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CHARACTER INNER AMENABILITY OF CERTAIN BANACH ALGEBRAS

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Abstract. Character inner amenability for a certain class of Banach algebras including projective tensor products, Lau products and module extensions is investigated. Some illuminating examples are given.

1. Introduction. The concept of left amenability for a Lau algebra (a predual of a von Neumann algebra for which the identity of the dual is a multiplicative linear functional, [L]) has been extended to arbitrary Banach algebras by introducing the notion of φ-amenability in Kaniuth et al. [KLP1]. A Banach algebra A is called *φ-amenable* ($\varphi \in \triangle(A) =$ the spectrum of A) if there exists an $m \in A^{**}$ satisfying $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ $(a \in A,\ f \in A^*)$, and *character amenable* if it is φ-amenable for each $\varphi \in \triangle(A)$. Many aspects of φ-amenability have been investigated in [KLP2, M2, HMT]. Recently Jabbari et al. [JMZ] have introduced the φ-version of inner amenability. A Banach algebra A is said to be *φ-inner amenable* if there exists an $m \in A^{**}$ satisfying $m(\varphi) = 1$ and $m(f \cdot a) = m(a \cdot f)$ $(f \in A^*, a \in A)$. Such an m will sometimes be referred to as a *φ-inner mean*, and A is said to be *character inner amenable* if it is φ-inner amenable for every $\varphi \in \triangle(A)$. As remarked in [JMZ, Remark 2.4], this concept considerably generalizes the notion of inner amenability for Lau algebras which was introduced by Nasr-Isfahani [N]. The authors of [JMZ] also gave several characterizations of φ-inner amenability. For instance, as in the case of φ-amenability in [KLP1, Theorem 1.4], they showed that a φ-inner mean is in fact some w^*-cluster point of a bounded net (a_α) in A satisfying $\|a_\alpha a - a a_\alpha\| \to 0$ for all $a \in A$ and $\varphi(a_\alpha) = 1$ for all α [JMZ, Theorem 2.1].

In this paper, we are going to investigate the character inner amenability for certain products of Banach algebras including the projective tensor product $A \hat{\otimes} B$, Lau product $A \times_\theta B$ and the module extension $A \oplus X$. For

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instance, we show that the projective tensor product $A \hat{\otimes} B$ is character inner amenable if and only if both A and B have this property.

2. Preliminary results and examples. Before we proceed to the results we need some preliminaries. The second dual A^{**} of a Banach algebra A can be made into a Banach algebra under each of the Arens products $\Box$ and $\Diamond$ which are defined as follows. For $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$,

$$\langle m \Box n, f \rangle = \langle m, n \cdot f \rangle, \quad \langle n \cdot f, a \rangle = \langle n, f \cdot a \rangle, \quad \langle f \cdot a, b \rangle = \langle f, ab \rangle;$$

and

$$\langle f, m \Diamond n \rangle = \langle f \cdot m, n \rangle, \quad \langle a, f \cdot m \rangle = \langle a \cdot f, m \rangle, \quad \langle b, a \cdot f \rangle = \langle ba, f \rangle.$$

We commence with a definition from [JMZ].

Definition 2.1. Let A be a Banach algebra and let $\varphi \in \triangle(A)$. Then A is called *φ-inner amenable* if there exists an $m \in A^{**}$ such that $m(\varphi) = 1$ and $m \Box a = a \Box m$ $(a \in A)$. We call such an m a *φ-inner mean.* A Banach algebra A is called *character inner amenable* if it is φ-inner amenable for all $\varphi \in \triangle(A)$.

The next straightforward characterization of φ-inner amenability (see [JMZ, Theorem 2.1] which is inspired from [KLP1, Theorem 1.4] and [E, LP1, LP2]) will be frequently used.

Proposition 2.2. *Let A be a Banach algebra and $\varphi \in \triangle(A)$. Then the following statements are equivalent.*

(i) *A is φ-inner amenable.*
(ii) *There exists a bounded net (a_α) in A such that $\|aa_\alpha - a_\alpha a\| \to 0$ for all $a \in A$ and $\varphi(a_\alpha) = 1$ for all α.*
(iii) *There exists a bounded net (a_α) in A such that $\|aa_\alpha - a_\alpha a\| \to 0$ for all $a \in A$ and $\varphi(a_\alpha) \to 1$.*

Examples 2.3. (i) Every Banach algebra with a bounded approximate identity (e_α) is character inner amenable. Indeed, one can verify that $\|ae_\alpha - e_\alpha a\| \to 0$ and $\varphi(e_\alpha) \to 1$ for each $\varphi \in \triangle(A)$.

(ii) Every commutative Banach algebra is character inner amenable.

(iii) Let $A = \left\{ \left(\begin{smallmatrix} 0 & a \\ 0 & b \end{smallmatrix} \right) : a, b \in \mathbb{C} \right\}$ and define $\varphi : A \to \mathbb{C}$ by $\varphi\left(\left(\begin{smallmatrix} 0 & a \\ 0 & b \end{smallmatrix} \right) \right) = b$. A direct verification shows that there is no bounded net (a_α) in A satisfying the conditions of Proposition 2.2. Therefore A is not φ-inner amenable.

(iv) Given a Banach space A, fix a non-zero $\varphi \in A^*$ with $\|\varphi\| \leq 1$. Then the product $a \cdot b = \varphi(a)b$ turns A into a Banach algebra with $\triangle(A) = \{\varphi\}$. Trivially A has a left identity (indeed, every $e \in A$ with $\varphi(e) = 1$ is a left identity), while it has no bounded approximate identity in the case where $\dim(A) > 1$. In this case A is not φ-inner amenable. Indeed, if m is a φ-inner mean for A then $m(\varphi) = 1$ and $m \Box a = a \Box m$ for all $a \in A$. But a simple calculation shows that $m\Box a = m(\varphi)a$ and $a\Box m = \varphi(a)m$. Therefore $a = \varphi(a)m$ for each $a \in A$, that is, $\dim(A) = 1$.

(v) Let the Banach algebra A of (iv) be generated by two elements a and b, that is, $\dim(A) = 2$, and let $\varphi \in A^*$ be such that $\varphi(a) = 1$ and $\varphi(b) = 0$. If I is the subspace generated by b then I is a closed ideal for which I and A/I are character inner amenable; however, A itself is not.

Note that, as [JMZ, Theorem 2.8] demonstrates, if A is character inner amenable then so is A/I for each closed ideal I of A. However, I may not be character inner amenable; for example, the unitization of a non-character inner amenable Banach algebra is character inner amenable.

(vi) For a reflexive Banach algebra A with $\varphi \in \triangle(A)$ it is easy to verify that A is φ-inner amenable if and only if $Z(A) \cap (A - \ker\varphi) \neq \emptyset$, where $Z(A)$ is the algebraic center of A.

3. Projective tensor product $A \hat{\otimes} B$. Let $A \hat{\otimes} B$ be the projective tensor product of two Banach algebras A and B. For $f \in A^*$ and $g \in B^*$, let $f \otimes g$ denote the element of $(A\hat{\otimes}B)^*$ satisfying $(f \otimes g)(a \otimes b) = f(a)g(b)$ $(a \in A,\ b \in B)$. Recall that

$$\triangle(A \hat{\otimes} B) = \{\varphi \otimes \psi : \varphi \in \triangle(A),\ \psi \in \triangle(B)\}.$$

In the next result, as in the case of character amenability in [KLP1, Theorem 3.3], we investigate the character inner amenability of $A \hat{\otimes} B$. It is worth mentioning that our method of proof provides an alternative proof for [KLP1, Theorem 3.3] which does not rely on derivation techniques.

THEOREM 3.1. *Let A and B be Banach algebras and let $\varphi \in \triangle(A)$, $\psi \in \triangle(B)$. Then $A \hat{\otimes} B$ is $(\varphi \otimes \psi)$-inner amenable if and only if A is φ-inner amenable and B is ψ-inner amenable. In particular, $A\hat{\otimes}B$ is character inner amenable if and only if both A and B are character inner amenable.*

Proof. Let $m \in (A \hat{\otimes} B)^{**}$ be a $(\varphi \otimes \psi)$-inner mean. Then $m(\varphi \otimes \psi) = 1$ and

$$m((f \otimes \psi) \cdot (a \otimes b)) = m((a \otimes b) \cdot (f \otimes \psi)) \quad (f \in A^*,\ a \in A,\ b \in B).$$

Define $m_\varphi : A^* \to \mathbb{C}$ by $m_\varphi(f) = m(f \otimes \psi)$. Then $m_\varphi(\varphi) = m(\varphi \otimes \psi) = 1$. Choose $b_0 \in B$ such that $\psi(b_0) = 1$ and let $f \in A^*$ and $a \in A$. Then

$$\begin{aligned} m_\varphi(f \cdot a) &= m(f \cdot a \otimes \psi) = m(f \cdot a \otimes \psi \cdot b_0) \\ &= m((f \otimes \psi) \cdot (a \otimes b_0)) = m((a \otimes b_0) \cdot (f \otimes \psi)) \\ &= m(a \cdot f \otimes b_0 \cdot \psi) = m(a \cdot f \otimes \psi) = m_\varphi(a \cdot f). \end{aligned}$$

It follows that A is φ-inner amenable, and similarly B is ψ-inner amenable.

For the converse, let A be φ-inner amenable and B ψ-inner amenable. Then there exist bounded nets (a_α) in A and (b_β) in B such that $\varphi(a_\alpha) = 1$, $\|aa_\alpha - a_\alpha a\| \to 0$ $(a \in A)$ and $\psi(b_\beta) = 1$ $\|bb_\beta - b_\beta b\| \to 0$, $(b \in B)$. The net

$(a_\alpha \otimes b_\beta)$ is bounded in $A \hat{\otimes} B$ and $(\varphi\otimes\psi)(a_\alpha\otimes b_\beta) = \varphi(a_\alpha)\psi(b_\beta) = 1$. Now suppose $\|a_\alpha\| \le M_1$, $\|b_\beta\| \le M_2$ and let $F = \sum_{j=1}^N c_j \otimes d_j \in A \otimes B$. Then

$$\begin{aligned}&\|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F\| \\ &\qquad = \Big\| \sum_{j=1}^N [(c_j a_\alpha - a_\alpha c_j) \otimes d_j b_\beta + a_\alpha c_j \otimes (d_j b_\beta - b_\beta d_j)]\Big\| \\ &\qquad \le \sum_{j=1}^N M_2 \|d_j\|\,\|c_j a_\alpha - a_\alpha c_j\| + \sum_{j=1}^N M_1\|c_j\|\,\|d_j b_\beta - b_\beta d_j\|.\end{aligned}$$

Since $\|c_j a_\alpha - a_\alpha c_j\| \to 0$ and $\|d_j b_\beta - b_\beta d_j\| \to 0$ $(1 \le j \le N)$, it follows that $\|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F\| \to 0$.

Now let $w \in A \hat{\otimes} B$, so there exist sequences $\{c_j\} \subseteq A$ and $\{d_j\} \subseteq B$ such that $w = \sum_{j=1}^\infty c_j \otimes d_j$ with $\sum_{j=1}^\infty \|c_j\|\,\|d_j\| < \infty$. Let $\epsilon > 0$ be given, and choose $N \in \mathbb{N}$ such that $\sum_{j=N+1}^\infty \|c_j\|\,\|d_j\| < \epsilon/4M_1M_2$. Put $F = \sum_{j=1}^N c_j \otimes d_j$. As $\|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F\| \to 0$, there exists (α_0, β_0) such that

$$\|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F\| < \epsilon/2 \quad \text{for all } (\alpha,\beta) \ge (\alpha_0,\beta_0).$$

Now for such (α,β),

$$\begin{aligned}&\|w(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)w\| \\ &\quad = \Big\| F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F + \sum_{j=N+1}^\infty [c_j a_\alpha \otimes d_j b_\beta - a_\alpha c_j \otimes b_\beta d_j]\Big\| \\ &\quad \le \|F(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)F\| + 2M_1M_2 \sum_{j=N+1}^\infty \|c_j\|\,\|d_j\| \\ &\quad < \epsilon/2 + (2M_1M_2 \cdot \epsilon/4M_1M_2) = \epsilon.\end{aligned}$$

Hence $\|w(a_\alpha \otimes b_\beta) - (a_\alpha \otimes b_\beta)w\| \to 0$. Applying Proposition 2.2 shows that $A \hat{\otimes} B$ is $(\varphi\otimes\psi)$-inner amenable. ∎

4. The Lau product $A \times_\theta B$. Let A and B be two Banach algebras with $\triangle(B) \ne \emptyset$. For a $\theta \in \triangle(B)$ the *θ-Lau product* $A \times_\theta B$ is defined as the cartesian product $A \times B$ with the algebra multiplication

$$(a,b)\cdot(c,d) = (ac + \theta(d)a + \theta(b)c, bd)$$

and with the norm $\|(a,b)\| = \|a\| + \|b\|$. This product was first introduced by Lau [L] for Lau algebras and then by Monfared [M1] for the general case. $A \times_\theta B$ is a Banach algebra and it is shown in [M1, Proposition 2.4] that

$$\triangle(A \times_\theta B) = (\triangle(A) \times \{\theta\}) \cup (\{0\} \times \triangle(B)).$$

In a natural way the dual space $(A \times_\theta B)^*$ can be identified with $A^* \times B^*$ via $(f,g)((a,b)) = f(a) + g(b)$. Recall that the dual norm on $A^* \times B^*$ is $\|(f,g)\| = \max\{\|f\|, \|g\|\}$. Also if A^{**}, B^{**} and $(A \times_\theta B)^{**}$ are equipped with their first Arens products then $(A \times_\theta B)^{**} = A^{**} \times_\theta B^{**}$ as an isometric isomorphism. Also for $(m,n),(p,q) \in (A \times_\theta B)^{**}$ we have $(m,n) \square (p,q) = (m \square p + n(\theta)p + q(\theta)m, n \square q)$; see [M1, Proposition 2.12].

The next result, which extends [N, Proposition 4.2], studies character inner amenability of $A \times_\theta B$.

THEOREM 4.1. *Let $\varphi \in \triangle(A)$ and $\psi \in \triangle(B)$. Then*

- (i) *$A \times_\theta B$ is (φ, θ)-inner amenable if and only if either A is φ-inner amenable or B is θ-inner amenable.*
- (ii) *$A \times_\theta B$ is $(0, \psi)$-inner amenable if and only if B is ψ-inner amenable.*
- (iii) *$A \times_\theta B$ is character inner amenable if and only if B is character inner amenable.*

Proof. (i) Let $A \times_\theta B$ be (φ, θ)-inner amenable. Then there exists $(m,n) \in \mathcal{A}^{**} \times_\theta B^{**}$ such that $(m,n)((\varphi,\theta)) = 1$ and $(m,n)\square(a,b) = (a,b)\square(m,n)$ for all $(a,b) \in A \times_\theta B$. It follows that $m(\varphi) + n(\theta) = 1$, $m \square a = a \square m$ and $n \square b = b \square n$ for all $a \in A$ and $b \in B$. Now if $n(\theta) = 0$ then $m(\varphi) = 1$ and so m is a φ-inner mean for A. If $n(\theta) \neq 0$ then $\frac{n}{n(\theta)} \square b = b \square \frac{n}{n(\theta)}$, that is, $\frac{n}{n(\theta)}$ is a θ-inner mean for B.

For the converse, suppose that m is a φ-inner mean for A; then trivially $(m,0)$ is a (φ,θ)-inner mean for $A \times_\theta B$. The same argument is used for the case where B is ψ-inner amenable. (ii) needs a similar proof, and (iii) follows trivially from (i) and (ii). ∎

Now we turn to the question of character inner amenability of the Banach algebras $A \oplus_\infty B$ and $A \oplus_p B$. Recall that these are equipped with the usual direct product multiplication and the norms $\|(a,b)\| = \max\{\|a\|, \|b\|\}$ and $\|(a,b)\| = (\|a\|^p + \|b\|^p)^{1/p}$, respectively. A direct verification shows that

$$\triangle(A \oplus_p B) = (\triangle(A) \times \{0\}) \cup (\{0\} \times \triangle(B)), \quad 1 \leq p \leq \infty,$$

from which we get the next result.

PROPOSITION 4.2. *Let A and B be Banach algebras and let $1 \leq p \leq \infty$. Then $A \oplus_p B$ is character inner amenable if and only if both A and B are character inner amenable.*

5. Module extension and triangular Banach algebras. For a Banach algebra A and a Banach A-module X let $A \oplus X$ be the module extension Banach algebra which is equipped with the algebra product $(a,x)\cdot(b,y) = (ab, ay + xb)$ $(a,b \in A,\ x,y \in X)$ and the norm $\|(a,x)\| = \|a\| + \|x\|$. The second dual $(A \oplus X)^{**}$ can be identified with $A^{**} \oplus_1 X^{**}$ as a Banach space, and it is not difficult to verify that the first Arens product on $(A \oplus X)^{**}$ is given by $(m, \lambda) \,\Box\, (n, \mu) = (m \,\Box\, n, m\mu + \lambda n)$. Some aspects of module extension Banach algebras have been discussed in [Z].

Let A and B be Banach algebras and let X be a Banach A-B-bimodule, that is, a left A-module and a right B-module satisfying

$$\|axb\| \leq \|a\| \, \|x\| \, \|b\| \quad (a \in A,\ b \in B\ x \in X).$$

The corresponding triangular Banach algebra

$$\tau = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : a \in A,\, x \in X,\, b \in B \right\}$$

is equipped with the norm $\left\| \left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix} \right) \right\| = \|a\| + \|x\| + \|b\|$ and the usual 2×2-matrix operations. The Arens products on the second dual of τ are studied in [FM]. Recall that the class of module extension Banach algebras includes the triangular Banach algebras. Indeed, τ can be identified with the module extension $(A \oplus_1 B) \oplus X$ in which X is considered as an $A \oplus_1 B$-module under the operations $(a, b) \cdot x = ax$ and $x \cdot (a, b) = xb$.

Proposition 5.1. *Let A be a Banach algebra and X be a Banach A-module. Then for the module extension Banach algebra $A \oplus X$, $\triangle(A \oplus X) = \triangle(A) \times \{0\}$. In particular, for the triangular Banach algebra τ, $\triangle(\tau) = \triangle(A \oplus_1 B) \times \{0\}$.*

Proof. Trivially $\triangle(A) \times \{0\} \subseteq \triangle(A \oplus X)$. Let $(\varphi, \psi) \in \triangle(A \oplus X)$. Then for $a, b \in A$, $(\varphi, \psi)((a, 0)(b, 0)) = (\varphi, \psi)((a, 0))(\varphi, \psi)((b, 0))$. It follows that $\varphi(ab) = \varphi(a)\varphi(b)$ and also

$$0 = (\varphi, \psi)((0, x)(0, y)) = (\varphi, \psi)((0, x))(\varphi, \psi)((0, y)) = \psi(x)\psi(y) \quad (x, y \in X).$$

So $\psi = 0$ and finally $\varphi \in \triangle(A)$. Hence $\triangle(A \oplus X) = \triangle(A) \times \{0\}$. The second part is clear. ∎

The next result on the character amenability of $A \oplus X$ and τ is a direct application of [KLP1, Theorem 1.4] to the module extension $A \oplus X$.

Proposition 5.2. *Let A be a Banach algebra, X be a Banach A-module and let $\varphi \in \triangle(A)$. Then $A \oplus X$ is $(\varphi, 0)$-amenable if and only if there exists a bounded net (a_α, x_α) in $A \oplus X$ satisfying*

(i) $\|a a_\alpha - \varphi(a) a_\alpha\| \to 0$ *for all* $a \in A$ *and* $\varphi(a_\alpha) = 1$ *for all* α,
(ii) $\|a x_\alpha - \varphi(a) x_\alpha\| \to 0$ *for all* $a \in A$,
(iii) $\|x a_\alpha\| \to 0$ *for all* $x \in X$.

Corollary 5.3.

(i) *If $A \oplus X$ is character amenable then so is A. The converse also holds in the case where $XA = 0$.*

(ii) *If* τ *is character amenable then both* A *and* B *are character amenable. The converse also holds in the case where* $XB = 0$.

Similar to Proposition 5.2 we have the next result, which is based on Proposition 2.2, characterizing the character inner amenability of $A \oplus X$.

PROPOSITION 5.4. *Let* A *be a Banach algebra,* X *be a Banach* A*-module and let* $\varphi \in \triangle(A)$*. Then* $A \oplus X$ *is* $(\varphi, 0)$*-inner amenable if and only if there exists a bounded net* (a_α, x_α) *in* $A \oplus X$ *satisfying*

(i) $\|aa_\alpha - a_\alpha a\| \to 0$ *for all* $a \in A$ *and* $\varphi(a_\alpha) = 1$ *for all* α,
(ii) $\|xa_\alpha - a_\alpha x\| \to 0$ *for all* $x \in X$,
(iii) $\|ax_\alpha - x_\alpha a\| \to 0$ *for all* $a \in A$.

COROLLARY 5.5. *If* $A \oplus X$ *is character inner amenable then* A *is character inner amenable. In particular if* τ *is character inner amenable then both* A *and* B *are character inner amenable.*

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