbourhood \( M \) of \( y \) in \( G \) intersects both \( U \setminus (B \times S) = U \cap \bigcup_{i=k}^{\infty} B_i \times S \) and \( V \setminus (B \times S) = V \cap \bigcup_{i=k}^{\infty} B_i \times S \) by (7). Thus, by (6),

(8) both \( U \) and \( V \) contain infinitely many sets \( B_i \times S \).

One can easily check that (3) combined with (8) yields \( B \times S \subset \operatorname{cl} U \cap \operatorname{cl} V = N \), a contradiction with (1) and (2). This finishes the proof of (5).

By (1), (2), (5) and Lemma 4.1, there is a sequence \( y_n \) in \( L \setminus (A \times I) \) converging to \( y \) and a sequence \( C_i \ni y_i \), \( i = 1, 2, \ldots \), of continua in \( L \) converging to \( K \ni y \) with respect to the Hausdorff metric. Without loss of generality we can assume that no element of \( C_i \), \( i = 1, 2, \ldots \), intersects \( \{(0,0)\} \times I \). This combined with (4) implies that \( p(C_i) \), \( i = 1, 2, \ldots \), is a sequence of continua in \( X \) converging to \( p(K) = J \) with respect to the Hausdorff metric, such that \( (0,0) \not\in p(C_i) \) and \( p(y_i) \in p(C_i) \cap (X \setminus A) \neq \emptyset \) for \( i = 1, 2, \ldots \), a contradiction.

**Remark 4.3.** The continuum \( X \) described in Example 4.2 is a well-known example of a continuum without the property of Kelley. In fact, the following theorem is true.

**If** \( X \) **is a continuum such that there is a partition** \( L \) **in** \( X \times I \) **between the top and the bottom of the cylinder such that** \( L \) **is a continuum having the property of Kelley, then** \( X \) **has the property of Kelley; in particular, any continuum belonging to** \( K \) **has the property of Kelley.**

The proof of the theorem above uses some different methods and will be published elsewhere.
5. Another question of Krasinkiewicz. Let us recall that the suspension \( S(X) \) of a topological space \( X \) is the quotient space \( X \times I/R \), where \( R \) is the equivalence relation corresponding to the decomposition of the set \( X \times I \) into the sets \( X \times \{0\} \), \( X \times \{1\} \), and the singletons contained in \( X \times (0,1) \).

For every continuum \( X \), J. Krasinkiewicz [Kr, Section 5] constructed a dendroid (i.e., an arcwise connected hereditarily unicoherent non-degenerate continuum) \( Z \), an arc \( L \subset Z \) and a monotone surjection \( g \) such that

(i) \( g \) maps the suspension \( S(X) \) of \( X \) onto \( Z \),
(ii) the fibers of \( g \) are hereditarily indecomposable,
(iii) \( L \) joins \( g(v_0) \) to \( g(v_1) \), where \( v_0, v_1 \) are the vertices of \( S(X) \),
(iv) \( (g \circ j)^{-1}(z) \), for \( z \in \text{int} L = L \setminus \{g(v_0), g(v_1)\} \), cuts \( X \times I \) between \( X \times \{0\} \) and \( X \times \{1\} \), where \( j : X \times I \to S(X) \) is the quotient map.

He posed the following problem related to this construction.

Problem 5.1 ([Kr, Problem 6.2]). Let \( Z \) and \( L \) be as in Section 5 of [Kr]. Does there exist a point in \( \text{int} L \) which separates \( Z \) between the ends of \( L \)? Is \( L \) a monotone retract of \( Z \)?

The negative answer to Problem 1.5, given in Section 4, implies the negative answer to the first question in Problem 5.1. The second question remains open.

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