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A NOTE ON RINGS OF CONSTANTS OF DERIVATIONS IN INTEGRAL DOMAINS

BY

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Abstract. We observe that the characterization of rings of constants of derivations in characteristic zero as algebraically closed subrings also holds in positive characteristic after some natural adaptation. We also present a characterization of such rings in terms of maximality in some families of rings.

1. Introduction. Throughout this paper by a ring we mean a commutative ring with unity, and by an integral domain (briefly, a domain) we mean a ring without nonzero zero divisors. If R is a domain, then R_0 denotes its field of fractions. If $K \subseteq L$ is a field extension, then $(L : K)$ is its degree.

Let A be a domain and let $d \colon A \to A$ be a derivation. The kernel of d is denoted by A^d and is called the *ring of constants* of d. If $\operatorname{char} A = p > 0$, then $A^p \subseteq A^d$, where $A^p = \{a^p;\ a \in A\}$. Let K be a subring of A. If d is a K-derivation, then A^d is a K-subalgebra of A. If $\operatorname{char} A = p > 0$ and d is a K-derivation, then $KA^p \subseteq A^d$.

Let A be a finitely generated k-domain, where k is a field of characteristic zero. Let R be a k-subalgebra of A. Nowicki ([N1, Theorem 4.1.4]; [N2, Theorem 5.4]) proved that the following conditions are equivalent:

(1) $R = A^d$ for some k-derivation d of A,
(2) R is integrally closed in A and $R_0 \cap A = R$.

Daigle [D] observed that the condition (2) means that R is algebraically closed in A.

In [J3, Theorem 1.1] the present author obtained a positive characteristic analog of the above theorem: if $\operatorname{char} k = p > 0$, then $R = A^d$ for some k-derivation d if and only if $kA^p \subseteq R$ and $R_0 \cap A = R$. He also noted a generalization of this fact for B-derivations, where B is a subalgebra of A such that $A^p \subseteq B$ and A is finitely generated over B (not necessarily over a field) [J1, Theorem 2.5]. In [J2] the author obtained a characterization,

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which is quite similar in form to algebraic closedness: for every $g \in A$ and $a_0, a_1, \ldots, a_{p-1} \in R$, such that $a_i \neq 0$ for some i, if $a_{p-1}g^{p-1} + \cdots + a_1 g + a_0 = 0$, then $g \in R$. In this note we show how to introduce the notion of separable algebraic closedness of a subring so that Nowicki's characterization holds true in arbitrary characteristic in a more general form, presented in Theorem 3.1.

In Theorem 3.1 we also present a characterization of rings of constants as maximal elements in suitable families of rings. Nowicki and Nagata [NN, Lemma 3.1], [N1, Proposition 5.2.1] proved that for a polynomial $f \in k[x_1, \ldots, x_n]$, where k is a field of characteristic zero, the ring $k[f]$ is integrally closed in $k[x_1, \ldots, x_n]$ if and only if it is a maximal element of the family $\{k[g];\ g \in k[x_1, \ldots, x_n]\}$. The author observed in [J2] that there is no direct analog of this property in positive characteristic for rings of constants. However, if we consider a family of subrings $R \subseteq k[x_1, \ldots, x_n]$ such that $k[x_1^p, \ldots, x_n^p] \subseteq R$ and $(R_0 : k(x_1^p, \ldots, x_n^p)) = p$, then maximal elements of this family are exactly rings of constants of k-derivations. It turns out that such families may be constructed for all rings of constants of K-derivations in an arbitrary finitely generated K-domain A.

2. Separably algebraically closed subrings. Daigle [D] introduced the notions of algebraic elements and algebraic closedness for domains of characteristic zero. Let us extend them to domains of arbitrary characteristic introducing separably algebraic elements.

DEFINITION 2.1. Let A be a domain of characteristic $p \geq 0$, and let R be a subring of A. If $p = 0$, we put $T^p = 1$ and $R_0[T^p] = R_0$.

(a) An element $a \in A$ is called *separably algebraic* over R if $w(a) = 0$ for some irreducible polynomial $w(T) \in R_0[T] \setminus R_0[T^p]$.
(b) The set of all elements of A separably algebraic over R is called the *separable algebraic closure* of R in A and is denoted by $\overline{R}^A$.
(c) A subring R is called *separably algebraically closed* in A if $\overline{R}^A = R$.

The definition of a separably algebraic element of A over R means that this element is separably algebraic over the subfield R_0 as an element of the field A_0. Observe that irreducibility of the polynomial $w(T)$ is essential only in positive characteristic.

Recall that if $\operatorname{char} A = 0$, then $\overline{R}^A = \overline{R_0}^{A_0} \cap A$ ([D, Exercise 1.2]). In the case of positive characteristic we are interested in subrings containing A^p. In this particular case we have the following.

PROPOSITION 2.2. *Let A be a domain of characteristic $p > 0$. Let R be a subring of A such that $A^p \subseteq R$. Then:*

(a) $\overline{R}^A = R_0 \cap A$,
(b) *R is separably algebraically closed in A if and only if $R_0 \cap A = R$.*

Proof. (a) Every element $a \in R_0 \cap A$, is a root of the irreducible polynomial $w(T) = T - a \in R_0[T] \setminus R_0[T^p]$, so $R_0 \cap A \subseteq \overline{R}^A$.

Now, let $a \in \overline{R}^A$. Then a, as an element of the field A_0, is separably algebraic over R_0. On the other hand, a is purely inseparable over R_0, because $a^p \in R$. Then $a \in R_0$, by [ZS, II.5, Lemma 1]. Therefore, $\overline{R}^A \subseteq R_0 \cap A$, and finally, $\overline{R}^A = R_0 \cap A$.

(b) This follows directly from (a). ∎

3. A characterization of rings of constants. Now we present a generalization of Nowicki's characterization ([N1, Theorem 4.1.4]; [N2, Theorem 5.4]) to the case of arbitrary characteristic. In the case of $\operatorname{char} K = p > 0$, the equivalence (1)⇔(2) below follows from Proposition 2.2(b) and [J1, Theorem 2.5], so it would be enough to note that in the case of $\operatorname{char} K = 0$ the original proof is valid for K being a domain, not necessarily a field. However, it seems of interest to sketch the original proof in the case of arbitrary characteristic. We also present a characterization of rings of constants of derivations as maximal elements in some families of rings.

THEOREM 3.1 ([N1, 4.1.4]; [N2, 5.4]; [D, 1.4]). *Let A be a finitely generated K-domain, where K is a domain (of arbitrary characteristic). Let R be a K-subalgebra of A. If $\operatorname{char} K = p > 0$, assume additionally that $A^p \subseteq R$ and put $B = KA^p$. The following conditions are equivalent:*

(1) *R is the ring of constants of some K-derivation of A,*
(2) *R is separably algebraically closed in A,*
(3) *R is a maximal element in one of the following families of rings:*

$$\begin{cases} \Phi_m = \{R;\ K \subseteq R \subseteq A,\ \operatorname{tr\,deg}_K R \le m\} & \text{if } \operatorname{char} A = 0, \\ \Psi_m = \{R;\ B \subseteq R \subseteq A,\ (R_0 : B_0) \le p^m\} & \text{if } \operatorname{char} A = p > 0, \end{cases}$$

where $m = 0, 1, 2, \ldots$.

Proof. (1)⇒(2). Assume that $R = A^d$ for some K-derivation d of A. Consider an element $a \in A$ such that $w(a) = 0$ for some irreducible polynomial $w(T) \in R_0[T] \setminus R_0[T^p]$, where in the case of $p = 0$ we put $T^p = 1$ and $R_0[T^p] = R_0$. We have $w'(a)d(a) = d(w(a)) = 0$. The polynomial $w'(T)$ is nonzero and of smaller degree than $w(T)$, so $w'(a) \neq 0$. Then $d(a) = 0$, that is, $a \in R$.

(2)⇒(1). Assume that $\overline{R}^A = R$. Put $M = \overline{R_0}^{A_0}$ and consider a K_0-derivation $\delta\colon A_0 \to A_0$ such that $M = A_0^\delta$ ([N1, Theorem 3.3.2]; [N2, Theorem 4.2]; [G, Lemma 2]). Let $f_1, \ldots, f_s$ be generators of A over K. Take

$w \in A \setminus \{0\}$ such that $w\delta(f_i) \in A$ for every i. Put $\delta' = w\delta$; then $\delta'(A) \subseteq A$. Let $d\colon A \to A$ be the restriction of δ' to A. Then $A^d = A^{\delta'} \cap A = M \cap A$.

We will show that $M \cap A = R$. Obviously $R \subseteq M \cap A$. On the other hand, any element $a \in M \cap A$ is (as an element of A_0) separably algebraic over R_0. In the sense of Definition 2.1 this means that a is (as an element of A) separably algebraic over R, hence it belongs to R. Finally, we see that $A^d = R$.

(2)⇒(3). Assume that $\overline{R}^A = R$. If $\operatorname{char} A = 0$, put $\operatorname{tr\,deg}_K R = m$, so $R \in \Phi_m$, and consider a ring $S \in \Phi_m$ such that $R \subseteq S$. Then $\overline{R}^A \subseteq \overline{S}^A$. Suppose that $\overline{R}^A \neq \overline{S}^A$ and consider an element $a \in \overline{S}^A \setminus \overline{R}^A$. Since $a \in A$ and $a \notin \overline{R}^A$, a is transcendental over R_0, so $\operatorname{tr\,deg}_K S = \operatorname{tr\,deg}_K \overline{S}^A > \operatorname{tr\,deg}_K R = m$; a contradiction with the assumption that $S \in \Phi_m$.

If $\operatorname{char} A = p > 0$, put $(R_0 : B_0) = p^m$, so $R \in \Psi_m$, and consider a ring $S \in \Psi_m$ such that $R \subseteq S$. Then $\overline{R}^A \subseteq \overline{S}^A$. Suppose that $\overline{R}^A \neq \overline{S}^A$ and consider an element $a \in \overline{S}^A \setminus \overline{R}^A$. Then, by Proposition 2.2, $a \in A$, $a \in S_0$ and $a \notin R_0$, so R_0 is properly included in S_0 and $(S_0 : B_0) > (R_0 : B_0) = p^m$; a contradiction with the assumption that $S \in \Psi_m$.

In both cases we obtain $\overline{R}^A = \overline{S}^A$. Hence

$$R \subseteq S \subseteq \overline{S}^A = \overline{R}^A = R,$$

so $R = S$, that is, R is a maximal element of the family Φ_m, resp. Ψ_m.

(3)⇒(2). If $\operatorname{char} A = 0$, assume that R is a maximal element of Φ_m. Recall that $\overline{R}^A = \overline{R_0}^{A_0} \cap A$. Since the field extension $R_0 \subseteq \overline{R_0}^{A_0}$ is algebraic, we have $\operatorname{tr\,deg}_K \overline{R}^A = \operatorname{tr\,deg}_K R$, so $\overline{R}^A$ is also an element of Φ_m.

If $\operatorname{char} A = p > 0$, assume that R is a maximal element of Ψ_m. We have $\overline{R}^A = R_0 \cap A$ (Proposition 2.2). Therefore $\overline{R}^A \subseteq R_0$ and $(\overline{R}^A)_0 \subseteq R_0$. On the other hand, $R \subseteq \overline{R}^A$, so $R_0 \subseteq (\overline{R}^A)_0$. We see that $(\overline{R}^A)_0 = R_0$, so $\overline{R}^A$ is also an element of Ψ_m.

In both cases, since $R \subseteq \overline{R}^A$, we have $R = \overline{R}^A$, by the maximality of R. ■

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