A NOTE ON RINGS OF CONSTANTS OF DERIVATIONS IN INTEGRAL DOMAINS

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Abstract. We observe that the characterization of rings of constants of derivations in characteristic zero as algebraically closed subrings also holds in positive characteristic after some natural adaptation. We also present a characterization of such rings in terms of maximality in some families of rings.

1. Introduction. Throughout this paper by a ring we mean a commutative ring with unity, and by an integral domain (briefly, a domain) we mean a ring without nonzero zero divisors. If $R$ is a domain, then $R_0$ denotes its field of fractions. If $K \subseteq L$ is a field extension, then $(L : K)$ is its degree.

Let $A$ be a domain and let $d: A \rightarrow A$ be a derivation. The kernel of $d$ is denoted by $A^d$ and is called the ring of constants of $d$. If $\text{char} A = p > 0$, then $A^p \subseteq A^d$, where $A^p = \{ a^p ; a \in A \}$. Let $K$ be a subring of $A$. If $d$ is a $K$-derivation, then $A^d$ is a $K$-subalgebra of $A$. If $\text{char} A = p > 0$ and $d$ is a $K$-derivation, then $KA^p \subseteq A^d$.

Let $A$ be a finitely generated $k$-domain, where $k$ is a field of characteristic zero. Let $R$ be a $k$-subalgebra of $A$. Nowicki ([N1, Theorem 4.1.4]; [N2, Theorem 5.4]) proved that the following conditions are equivalent:

1. $R = A^d$ for some $k$-derivation $d$ of $A$,
2. $R$ is integrally closed in $A$ and $R_0 \cap A = R$.

Daigle [D] observed that the condition (2) means that $R$ is algebraically closed in $A$.

In [J3, Theorem 1.1] the present author obtained a positive characteristic analog of the above theorem: if $\text{char} k = p > 0$, then $R = A^d$ for some $k$-derivation $d$ if and only if $kA^p \subseteq R$ and $R_0 \cap A = R$. He also noted a generalization of this fact for $B$-derivations, where $B$ is a subalgebra of $A$ such that $A^p \subseteq B$ and $A$ is finitely generated over $B$ (not necessarily over a field) [J1, Theorem 2.5]. In [J2] the author obtained a characterization,
which is quite similar in form to algebraic closedness: for every $g \in A$ and $a_0, a_1, \ldots, a_{p-1} \in R$, such that $a_i \neq 0$ for some $i$, if $a_{p-1}g^{p-1} + \cdots + a_1g + a_0 = 0$, then $g \in R$. In this note we show how to introduce the notion of separable algebraic closedness of a subring so that Nowicki’s characterization holds true in arbitrary characteristic in a more general form, presented in Theorem 3.1.

In Theorem 3.1 we also present a characterization of rings of constants as maximal elements in suitable families of rings. Nowicki and Nagata [NN, Lemma 3.1], [N1, Proposition 5.2.1] proved that for a polynomial $f \in k[x_1, \ldots, x_n]$, where $k$ is a field of characteristic zero, the ring $k[f]$ is integrally closed in $k[x_1, \ldots, x_n]$ if and only if it is a maximal element of the family \{ $k[g]; g \in k[x_1, \ldots, x_n]$ \}. The author observed in [J2] that there is no direct analog of this property in positive characteristic for rings of constants. However, if we consider a family of subrings $R \subseteq k[x_1, \ldots, x_n]$ such that $k[x_1^p, \ldots, x_n^p] \subseteq R$ and $(R_0 : k(x_1^p, \ldots, x_n^p)) = p$, then maximal elements of this family are exactly rings of constants of $k$-derivations. It turns out that such families may be constructed for all rings of constants of $K$-derivations in an arbitrary finitely generated $K$-domain $A$.

2. Separably algebraically closed subrings. Daigle [D] introduced the notions of algebraic elements and algebraic closedness for domains of characteristic zero. Let us extend them to domains of arbitrary characteristic introducing separably algebraic elements.

**Definition 2.1.** Let $A$ be a domain of characteristic $p \geq 0$, and let $R$ be a subring of $A$. If $p = 0$, we put $T^p = 1$ and $R_0[T^p] = R_0$.

(a) An element $a \in A$ is called *separably algebraic* over $R$ if $w(a) = 0$ for some irreducible polynomial $w(T) \in R_0[T] \setminus R_0[T^p]$.

(b) The set of all elements of $A$ separably algebraic over $R$ is called the *separable algebraic closure* of $R$ in $A$ and is denoted by $\overline{R}^A$.

(c) A subring $R$ is called *separably algebraically closed* in $A$ if $\overline{R}^A = R$.

The definition of a separably algebraic element of $A$ over $R$ means that this element is separably algebraic over the subfield $R_0$ as an element of the field $A_0$. Observe that irreducibility of the polynomial $w(T)$ is essential only in positive characteristic.

Recall that if $\text{char } A = 0$, then $\overline{R}^A = \overline{R}_{0}^{A_0} \cap A$ ([D, Exercise 1.2]). In the case of positive characteristic we are interested in subrings containing $A^p$. In this particular case we have the following.

**Proposition 2.2.** Let $A$ be a domain of characteristic $p > 0$. Let $R$ be a subring of $A$ such that $A^p \subseteq R$. Then:
(a) $\overline{R}^A = R_0 \cap A$,
(b) $R$ is separably algebraically closed in $A$ if and only if $R_0 \cap A = R$.

Proof. (a) Every element $a \in R_0 \cap A$, is a root of the irreducible polynomial $w(T) = T - a \in R_0[T] \setminus R_0[T^p]$, so $R_0 \cap A \subseteq \overline{R}^A$.

Now, let $a \in \overline{R}^A$. Then $a$, as an element of the field $A_0$, is separably algebraic over $R_0$. On the other hand, $a$ is purely inseparable over $R_0$, because $a^p \in R$. Then $a \in R_0$, by [ZS II.5, Lemma 1]. Therefore, $\overline{R}^A \subseteq R_0 \cap A$, and finally, $\overline{R}^A = R_0 \cap A$.

(b) This follows directly from (a).

3. A characterization of rings of constants. Now we present a generalization of Nowicki’s characterization ([N1 Theorem 4.1.4]; [N2 Theorem 5.4]) to the case of arbitrary characteristic. In the case of $\text{char} K = p > 0$, the equivalence (1) $\iff$ (2) below follows from Proposition 2.2(b) and [J1, Theorem 2.5], so it would be enough to note that in the case of $\text{char} K = 0$ the original proof is valid for $K$ being a domain, not necessarily a field. However, it seems of interest to sketch the original proof in the case of arbitrary characteristic. We also present a characterization of rings of constants of derivations as maximal elements in some families of rings.

**Theorem 3.1** ([N1 4.1.4]; [N2 5.4]; [D 1.4]). Let $A$ be a finitely generated $K$-domain, where $K$ is a domain (of arbitrary characteristic). Let $R$ be a $K$-subalgebra of $A$. If $\text{char} K = p > 0$, assume additionally that $A^p \subseteq R$ and put $B = KA^p$. The following conditions are equivalent:

1. $R$ is the ring of constants of some $K$-derivation of $A$,
2. $R$ is separably algebraically closed in $A$,
3. $R$ is a maximal element in one of the following families of rings:

$$ \Phi_m = \{ R; K \subseteq R \subseteq A, \text{tr deg}_K R \leq m \} \text{ if char } A = 0,$$
$$ \Psi_m = \{ R; B \subseteq R \subseteq A, (R_0 : B_0) \leq p^m \} \text{ if char } A = p > 0,$$

where $m = 0, 1, 2, \ldots$.

Proof. (1)$\Rightarrow$(2). Assume that $R = A^d$ for some $K$-derivation $d$ of $A$. Consider an element $a \in A$ such that $w(a) = 0$ for some irreducible polynomial $w(T) \in R_0[T] \setminus R_0[T^p]$, where in the case of $p = 0$ we put $T^p = 1$ and $R_0[T^p] = R_0$. We have $w'(a)d(a) = d(w(a)) = 0$. The polynomial $w'(T)$ is nonzero and of smaller degree than $w(T)$, so $w'(a) \neq 0$. Then $d(a) = 0$, that is, $a \in R$.

(2)$\Rightarrow$(1). Assume that $\overline{R}^A = R$. Put $M = \overline{R_0} A_0$ and consider a $K_0$-derivation $\delta: A_0 \rightarrow A_0$ such that $M = A_0^\delta$ ([N1 Theorem 3.3.2]; [N2 Theorem 4.2]; [G, Lemma 2]). Let $f_1, \ldots, f_s$ be generators of $A$ over $K$. Take
Consider an element \( w \in A \setminus \{0\} \) such that \( w \delta(f_i) \in A \) for every \( i \). Put \( \delta' = w\delta \); then \( \delta'(A) \subseteq A \). Let \( d : A \to A \) be the restriction of \( \delta' \) to \( A \). Then \( A^d = A^{\delta'} \cap A = M \cap A \).

We will show that \( M \cap A = R \). Obviously \( R \subseteq M \cap A \). On the other hand, any element \( a \in M \cap A \) is (as an element of \( A_0 \)) separably algebraic over \( R_0 \). In the sense of Definition 2.1 this means that \( a \) is (as an element of \( A \)) separably algebraic over \( R \), hence it belongs to \( R \). Finally, we see that \( A^d = R \).

(2) \( \Rightarrow \) (3). Assume that \( \overline{R}^A = R \). If \( \text{char } A = 0 \), put \( \text{tr deg}_K R = m \), so \( R \in \Phi_m \), and consider a ring \( S \in \Phi_m \) such that \( R \subseteq S \). Then \( \overline{R}^A \subseteq S^A \). Suppose that \( \overline{R}^A \neq S^A \) and consider an element \( a \in S^A \setminus \overline{R}^A \). Since \( a \in A \) and \( a \notin \overline{R}^A \), \( a \) is transcendental over \( R_0 \), so \( \text{tr deg}_K S = \text{tr deg}_K S^A > \text{tr deg}_K R = m \); a contradiction with the assumption that \( S \in \Phi_m \).

If \( \text{char } A = p > 0 \), put \( (R_0 : B_0) = p^m \), so \( R \in \Psi_m \), and consider a ring \( S \in \Psi_m \) such that \( R \subseteq S \). Then \( \overline{R}^A \subseteq S^A \). Suppose that \( \overline{R}^A \neq S^A \) and consider an element \( a \in S^A \setminus \overline{R}^A \). Then, by Proposition 2.2 \( a \in A \), \( a \in S_0 \) and \( a \notin R_0 \), so \( R_0 \) is properly included in \( S_0 \) and \( (S_0 : B_0) > (R_0 : B_0) = p^m \); a contradiction with the assumption that \( S \in \Psi_m \).

In both cases we obtain \( \overline{R}^A = S^A \). Hence
\[
R \subseteq S \subseteq S^A = \overline{R}^A = R,
\]
so \( R = S \), that is, \( R \) is a maximal element of the family \( \Phi_m \), resp. \( \Psi_m \).

(3) \( \Rightarrow \) (2). If \( \text{char } A = 0 \), assume that \( R \) is a maximal element of \( \Phi_m \). Recall that \( \overline{R}^A = \overline{R}_0^{A_0} \cap A \). Since the field extension \( R_0 \subseteq \overline{R}_0^{A_0} \) is algebraic, we have \( \text{tr deg}_K \overline{R}^A = \text{tr deg}_K R \), so \( \overline{R}^A \) is also an element of \( \Phi_m \).

If \( \text{char } A = p > 0 \), assume that \( R \) is a maximal element of \( \Psi_m \). We have \( \overline{R}^A = R_0 \cap A \) (Proposition 2.2). Therefore \( \overline{R}^A \subseteq R_0 \) and \( (\overline{R}^A)_0 \subseteq R_0 \). On the other hand, \( R \subseteq \overline{R}^A \), so \( R_0 \subseteq (\overline{R}^A)_0 \). We see that \( (\overline{R}^A)_0 = R_0 \), so \( \overline{R}^A \) is also an element of \( \Psi_m \).

In both cases, since \( R \subseteq \overline{R}^A \), we have \( R = \overline{R}^A \), by the maximality of \( R \). ■

REFERENCES

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