COLLOQUIUM MATHEMATICUM

VOL. 122 2011 NO. 2

A GENERALIZATION OF A THEOREM OF MAMMANA

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Abstract. We prove that any linear ordinary differential operator with complex-valued coefficients continuous in an interval I can be factored into a product of first-order operators globally defined on I. This generalizes a theorem of Mammana for the case of real-valued coefficients.

1. Introduction. Let L be a linear ordinary differential operator of order n,

$$L = \left(\frac{d}{dx}\right)^n + a_1(x)\left(\frac{d}{dx}\right)^{n-1} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x), \tag{1.1}$$

where the coefficients $a_1, \ldots, a_n$ are *real-valued* continuous functions in an interval I, $a_j \in C^0(I)$. Mammana [4, 5] proved that L always admits a factorization of the form

$$L = \left(\frac{d}{dx} - \alpha_1(x)\right) \cdots \left(\frac{d}{dx} - \alpha_n(x)\right), \tag{1.2}$$

where the functions $\alpha_1, \ldots, \alpha_n$ are in general *complex-valued* and continuous in the entire interval I and such that $\alpha_j \in C^{j-1}(I, \mathbb{C})$ $(1 \le j \le n)$. (See [5, Teorema generale, p. 207].)

A *local* factorization of the form (1.2) had been known for some time (see, for instance, [3, p. 121]). The new point established in [4, 5] is that one can always find a *global* decomposition of the form (1.2) (i.e., valid on the whole interval I) if one allows the α_j to be complex-valued. The proof is based on the existence of a fundamental system of solutions of the homogeneous equation $Ly = 0$ (L given by (1.1)) whose complete chain of Wronskians is never zero in I.

More specifically, let $z_1, \ldots, z_n$ be a fundamental system of solutions with the property that the sequence of Wronskian determinants

2010 *Mathematics Subject Classification*: Primary 34A30.
Key words and phrases: linear differential operators, Pólya–Mammana factorization.

DOI: 10.4064/cm122-2-6

$$w_0 = 1, \quad w_1 = z_1,$$

$$w_2 = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix}, \; \ldots, \; w_j = \begin{vmatrix} z_1 & z_2 & \cdots & z_j \\ z_1' & z_2' & \cdots & z_j' \\ \vdots & \vdots & & \vdots \\ z_1^{(j-1)} & z_2^{(j-1)} & \cdots & z_j^{(j-1)} \end{vmatrix} \qquad (1 \le j \le n)$$

never vanishes in the interval I. A generic fundamental system does not have this property. Recall that $z_1, \ldots, z_n$ are linearly independent solutions of $Ly = 0$ if and only if their Wronskian w_n is nonzero at some point of I, in which case $w_n(t) \neq 0$ for all $t \in I$. However, the lower dimensional Wronskians w_j, $j < n$, can vanish in I. Mammana proves that a fundamental system with $w_j(x) \neq 0$ for all $x \in I$ and j always exists, with z_1 (generally) complex-valued, while $z_2, \ldots, z_n$ can be taken to be real-valued. The functions α_j in (1.2) are then the logarithmic derivatives of ratios of Wronskians, namely

$$\alpha_j = \frac{d}{dx} \log \frac{w_{n-j+1}}{w_{n-j}} \qquad (1 \le j \le n). \tag{1.3}$$

The purpose of this paper is to generalize the result of Mammana to linear ordinary differential operators (1.1) with *complex-valued* coefficients $a_j \in C^0(I, \mathbb{C})$ $(1 \le j \le n)$. We prove that any such operator can be written in the form (1.2) with $\alpha_j \in C^{j-1}(I, \mathbb{C})$, by showing that there exists a fundamental system with a nowhere-vanishing complete chain of Wronskians (this condition being equivalent to factorization).

Our proof is quite different from the proof of Mammana in the real case. It is more of a topological or differential-geometric nature. For example for $n = 2$ we use the fact that a differentiable map $f : I \to \mathbb{CP}^1$ cannot be surjective (by Sard's theorem) to prove the existence of a nowhere-vanishing complex linear combination of any given fundamental system. This implies the factorization of L. The case $n > 2$ is handled by induction on n using similar ideas.

2. The case $n = 2$.

We start with the following result, whose proof is elementary.

Proposition 2.1. *Let L be a second-order linear ordinary differential operator*

$$L = \left(\frac{d}{dx}\right)^2 + a_1(x)\frac{d}{dx} + a_2(x),$$

where $a_1, a_2 \in C^0(I, \mathbb{C})$, I an interval. Then the following conditions are equivalent:

(i) *L admits the factorization*

$$L = \left(\frac{d}{dx} - \gamma(x) \right) \left(\frac{d}{dx} - \beta(x) \right) \tag{2.1}$$

for some $\gamma \in C^0(I, \mathbb{C})$ *and* $\beta \in C^1(I, \mathbb{C})$.

(ii) *There exists a solution* $\beta \in C^1(I, \mathbb{C})$ *of the complex Riccati equation*

$$\beta' + \beta^2 + a_1 \beta + a_2 = 0.$$

(iii) *There exists a solution* $\alpha : I \to \mathbb{C}$ *of* $Ly = 0$ *such that* $\alpha(x) \neq 0$ *for all* $x \in I$. *The relation between the functions* α, β *and* γ *is then*

$$\beta = \alpha'/\alpha, \qquad \alpha = e^{\int \beta \, dx}, \qquad \gamma = -a_1 - \beta.$$

If a_1 and a_2 are real-valued, then conditions (i)–(iii) above can always be satisfied with α, β and γ complex-valued. Indeed let y_1, y_2 be two linearly independent real solutions of $Ly = 0$. Then the function $\alpha = y_1 + iy_2$ is never zero in I, and we get the factorization (2.1) with $\beta = \alpha'/\alpha$ [4]. It is natural to ask in the real case if there exists a factorization of the form (2.1) with β and γ real-valued. The answer is no, in general. Indeed, for I open or compact and a_1, a_2 real-valued, conditions (i)–(iii) with α, β, γ real-valued are equivalent to

(iv) *L is disconjugate on I, i.e., every nontrivial real solution of* $Ly = 0$ *has at most one zero in* I.

See, for instance, [2, Corollary 6.1, p. 351], or [1, Theorem 1, p. 5]. This is also proved in [4] (for I compact), but the connection between disconjugacy and the factorization of a real linear differential operator of order n into a product of first-order real operators was first discussed by Pólya in [6]. (The so-called Pólya factorization [6, formula (18)] is equivalent to the Mammana factorization (1.2); see [1, formula (8), p. 92].) In general, a real L is not disconjugate on I. For example if the differential equation $Ly = 0$ is *oscillatory* on I, then every solution has infinitely many zeros in I.

When we move from real-valued coefficients to complex-valued coefficients, the equivalence between disconjugacy and factorization breaks down. (The definition of disconjugacy in the complex case is similar to the one in the real case.) A technical reason for this is that there is no analogue of Rolle's theorem in the complex case. Rolle's theorem is used, in the real case, for proving one of the implications in the above mentioned equivalence, as well as a number of important results, such as Sturm's separation theorem (see e.g., [1, Proposition 1, p. 4]).

This brings us to the question whether conditions (i)–(iii) in Proposition 2.1 always hold for complex differential operators. We shall now see that this is indeed the case. Thus for a_1 and a_2 complex-valued, we can always arrange a factorization of the form (2.1) (even if L is not disconjugate on I). Of course such a factorization is not unique, in fact we shall see that the functions α as in (iii) are quite abundant.

The proof is, however, quite different from the proof of Mammana in the real case. Indeed, if y_1, y_2 is a fundamental system, it is not clear how to exhibit a nowhere-vanishing linear combination of y_1 and y_2 as in the real case. Intuitively, one can reason by contradiction as follows. Suppose such a linear combination does not exist. Then every solution of $Ly = 0$ has at least one zero in I. This implies that in order to specify a given solution, it is enough to know one of its zeros and the derivative at that point. This is one real parameter plus one complex parameter, for a total of three real parameters. But we know that the vector space of solutions is isomorphic to $\mathbb{C}^2 \simeq \mathbb{R}^4$. This argument would allow us to construct an injective map $\mathbb{R}^4 \to I \times \mathbb{R}^2 \subset \mathbb{R}^3$, and one would have to prove its continuity to get a contradiction.

Instead of making this argument more precise, we will proceed in a different (and simpler) way. We will actually get the result as a corollary of the following proposition about the impossibility of filling the sphere S^2 with a differentiable curve.

Recall that the complex projective space $\mathbb{CP}^1$ is the compactification of $\mathbb{C}$ and can be identified with the Riemann sphere S^2.

PROPOSITION 2.2. *Let $I \subset \mathbb{R}$ be an interval. A differentiable map $f : I \to \mathbb{CP}^1$ cannot be surjective.*

Proof. Sard's theorem implies that the image under f of the set of critical values has measure zero. Since all points in I are critical, $f(I)$ has measure zero and must be different from S^2.

COROLLARY 2.3. *Let $y_1, y_2 : I \to \mathbb{C}$ be two differentiable functions without common zeros in I. Then there exists a linear combination of y_1 and y_2 that vanishes nowhere in I.*

Proof. Since y_1 and y_2 do not vanish simultaneously, there is a well defined map

$$f : I \to \mathbb{CP}^1, \qquad x \mapsto f(x) = [y_1(x) : y_2(x)].$$

Assume for contradiction that any linear combination has a zero in I. If $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0\}$, then the determinant

$$\begin{vmatrix} \alpha & y_1(x) \\ \beta & y_2(x) \end{vmatrix}$$

vanishes at some point $x_0 \in I$. This implies that $(y_1(x_0), y_2(x_0))$ is proportional to (α, β). Thus the map f is surjective, which contradicts Proposition 2.2.

THEOREM 2.4. *Let $I \subset \mathbb{R}$ be an interval, and let*

$$L = \left(\frac{d}{dx}\right)^2 + a_1(x)\frac{d}{dx} + a_2(x)$$

be a second-order linear differential operator, where $a_1, a_2 : I \to \mathbb{C}$ are continuous functions. Then there exists a solution $\alpha : I \to \mathbb{C}$ of $Ly = 0$ that vanishes nowhere in I. As a consequence, L admits a factorization of the form (2.1).

Proof. Let y_1, y_2 be two linearly independent solutions of $Ly = 0$. Then y_1, y_2 have no common zero in I, and the result follows from Corollary 2.3 and Proposition 2.1.

3. The general case. Let $f_1, \dots, f_n : I \to \mathbb{C}$ be some functions, and let $\mathcal{L}(f_1, \dots, f_n)$ be their linear span (over $\mathbb{C}$), that is, $f \in \mathcal{L}(f_1, \dots, f_n)$ if and only if f is a linear combination of $f_1, \dots, f_n$ with complex coefficients.

LEMMA 3.1. *Let $f_1, \dots, f_n : I \to \mathbb{C}$ be C^1 functions without a common zero in I, that is, for each $x \in I$ there is $j \in \{1, \dots, n\}$ such that $f_j(x) \neq 0$. Then there exists $f \in \mathcal{L}(f_1, \dots, f_n)$ such that*

$$f(x) \neq 0 \qquad \forall x \in I.$$

Proof. We proceed by induction on n. If $n = 1$ the assertion is obvious. Assume it is true for n and let $f_1, \dots, f_n, f_{n+1} : I \to \mathbb{C}$ be C^1 functions without a common zero in I. Consider the map $F : I \to \mathbb{C}^{n+1}$ given by

$$F(x) = (f_1(x), \dots, f_n(x), f_{n+1}(x)).$$

Let $\mathbb{CP}^n$ be the complex projective space, i.e. the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the action of $\mathbb{C}^*$. It is standard to denote the projection $\mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{CP}^n$ by

$$\pi((x_1, \dots, x_{n+1})) = [x_1 : \dots : x_{n+1}].$$

Note that $\pi(F(x))$ is well defined since the functions $f_1, \dots, f_n$ do not vanish simultaneously at any $x \in I$. Since F is C^1, the composition $\pi \circ F : I \to \mathbb{CP}^n$ cannot be surjective by Sard's theorem. Thus there exists $a = [a_1 : \dots : a_{n+1}]$ such that $\pi(F(x)) \neq a$ for all $x \in I$. Let $M = (m_{ij})$ be an $(n+1) \times (n+1)$ invertible matrix such that

$$M \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} =: e_{n+1}.$$

Regarding M as a linear map from $\mathbb{C}^{n+1}$ into itself, we see that $(M \circ F)(x)$ is not proportional to e_{n+1} at any point $x \in I$. Hence the n functions

$$z_1 = \sum_{k=1}^{n+1} m_{1k} f_k, \ \dots \ , z_n = \sum_{k=1}^{n+1} m_{nk} f_k$$

do not vanish simultaneously at any $x \in I$. Since $z_1, \dots, z_n$ are C^1, we can use the inductive hypothesis to get $f \in \mathcal{L}(z_1, , \dots, z_n)$ such that $f(x) \neq 0$ for all $x \in I$. But since $\mathcal{L}(z_1, \dots, z_n) \subset \mathcal{L}(f_1, \dots, f_n, f_{n+1})$, we get $f \in \mathcal{L}(f_1, \dots, f_n, f_{n+1})$, which proves the lemma.

Let $f_1, \dots, f_n : I \to \mathbb{C}$ be functions of class C^n. Their *Wronskian* is defined to be the following determinant:

$$\mathrm{W}(f_1, \dots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

THEOREM 3.2. *Assume the Wronskian* $\mathrm{W}(f_1, \dots, f_n)$ *has no zeros in* I. *Then there exist* $z_1, \dots, z_n \in \mathcal{L}(f_1, \dots, f_n)$ *such that*

$$\begin{cases} \mathrm{W}(z_1)(x) = z_1(x) \neq 0 \ \forall x \in I, \\ \mathrm{W}(z_1, z_2)(x) \neq 0 \ \forall x \in I, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{W}(z_1, \dots, z_n)(x) \neq 0 \ \forall x \in I. \end{cases} \tag{3.1}$$

Proof. The existence of z_1 follows from Lemma 3.1. To construct $z_2 \in \mathcal{L}(f_1, \dots, f_n)$, we need constants $\alpha_1, \dots, \alpha_n$ such that

$$\begin{cases} z_2(x) := \alpha_1 f_1(x) + \cdots + \alpha_n f_n(x), \\ \mathrm{W}(z_1, z_2)(x) \neq 0 \ \forall x \in I. \end{cases} \tag{3.2}$$

Observe that

$$\begin{aligned} \mathrm{W}(z_1, z_2)(x) &= \mathrm{W}(z_1, \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n)(x) \\ &= \sum_{k=1}^{n} \alpha_k \mathrm{W}(z_1, f_k)(x) = \sum_{k=1}^{n} \alpha_k \begin{vmatrix} z_1(x) & f_k(x) \\ z_1'(x) & f_k'(x) \end{vmatrix}. \end{aligned}$$

Thus the existence of $z_2(x)$ with the desired properties (3.2) is equivalent to the existence of a linear combination of the 2×2 determinants

$$D_k(x) := \begin{vmatrix} z_1(x) & f_k(x) \\ z_1'(x) & f_k'(x) \end{vmatrix}, \qquad k = 1, \dots, n,$$

without zeros in I. Notice that $D_1, \dots, D_n$ are C^1. If we show that they do not have a common zero on I, then Lemma 3.1 yields the existence of z_2.

Suppose $x_0 \in I$ and $D_1(x_0) = \cdots = D_n(x_0) = 0$. Then the rank of the matrix

$$\begin{pmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\ f_1'(x_0) & f_2'(x_0) & \cdots & f_n'(x_0) \end{pmatrix}$$

is one since each column $\binom{f_j(x_0)}{f_j'(x_0)}$ is proportional to the nonzero column $\binom{z_1(x_0)}{z_1'(x_0)}$. That is, the first two rows of the Wronskian $\mathrm{W}(f_1, \ldots, f_n)(x_0)$ are proportional. Since the Wronskian was assumed to be nonzero on I, we have a contradiction. Thus indeed z_2 exists.

Suppose now that we have constructed $z_j \in \mathcal{L}(f_1, \ldots, f_n)$ $(j < n)$ such that

$$\begin{cases} \mathrm{W}(z_1)(x) = z_1(x) \neq 0 \ \forall x \in I, \\ \mathrm{W}(z_1, z_2)(x) \neq 0 \ \forall x \in I, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{W}(z_1, \ldots, z_j)(x) \neq 0 \ \forall x \in I, \end{cases}$$

and let us show the existence of $z_{j+1} \in \mathcal{L}(f_1, \ldots, f_n)$ such that

$$\mathrm{W}(z_1, \ldots, z_j, z_{j+1})(x) \neq 0 \qquad \forall x \in I.$$

As for z_2, we look for constants $\beta_1, \ldots, \beta_n$ such that

$$\mathrm{W}(z_1, \ldots, z_j, \beta_1 f_1 + \cdots + \beta_n f_n)(x) \neq 0 \qquad \forall x \in I.$$

That is, we look for a linear combination of the determinants

$$E_k(x) := \mathrm{W}(z_1, \ldots, z_j, f_k)(x) \qquad (k = 1, \ldots, n)$$

without zeros in I. Note that the functions E_k are C^1, so again by Lemma 3.1 it suffices to show that they do not have a common zero in I. Assume on the contrary that there exists $x_0 \in I$ such that $E_1(x_0) = \cdots = E_n(x_0) = 0$. Then the matrix

$$\begin{pmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ f_1'(x_0) & \cdots & f_n'(x_0) \\ \vdots & \cdots & \vdots \\ f_1^{(j)}(x_0) & \cdots & f_n^{(j)}(x_0) \end{pmatrix}$$

has rank $\leq j$ because all its columns are linear combinations of the columns of the matrix

$$\begin{pmatrix} z_1(x_0) & \cdots & z_j(x_0) \\ z_1'(x_0) & \cdots & z_j'(x_0) \\ \vdots & \cdots & \vdots \\ z_1^{(j)}(x_0) & \cdots & z_j^{(j)}(x_0) \end{pmatrix}$$

whose rank is j because $\mathrm{W}(z_1, \dots, z_j)(x) \neq 0$ for all $x \in I$. It follows that the first $j+1$ rows of the Wronskian $\mathrm{W}(f_1, \dots, f_n)(x_0)$ are linearly dependent and so $\mathrm{W}(f_1, \dots, f_n)(x_0) = 0$. This is a contradiction because we assume the Wronskian $\mathrm{W}(f_1, \dots, f_n)$ has no zeros in I.

This completes the proof of the theorem.

THEOREM 3.3. *Let L be a linear ordinary differential operator of order n,*

$$L = \left(\frac{d}{dx}\right)^n + a_1(x)\left(\frac{d}{dx}\right)^{n-1} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x),$$

with coefficients $a_j \in C^0(I, \mathbb{C})$, I an interval. Then L has the property W, *i.e., there exists a fundamental system $z_1, \dots, z_n$ of solutions of $Ly = 0$ such that* (3.1) *holds. Consequently, L admits the factorization*

$$L = \left(\frac{d}{dx} - \alpha_1(x)\right) \cdots \left(\frac{d}{dx} - \alpha_n(x)\right),$$

where $\alpha_j \in C^{j-1}(I, \mathbb{C})$ $(1 \leq j \leq n)$ is given by (1.3) *with $w_0 = 1$ and $w_j = W(z_1, \dots, z_j)$.*

Proof. Let $f_1, \dots, f_n$ be any fundamental system of solutions of $Ly = 0$. We apply Theorem 3.2 to find $z_1, \dots, z_n$ with the required property. The equivalence (in the real case) between the property W of L and the factorization of L into first-order factors is proved in [1, Theorem 2, p. 91]. Note that the proof remains unchanged in the case of complex-valued coefficients. (The condition that the partial Wronskians are all positive throughout I is replaced by the condition that they vanish nowhere in I.) See also [6], and [5, Lemma II, p. 198].

Acknowledgements. The authors would like to thank Uri Elias, Luciano Pandolfi, and Paolo Tilli for interesting conversations.

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Received 22 September 2010 (5429)