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ON PARTITIONS IN CYLINDERS OVER CONTINUA AND A QUESTION OF KRASINKIEWICZ

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Abstract. We show that a metrizable continuum X is locally connected if and only if every partition in the cylinder over X between the bottom and the top of the cylinder contains a connected partition between these sets.

J. Krasinkiewicz asked whether for every metrizable continuum X there exists a partiton L between the top and the bottom of the cylinder $X \times I$ such that L is a hereditarily indecomposable continuum. We answer this question in the negative. We also present a construction of such partitions for any continuum X which, for every $\epsilon > 0$, admits a confluent ϵ-mapping onto a locally connected continuum.

1. Introduction. Our terminology follows [Ku]. All spaces are meant to be metrizable unless otherwise stated. All mappings are continuous. A closed subset $L \subset X$ is a *partition* in X between the sets $A, B \subset X$ if there exist open disjoint subsets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $X \setminus L = U \cup V$.

THEOREM 1.1 ([Ku, Ch. VIII, §57, I, Theorem 9, and III, Theorem 1]). *If X is a locally connected continuum, then any partition in the cylinder over X between the bottom and the top of the cylinder contains a connected partition between these sets.*

We shall show that the converse is also true, i.e., we will prove the following theorem.

THEOREM 1.2. *If X is a non-locally connected metrizable continuum, then there exists a partition L in $X \times I$ between the top and the bottom of the cylinder such that L does not contain any connected partition between these sets.*

COROLLARY 1.3. *A metrizable continuum X is locally connected if and only if every partition in the cylinder over X between the bottom and the top of the cylinder contains a connected partition between these sets.*

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Let us recall that a continuum is *indecomposable* if it is not the union of two proper subcontinua. A continuum X is *hereditarily indecomposable* if any subcontinuum of X is indecomposable. Bing [B] proved that for any continuum X and any disjoint closed subsets A, B of X there is a partition L between A and B such that every component of L is hereditarily indecomposable. Bing's theorem combined with Theorem 1.1 yields the following corollary.

COROLLARY 1.4. *If X is a locally connected continuum, then there is a partition L in $X \times I$ between the top and the bottom of the cylinder such that L is a hereditarily indecomposable continuum.*

Let us recall that a closed set $F \subset X$ disjoint from $A, B \subset X$ *cuts* X between the sets A and B if F intersects any continuum $K \subset X$ such that $K \cap A \neq \emptyset \neq K \cap B$. Every partition between A and B in X cuts X between A and B, and the converse is true for locally connected continua.

For any continuum X, J. Krasinkiewicz [Kr] constructed an "arc of hereditarily indecomposable continua" which cut $X \times I$ between the top and the bottom. More precisely, he constructed a continuum $Y \subset X \times I$ with a monotone surjection $s : Y \to I$ such that every fiber $s^{-1}(t)$, $t \in (0,1)$, is a hereditarily indecomposable continuum which cuts $X \times I$ between $X \times \{0\}$ and $X \times \{1\}$. He posed the following problem.

PROBLEM 1.5 ([Kr, Problem 6.1]). *Does there exist, for any continuum X, a partition L in $X \times I$ between the top and the bottom of the cylinder such that L is a hereditarily indecomposable continuum?*

Let us denote by $\mathcal{K}$ the class of all continua X satisfying the condition described in Problem 1.5. In Section 4 we will give an example of a continuum which does not belong to $\mathcal{K}$ and answers negatively the question of Krasinkiewicz (see Example 4.2). This also implies a negative answer to the first question of Problem 6.2 in [Kr] (see Section 5 for more details).

Let us recall that a surjective mapping $f : X \to Y$ between compacta is *confluent* if for any continuum $K \subset Y$ and any component C of the set $f^{-1}(K)$ we have $f(C) = K$. The confluent mappings between compacta, defined by J. J. Charatonik in [JCh], form a class of mappings including the classes of open mappings and of monotone mappings.

Let us recall that a continuum X is *confluently* $\mathbb{LC}$-*like* if, for every $\epsilon > 0$, X admits a confluent ϵ-mapping onto a locally connected continuum. The class of confluently $\mathbb{LC}$-like continua was defined and investigated by L. G. Oversteegen and J. R. Prajs in [O-P]. Non-locally connected examples of confluently $\mathbb{LC}$-like continua include Knaster type continua (i.e., the inverse limits of arcs with open bonding mappings), solenoids, and fans that are cones over compact zero-dimensional sets.

Slightly modifying an argument by Bing [B], we shall prove the following theorem.

THEOREM 1.6. *Any confluently* $\mathbb{LC}$*-like continuum* X *belongs to* $\mathcal{K}$. *Moreover, there exists a partition* L *in* $X \times I$ *between the top and the bottom of the cylinder such that*

(i) L *is a hereditarily indecomposable continuum,*
(ii) $(X \times I) \setminus L = U \cup V$ *where* U, V *are disjoint open connected subsets in* $X \times I$.

2. Proof of Theorem 1.2. Suppose that X is a continuum which is not locally connected at $x \in X$. Then x has a closed neighbourhood Z such that

(1) $x \notin \operatorname{int}_Z C$, where C is a component of Z and $x \in Z$.

Let E be the space of components of Z equipped with the quotient topology and let $q : Z \to E$ be the quotient map. Since Z is a metrizable compact space we have

(2) E is a metrizable compact zero-dimensional space.

By (1), (2) and the sequential continuity of q we can find a sequence x_n of points of $\operatorname{int}_X Z$ converging to x and a sequence $C_n \ni x_n$ of pairwise different components of Z such that the sequence $q(C_n)$ converges to $q(C)$ in E. Since the sets $S' = \{q(C_{2i}) \mid i = 1, 2, \ldots\}$ and $S'' = \{q(C_{2i-1}) \mid i = 1, 2, \ldots\}$ are closed subsets of the subspace $E \setminus \{q(C)\}$, there exists a pair of disjoint, clopen subsets E' and E'' of $E \setminus \{q(C)\}$ such that $E \setminus \{q(C)\} = E' \cup E''$, $S' \subset E'$ and $S'' \subset E''$ by (2). Let us define $Z' = q^{-1}(E')$, $Z'' = q^{-1}(E'')$.

Obviously,

(3) Z' and Z'' are pairwise disjoint open subsets of Z and $Z \setminus C = Z' \cup Z''$.

Observe that

(4) x is a limit point of both $\operatorname{int}_X Z'$ and $\operatorname{int}_X Z''$.

By (2), the set E'' is the union of clopen subsets of E and hence

(5) $Z'' = q^{-1}(E'')$ is the union of clopen subsets W_j, $j \in J$, of Z.

Let Y be the partition in $X \times I$ between $X \times \{0\}$ and $X \times \{1\}$ defined by

(6) $Y = \operatorname{bd}_{X \times I} B$ where $B = (X \times [0, 1/4]) \cup ((Z' \cup C) \times [1/4, 1/2]) \cup (Z \times [1/2, 3/4])$.

Since the set B is compact, we have $Y \subset B$.

By (5) we have $Z'' \times [1/2, 3/4] = \bigcup_{j\in J} W_j \times [1/2, 3/4]$. One can easily check that the sets $W_j \times [1/2, 3/4]$ are clopen subsets of B and hence

(7) the set $A = Z'' \times [1/2, 3/4]$ is the union of clopen subsets of B.

Let us assume that Y contains a connected set L that is a partition in $X \times I$ between $X \times \{0\}$ and $X \times \{1\}$. Then by (7) we have

(8) $L \subset B \setminus A \subset (X \times [0, 1/4]) \cup ((Z' \cup C) \times [1/4, 3/4]) = D$.

Let $(X \times I) \setminus L = U \cup V$, where U and V are non-empty open subsets of $X \times I$ such that $X \times \{0\} \subset U$ and $X \times \{1\} \subset V$. Observe that by (6) any point of $G = \operatorname{int}_X Z' \times [0, 3/4) \cup X \times [0, 1/4)$ can be connected with $X \times \{0\}$ by a "vertical" interval contained in $(X \times I) \setminus L$. Similarly, by (8) any point of $(X \times I) \setminus D$ can be connected with $X \times \{1\}$ by a vertical interval contained in $(X \times I) \setminus L$. Therefore we have

(9) $G \subset U$ and $(X \times I) \setminus D \subset V$.

The definition of Y implies that the point $z = (x, 5/8)$ does not belong to L. By (3), (4) and (9), $z \in \operatorname{cl}_{X\times I} U \cap \operatorname{cl}_{X\times I} V$, a contradiction.

Remark 2.1. The arguments in the proof of Theorem 1.2 extend to the case of non-locally connected perfectly normal continua. Indeed, in this case the space E defined as in the proof of Theorem 1.2 is a perfectly normal (hence Fréchet) zero-dimensional compact space. One can easily check the remaining details of the proof.

3. Proof of Theorem 1.6

Lemma 3.1 ([L-R, Corollaries 4.3 and 5.2]). *If Y is a locally connected continuum, then for any mapping $f : X \to Y$ the following conditions are equivalent:*

(i) *f is confluent,*
(ii) *f is the composition $g \circ k$ of an open mapping g and a monotone mapping k.*

Corollary 3.2. *If $f_i : X_i \to Y_i$, for $i = 1, 2$, are confluent mappings of continua onto locally connected continua, then the mapping $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is also confluent.*

In the proof of Theorem 1.6 we shall use the following well-known lemma, the proof of which we recall for the reader's convenience.

Lemma 3.3. *If $f : X \to Y$ is a confluent mapping between continua and K, L are continua in Y such that $L \subset K$ and the set $f^{-1}(L)$ is connected, then the set $f^{-1}(K)$ is also connected.*

Proof. Since f is confluent, any component of $f^{-1}(K)$ intersects the continuum $f^{-1}(L) \subset f^{-1}(K)$ and hence the set $f^{-1}(K)$ is connected.

We shall prove Theorem 1.6 slightly modifying a proof of Bing from [B]. Let us recall that an embedding $f : I \to X$ is *ϵ-crooked* if there exist $0 < a < b < 1$ with $d(f(0), f(b)) < \epsilon$ and $d(f(1), f(a)) < \epsilon$. In our reasoning we will need the following lemma proved in [B].

LEMMA 3.4. *For any disjoint continua K_1, K_2 in the Hilbert cube E and any $\epsilon > 0$ there exist disjoint closed subsets F_1, F_2 of E such that $K_i \subset F_i$ for $i = 1, 2$ and any embedding $f : I \to X \setminus (F_1 \cup F_2)$ is ϵ-crooked.*

We will say that a sequence (A_i, B_i), $i = 0, 1, 2, \ldots$, of pairs of subcontinua of a space Y is a *Bing sequence* if the following conditions are satisfied:

(1) $A_i \cap B_i = \emptyset$, $\operatorname{int} A_{i+1} \supset A_i$ and $\operatorname{int} B_{i+1} \supset B_i$ for $i = 0, 1, \ldots$,
(2) $Y \setminus (A_i \cup B_i)$ is contained in an open subset W_i of a locally connected continuum $Z \supset Y$ such that any embedding $f : I \to W_i$ is $(1/i)$-crooked for $i = 1, 2, \ldots$.

Let us recall that a continuum X is *unicoherent* if for any continua A, B in X such that $X = A \cup B$ the set $A \cap B$ is a continuum. Any contractible continuum is unicoherent (cf. [Ku, §57, II, Theorem 2]).

LEMMA 3.5. *If X is a continuum such that there exists a Bing sequence in $X \times I$ with $A_0 = X \times \{0\}$ and $B_0 = X \times \{1\}$, then X belongs to $\mathcal{K}$. Moreover, there exists a partition L in $X \times I$ between the top and the bottom of the cylinder such that*

(i) *L is a hereditarily indecomposable continuum,*
(ii) *$(X \times I) \setminus L = U \cup V$ where U, V are disjoint open connected subsets in $X \times I$.*

Proof. Let (A_i, B_i), $i = 0, 1, \ldots$, be a Bing sequence in $X \times I$ such that $A_0 = X \times \{0\}$ and $B_0 = X \times \{1\}$ and let W_i and Z be as in the definition of a Bing sequence. Let U', V' be subsets of $X \times I$ defined by

(3) $U' = \bigcup_{i=0}^{\infty} A_i$, $V' = \bigcup_{i=0}^{\infty} B_i$

and let

(4) $K = (X \times I) \setminus (U' \cup V') = \bigcap_{i=0}^{\infty} [(X \times I) \setminus (A_i \cup B_i)]$.

By the definition and (1),

(5) U' and V' are open connected disjoint subsets of $X \times I$

and K is a closed subset of $X \times I$. Slightly modifying a reasoning in [B], we shall show that

(6) for any pair of intersecting continua M_1, M_2 in K, either $M_1 \subset M_2$ or $M_2 \subset M_1$.

Aiming at a contradiction, assume that there are continua M_1, M_2 in K such that $M_1 \cap M_2 \neq \emptyset$ and $M_1 \setminus M_2 \neq \emptyset \neq M_2 \setminus M_1$. Let $x_1 \in M_2 \setminus M_1$, $x_2 \in M_1 \setminus M_2$ and let k be such that $1/k$ is less than the distance between x_j and M_j for $j = 1, 2$. By (2) and (4) we have $M_1, M_2 \subset W_k$. Let $V_i \supset M_i$ be an open subset of W_k such that

(7) the distance between x_i and V_i is greater than $1/k$

for $i = 1, 2$, and let

(8) $U_i \subset V_i \subset W_k$

be a component of V_i such that $M_i \subset U_i$. By condition (2) and the Mazurkiewicz–Moore Theorem, U_i is an open, arcwise connected subset of Z for $i = 1, 2$. One can easily check that there is an embedding $f : I \to U_1 \cup U_2$ such that $f(0) = x_1$, $f(1) = x_2$, $f([0, 1/2]) \subset U_2$ and $f([1/2, 1]) \subset U_1$. By (7) and (8), f is not $(1/k)$-crooked, a contradiction with (2) and (8). This finishes the proof of (6).

From (6) it follows that K does not contain any non-trivial arc and hence K is a boundary subset of the cylinder $X \times I$. Thus

(9) $X \times I = \operatorname{cl} U' \cup \operatorname{cl} V'$

by (4). The cone over X is unicoherent by Theorem 2 in [Ku, §57, II], and hence the set

(10) $L = \operatorname{cl} U' \cap \operatorname{cl} V'$

is a connected partition in $X \times I$ contained in K by (4), (5) and (9). It follows that

(11) L is a hereditarily indecomposable continuum

by (6). Moreover, from (9) and (10) we have

(12) $(X \times I) \setminus L = U \cup V$,

where

(13) $U = \operatorname{cl} U' \setminus L = (X \times I) \setminus \operatorname{cl} V'$, $V = \operatorname{cl} V' \setminus L = (X \times I) \setminus \operatorname{cl} U'$.

Since $U' \subset U \subset \operatorname{cl} U'$ and $V' \subset V \subset \operatorname{cl} V'$ by (5) and (13), the sets U and V are open connected disjoint subsets of $X \times I$ such that $A_0 = X \times \{0\} \subset U$ and $B_0 = X \times \{1\} \subset V$, by (3), (5) and (13). This completes the proof of Lemma 3.5 by (11) and (12).

LEMMA 3.6. *If $Z_0, Z_1, \ldots, Z_n$, $n \geq 2$, are non-empty pairwise disjoint subcontinua of a locally connected continuum Y, then there is a pair K_0, K_1 of disjoint continua in Y such that $Z_i \subset K_i$ for $i = 0, 1$ and $\bigcup_{i=0}^{n} Z_i \subset K_0 \cup K_1$.*

Proof. We proceed by induction on n. For $n = 1$ we can put $K_i = Z_i$ for $i = 0, 1$. Let us suppose that the statement of Lemma 3.6 is valid for $n = 1, \ldots, k$, $k \geq 1$, and let $Z_0, Z_1, \ldots, Z_{k+1}$ be non-empty pairwise disjoint subcontinua of a locally connected continuum Y. Since Y is arcwise connected, there is an arc K in Y intersecting both Z_0 and Z_{k+1}. One can easily check that there is a subarc K' of K intersecting Z_{k+1} and such that $K' \cap Z_l \neq \emptyset$ for some $l \in \{0, 1, \ldots, k\}$ and $K \cap Z_i = \emptyset$ for $i \in \{0, 1, \ldots, k\} \setminus \{l\}$. Let us define

(14) $Z'_l = Z_l \cup K' \cup Z_{k+1}$ and $Z'_i = Z_i$ for $i \in \{0, 1, \ldots, k\} \setminus \{l\}$.

One can easily check that $Z'_0, Z'_1, \ldots, Z'_k$ are non-empty pairwise disjoint subcontinua of Y. By the inductive hypothesis there is a pair K_0, K_1 of disjoint continua such that $Z'_i \subset K_i$ for $i = 0, 1$ and $\bigcup_{i=0}^{k} Z'_i \subset K_0 \cup K_1$. It follows that $Z_i \subset K_i$ for $i = 0, 1$ and $\bigcup_{i=0}^{k+1} Z_i \subset K_0 \cup K_1$ by (14), which finishes the proof of the inductive step and of Lemma 3.6.

LEMMA 3.7. *If Z is a locally connected continuum, then for any pair C_0, C_1 of closed disjoint subsets of Z and any continua D_0, D_1 such that $D_i \subset C_i$ for $i = 0, 1$, there is a pair of disjoint continua K_0, K_1 in Z such that $C_0 \cup C_1 \subset K_0 \cup K_1$ and $D_i \subset \operatorname{int} K_i$ for $i = 0, 1$.*

Proof. Since the space Z is locally connected, there exists a finite cover $\mathcal{A}$ of Z consisting of continua with diameter less than $\epsilon/3$, where ϵ is less than the distance between C_0 and C_1. Let us define $A_i = \bigcup\{A \in \mathcal{A} \mid A \cap C_i \neq \emptyset\}$ for $i = 0, 1$. One can easily check that

(15) A_0 and A_1 are disjoint closed sets having finitely many components such that

(16) $C_i \subset \operatorname{int} A_i$ for $i = 0, 1$.

Let Z_i be a component of A_i containing D_i for $i = 0, 1$, and let $Z_2, Z_3, \ldots, Z_k$ be the remaining components of $A_0 \cup A_1$. By (15) and (16) we have

(17) $D_i \subset \operatorname{int} Z_i$ for $i = 0, 1$.

By Lemma 3.6 there is a pair K_0, K_1 of disjoint continua in Z such that

(18) $Z_i \subset K_i$ for $i = 0, 1$ and $A_0 \cup A_1 = \bigcup_{i=0}^{k} Z_i \subset K_0 \cup K_1$.

By (16)–(18) we have $D_i \subset \operatorname{int} Z_i \subset \operatorname{int} K_i$ for $i = 0, 1$ and $C_0 \cup C_1 \subset K_0 \cup K_1$. This finishes the proof of Lemma 3.7.

Proof of Theorem 1.6. Let X be any confluently $\mathbb{LC}$-like continuum. Without loss of generality we can assume that $X \times I$ equipped with the product metric is a subset of the Hilbert cube E. By Lemma 3.5, to prove Theorem 1.6 it suffices to define a Bing sequence (A_i, B_i) in $X \times I$ with $A_0 = X \times \{0\}$, $B_0 = X \times \{1\}$. Let us suppose that we have defined elements $(A_0, B_0), \ldots, (A_k, B_k)$ satisfying the conditions from the definition of a Bing sequence. By Lemma 3.4 there is a pair of disjoint closed subsets A, B in E such that

(19) $A_k \subset A$, $B_k \subset B$

and

(20) any function $f : I \to U = E \setminus (A \cup B)$ is $(1/(k+1))$-crooked.

Let $\epsilon > 0$ be less than the distance between A and B and let $g : X \to Y$ be a confluent ϵ-mapping onto a locally connected continuum. By Corollary 3.2,

(21) $g \times \mathrm{Id}_I$ is also a confluent ϵ-mapping.

One can easily check that the sets $C_0 = (g \times \mathrm{Id}_I)(A \cap (X \times I))$, $C_1 = (g \times \mathrm{Id}_I)(B \cap (X \times I))$, $D_0 = (g \times \mathrm{Id}_I)(A_k)$, $D_1 = (g \times \mathrm{Id}_I)(B_k)$ satisfy the assumption of Lemma 3.7 with $Z = Y \times I$, hence there is a pair of disjoint continua K_0, K_1 in $Y \times I$ with $C_0 \cup C_1 \subset K_0 \cup K_1$ and

(22) $Y \times \{i\} \subset D_i \subset \operatorname{int} K_i$ for $i = 0, 1$.

By Lemma 3.3, (21) and (22) the sets

$$A_{k+1} = (g \times \mathrm{Id}_I)^{-1}(K_0) \supset (g \times \mathrm{Id}_I)^{-1}(Y \times \{0\}) = X \times \{0\}$$

and

$$B_{k+1} = (g \times \mathrm{Id}_I)^{-1}(K_1) \supset (g \times \mathrm{Id}_I)^{-1}(Y \times \{1\}) = X \times \{1\}$$

are connected and hence, by (19) and (20), A_{k+1}, B_{k+1} are disjoint continua in $X \times I$ satisfying conditions (1) and (2) in the definition of a Bing sequence. This finishes the proof of Theorem 1.6.

REMARK 3.8. From the proof of Theorem 1.6 it follows that if X is a continuum such that $X \times I$ satisfies the condition described in Lemma 3.7 with $D_i = X \times \{i\}$ for $i = 0, 1$, then X belongs to $\mathcal{K}$. We do not know, however, whether such continua form a class larger than the class of confluently $\mathbb{LC}$-like continua.

REMARK 3.9. Observe that if

(23) $U, V \subset X \times I$ are disjoint open sets

with

(24) $\operatorname{cl} U \cup \operatorname{cl} V = X \times I$,

then the set

(25) $N = \operatorname{cl} U \cap \operatorname{cl} V$

is a partition in $X \times I$ between U and V. Indeed, by (23)–(25), $(X \times I) \setminus N = U' \cup V'$, where the sets

(26) $U' = \operatorname{cl} U \setminus N = (X \times I) \setminus \operatorname{cl} V \supset U$, $V' = \operatorname{cl} V \setminus N = (X \times I) \setminus \operatorname{cl} U \supset V$

are disjoint open subsets of $X \times I$ with

(27) $\operatorname{cl} U' = U' \cup N = \operatorname{cl} U$, $\operatorname{cl} V' = V' \cup N = \operatorname{cl} V$.

By (25)–(27) and the openness of U' and V',

(28) $\operatorname{bd} U' = \operatorname{cl} U' \setminus U' = \operatorname{cl} U \cap \operatorname{cl} V = N$, and similarly, $\operatorname{bd} V' = N$.

If moreover U and V are connected, then the sets U', V' are also connected by (26), (27), and the set $N \subset (X \times I) \setminus (U \cap V)$ is a continuum by (24), (25) and the unicoherence of the cone over X.

It follows that the condition (ii) in Theorem 1.6 can be replaced by

(ii′) $(X \times I) \setminus L = U \cup V$ where U, V are disjoint open connected subsets in $X \times I$ such that $\operatorname{bd} U = \operatorname{bd} V = L$.

4. An example of a continuum not belonging to $\mathcal{K}$. Let us recall that a space X has the *property of Kelley* if for each $x \in X$, for each sequence x_n converging to x in X and for any continuum $C \ni x$ in X there exists a sequence C_n of continua in X converging to C with respect to the Hausdorff metric and such that $x_n \in C_n$ for $i = 1, 2, \ldots$.

LEMMA 4.1 ([W, Theorem 3.1], [Ke] and [Ch-P, Theorem 2.2]). *Each hereditarily indecomposable continuum and each $\mathbb{LC}$-like continuum has the property of Kelley.*

EXAMPLE 4.2. Let $X \subset \mathbb{R}^2$ be a continuum defined by $X = A \cup \bigcup_{i=1}^{\infty} A_i$, where $A \subset \mathbb{R}^2$ is a segment with endpoints $(0,0), (0,2)$ and A_i, $i = 1, 2, \ldots$, is a segment with endpoints $(0,0), (1/i, 1)$. We shall show that there is no partition L in $X \times I$ between the top and the bottom of the cylinder such that

(1) L is a hereditarily indecomposable continuum

and hence X does not belong to $\mathcal{K}$.

On the contrary, let us assume that such a partition L exists. Let W, Z be disjoint open subsets in $X \times I$ such that $(X \times I) \setminus L = W \cup Z$. By the same argument as in Remark 3.9, we can prove that the set

(2) $N = \operatorname{cl} W \cap \operatorname{cl} Z \subset L$

is a partition in $X \times I$ between the top and the bottom of the cylinder satisfying the following condition:

(3) $(X \times I) \setminus N = U \cup V$ for some open disjoint subsets U, V of $X \times I$ such that $\operatorname{bd} U = \operatorname{bd} V = N$.

Let J be the segment with endpoints $(0, 1/2), (0, 3/2)$. By Theorem 1.1 there is a continuum K in $N \cap (J \times I)$ such that

(4) $p(K) = J$, where p stands for the projection of $X \times I$ onto the first factor.

Let $x=(0,3/4)\in J$ and let $y\in K$ be such that $p(y)=x$. By the definition of p we have $y=(x,z)$ for some $z\in I\setminus\{0,1\}$.

We shall show that

(5) there is a sequence y_n in $N\setminus(A\times I)$ converging to y.

It suffices to prove that every closed neighbourhood G of y intersects $N\setminus(A\times I)$. We can consider only neighbourhoods G of y in $X\times I$ of the form $G=(B\cup\bigcup_{i=k}^{\infty}B_i)\times S$, where $B\ni x$ is a segment in A with endpoints $(0,a),(0,b)$, $1>b>3/4>a>0$, B_i is a segment in A_i with endpoints $((1/i)\cdot a,a),((1/i)\cdot b,b)$ for $i=k,k+1,\ldots$, and S is a segment in I.

Assume on the contrary that for some G as described above, we have $G\cap N\subset A\times I$. By (3) and the connectivity of the sets $B_i\times S$ we have

(6) $B_i\times S\subset U$ or $B_i\times S\subset V$ for $i=k,k+1,\ldots.$

From (3) it follows that

(7) for any neighbourhood M of y in G the sets $M\cap U$, $M\cap V$ are non-empty open subsets of G.

Since the set $G\setminus(\bigcup_{i=k}^{\infty}B_i\times S)=B\times S\ni y$ is a boundary set in G, any neighbourhood M of y in G intersects both $U\setminus(B\times S)=U\cap\bigcup_{i=k}^{\infty}B_i\times S$ and $V\setminus(B\times S)=V\cap\bigcup_{i=k}^{\infty}B_i\times S$ by (7). Thus, by (6),

(8) both U and V contain infinitely many sets $B_i\times S$.

One can easily check that (3) combined with (8) yields $B\times S\subset \operatorname{cl}U\cap\operatorname{cl}V=N$, a contradiction with (1) and (2). This finishes the proof of (5).

By (1), (2), (5) and Lemma 4.1, there is a sequence y_n in $L\setminus(A\times I)$ converging to y and a sequence $C_i\ni y_i$, $i=1,2,\ldots$, of continua in L converging to $K\ni y$ with respect to the Hausdorff metric. Without loss of generality we can assume that no element of C_i, $i=1,2,\ldots$, intersects $\{(0,0)\}\times I$. This combined with (4) implies that $p(C_i)$, $i=1,2,\ldots$, is a sequence of continua in X converging to $p(K)=J$ with respect to the Hausdorff metric, such that $(0,0)\notin p(C_i)$ and $p(y_i)\in p(C_i)\cap(X\setminus A)\neq\emptyset$ for $i=1,2,\ldots$, a contradiction.

Remark 4.3. The continuum X described in Example 4.2 is a well-known example of a continuum without the property of Kelley. In fact, the following theorem is true.

If X is a continuum such that there is a partition L in $X\times I$ between the top and the bottom of the cylinder such that L is a continuum having the property of Kelley, then X has the property of Kelley; in particular, any continuum belonging to $\mathcal{K}$ has the property of Kelley.

The proof of the theorem above uses some different methods and will be published elsewhere.

5. Another question of Krasinkiewicz. Let us recall that the suspension $S(X)$ of a topological space X is the quotient space $X \times I/R$, where R is the equivalence relation corresponding to the decomposition of the set $X \times I$ into the sets $X \times \{0\}$, $X \times \{1\}$, and the singletons contained in $X \times (0, 1)$.

For every continuum X, J. Krasinkiewicz [Kr, Section 5] constructed a dendroid (i.e., an arcwise connected hereditarily unicoherent non-degenerate continuum) Z, an arc $L \subset Z$ and a monotone surjection g such that

(i) g maps the suspension $S(X)$ of X onto Z,
(ii) the fibers of g are hereditarily indecomposable,
(iii) L joins $g(v_0)$ to $g(v_1)$, where v_0, v_1 are the vertices of $S(X)$,
(iv) $(g \circ j)^{-1}(z)$, for $z \in \operatorname{int} L = L \setminus \{g(v_0), g(v_1)\}$, cuts $X \times I$ between $X \times \{0\}$ and $X \times \{1\}$, where $j : X \times I \to S(X)$ is the quotient map.

He posed the following problem related to this construction.

PROBLEM 5.1 ([Kr, Problem 6.2]). *Let Z and L be as in Section* 5 *of* [Kr]. *Does there exist a point in* $\operatorname{int} L$ *which separates Z between the ends of L? Is L a monotone retract of Z?*

The negative answer to Problem 1.5, given in Section 4, implies the negative answer to the first question in Problem 5.1. The second question remains open.

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