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ON MULTIPLICATION IN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. We introduce and examine the notion of dense weak openness. In particular we show that multiplication in $C(X)$ is densely weakly open whenever X is an interval in $\mathbb{R}$.

Let X and Y be arbitrary topological spaces and $f\colon X \to Y$. We say that f is *open* (*weakly open*) [*densely weakly open*] if for every nonempty open set $U \subseteq X$, the image $f[U]$ is open (respectively, $\operatorname{int} f[U] \neq \emptyset$) [respectively, $\operatorname{int} f[U]$ is dense in $f[U]$]. Obviously every open mapping is densely weakly open and every densely weakly open mapping is weakly open.

The notion of openness is standard. Weak openness was considered by several authors (see, e.g., Burke [Bu]). We introduce dense weak openness to improve some results on multiplication in function spaces.

We will mostly consider spaces $C(X)$ of continuous real functions defined on a topological space X, with the metric of uniform convergence, i.e.,

$$d(f,g) := \min\{1, \sup\{|f(x) - g(x)| : x \in X\}\}.$$

We will focus on the operation $\Phi\colon C(X) \times C(X) \to C(X)$ of multiplication given by $\Phi(f,g) := fg$. For $U, V \subseteq C(X)$ we write $U \cdot V$ instead of $\Phi[U \times V]$. We use the symbol $B(f,r)$ to denote the open ball centered at f with radius r. Recall (cf. [BWW]) that in the Banach space $C[0,1]$ of real-valued functions which are continuous on $[0,1]$, multiplication is not an open mapping. Indeed, define $f(x) := x - 1/2$ for $x \in [0,1]$. Then every element of $B(f,1/2) \cdot B(f,1/2)$ has a zero, while every neighborhood of f^2 contains an element that is never zero. (This example is due to Fremlin.)

The main result of [BWW] states that multiplication in $C[0,1]$ is weakly open. This was generalized by Kowalczyk [K2] to other mappings from $C[0,1] \times C[0,1]$ to $C[0,1]$ generated by continuous functions $\varphi\colon [0,1]^2 \to [0,1]$, and by Wachowicz [W] to the case of multiplication in $C^{(n)}[0,1]$ (in $C^{(n)}[0,1]$, the space of all real-valued functions on $[0,1]$ whose nth derivative is continuous, we consider the standard norm making this space complete); see also [Ko] for a discussion of openness of multiplication in $C(K)$,

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where K is compact, and [K3] for some other generalizations. Recently, Kowalczyk [K1] has proved that multiplication in $C(0,1)$ is weakly open.

Recall that a mapping f between topological spaces X and Y is *quasi-continuous* [Ke] at a point $x \in X$ if for all open sets $U \subseteq X$, $V \subseteq Y$ with $x \in U$, $f(x) \in V$, there is a nonempty open set $W \subseteq U$ such that $f[W] \subseteq V$. We say that f is *open* at a point $x \in X$ if $f(x) \in \operatorname{int} f[U]$ whenever U is a neighborhood of x. We will denote by $\operatorname{Op}(f)$ the set of all points $x \in X$ at which f is open. Clearly f is open iff $\operatorname{Op}(f) = X$.

We start with the following proposition.

PROPOSITION 1. *Let $f\colon X \to Y$ be weakly open. If f is quasi-continuous at every point $x \in X \setminus \operatorname{Op}(f)$, then it is densely weakly open.*

Proof. Let $U \subseteq X$ be nonempty and open. Fix a $y \in f[U]$. Then $y = f(x)$ for some $x \in U$. If $x \in \operatorname{Op}(f)$, then $f(x) \in \operatorname{int} f[U]$, so every neighborhood of $f(x)$ intersects $\operatorname{int} f[U]$.

If $x \notin \operatorname{Op}(f)$, then f is quasi-continuous at x. Let V be an open neighborhood of y. There is a nonempty open set $W \subseteq U$ such that $f[W] \subseteq V$. Since f is weakly open, we conclude that

$$V \cap \operatorname{int} f[U] \supseteq \operatorname{int} f[W] \neq \emptyset. \ \blacksquare$$

One can easily see that the quasi-continuity assumption is not redundant.

EXAMPLE. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} x & \text{if } x \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then for each nonempty open set $U \subseteq \mathbb{R}$, $\operatorname{int} f[U] \supseteq U \setminus \{0\} \neq \emptyset$, whence f is weakly open. On the other hand, f is not densely weakly open since 1 is an isolated point of $f[(-1/2, 1/2)]$. Evidently, f is not quasi-continuous at 0.

COROLLARY 2. *Weakly open continuous functions are densely weakly open.*

From the above corollary, by [Ko] and [W], we can conclude that multiplication in $C(K)$ (where K is compact and $\dim K = 1$) and in $C^{(n)}[0,1]$ (for each $n \in \mathbb{N}$) is a densely weakly open mapping.

It was observed by Kowalczyk [K1] that multiplication in $C(0,1)$ is not continuous at (f_0, g_0), where $f_0(x) := 0$ and $g_0(x) := 1/x$ for $x \in (0,1)$. We can prove much more.

PROPOSITION 3. *For a topological space X, let Φ denote multiplication in $C(X)$, and let $f_0, g_0 \in C(X)$. The following conditions are equivalent*:

(i) *Φ is continuous at (f_0, g_0),*
(ii) *Φ is quasi-continuous at (f_0, g_0),*
(iii) *the functions f_0 and g_0 are bounded.*

Proof. The implication (i)⇒(ii) is obvious.

(ii)⇒(iii). Suppose that, e.g., g_0 is unbounded. For each $n \in \mathbb{N}$ pick an $x_n \in X$ such that $|g_0|(x_n) \geq n$.

Let $U := B(f_0, 1) \times B(g_0, 1)$. Fix nonempty open sets V and W with $V \times W \subseteq U$. We will show that $V \cdot W \not\subseteq B(f_0 g_0, 1/2)$.

Let $f_1 \in V$ and $g \in W$. Choose a $\delta > 0$ such that $B(f_1, 2\delta) \subseteq V$ and put $f_2 := f_1 + \delta$. Then $f_2 \in V$ as well.

From $g \in B(g_0, 1)$, we infer

$$\liminf_{n\to\infty} |g|(x_n) \geq \lim_{n\to\infty} |g_0|(x_n) - 1 = \infty.$$

Consequently,

$$d(f_1 g, f_2 g) \geq \min\{1, \delta \lim_{n\to\infty} |g|(x_n)\} = 1.$$

Hence $d(f_1 g, f_0 g_0) \geq 1/2$ or $d(f_2 g, f_0 g_0) \geq 1/2$.

(iii)⇒(i). Assume that f_0 and g_0 are bounded. Put

$$M := 3 \max\{\sup_{x\in X} |f_0|(x), \sup_{x\in X} |g_0|(x)\} + 3.$$

Fix an $\varepsilon \in (0, 1)$. If $f \in B(f_0, \varepsilon/M)$ and $g \in B(g_0, \varepsilon/M)$, then

$$\begin{aligned} d(fg, f_0 g_0) &\leq \sup_{x\in X} |f(x)g(x) - f_0(x)g_0(x)| \\ &\leq \sup_{x\in X} |f|(x) \sup_{x\in X} |g(x) - g_0(x)| + \sup_{x\in X} |g_0|(x) \sup_{x\in X} |f(x) - f_0(x)| \\ &< (M/3 + 1)\varepsilon/M + (M/3) \cdot \varepsilon/M \leq \varepsilon. \end{aligned}$$

This completes the proof. ■

Recall that a topological space K is *pseudocompact* if each real-valued continuous function on K is bounded. By Corollary 2 and Proposition 3, if K is pseudocompact, then multiplication in $C(K)$ is densely weakly open whenever it is open. We do not know of any exact characterization of pseudocompact spaces K for which multiplication in $C(K)$ is open.

Example. Consider $C(X)$ with $X := (0, 1)$. For $x \in (0, 1)$, define

$$f_0(x) := x(x - 1/2), \quad g_0(x) := \frac{x - 1/2}{x}$$

Then the multiplication Φ is neither quasi-continuous nor open at (f_0, g_0).

Proof. Indeed, since g_0 is unbounded, by Proposition 3, Φ is not quasi-continuous at (f_0, g_0).

Towards a contradiction, suppose $(f_0, g_0) \in \mathrm{Op}(\Phi)$. Then

$$f_0 g_0 \in \mathrm{int}(B(f_0, r) \cdot B(g_0, r)) \quad \text{for all } r > 0.$$

Let $r \leq 1/2$. There is a $\delta > 0$ such that

$$B(f_0 g_0, \delta) \subseteq B(f_0, r) \cdot B(g_0, r).$$

Notice that $(f_0 g_0 + \delta/2)(x) > 0$ for all $x \in (0,1)$. Since every function in $B(g_0, r)$ has a zero in $(0,1)$, we obtain a contradiction with the above inclusion. ∎

The above example shows that we cannot extend Corollary 2 to the case of $C(0,1)$ using just Proposition 1. Nevertheless, we will prove that multiplication in $C(0,1)$ is densely weakly open.

THEOREM 4. *If $X \subseteq \mathbb{R}$ is an interval, then multiplication in $C(X)$ is densely weakly open.*

Proof. Let $U, V \subseteq C(X)$ be nonempty and open. To prove that $\operatorname{int}(U \cdot V)$ is dense in $U \cdot V$ fix $f_0 \in U$, $g_0 \in V$, and $\varepsilon \in (0,1)$. Let $\tau \in (0, \sqrt{\varepsilon}/7)$ be such that $B(f_0, 11\tau) \subseteq U$ and $B(g_0, 26\tau) \subseteq V$.

For each $x \in X$ with $|f_0|(x) > 4\tau$, there is a maximal interval $I_x \ni x$ such that $|f_0| \geq 3\tau$ on I_x. Let $\mathcal{I}_0$ be the collection of all such intervals I_x. Observe that since $f_0 \in C(X)$, the set $A_0 := \bigcup_{I \in \mathcal{I}_0} \operatorname{bd} I$ has no accumulation point in X.

For each $x \in \bigcup \mathcal{I}_0$ with $|g_0|(x) > 3\tau$, there is a maximal interval $I'_x \ni x$ such that $I'_x \subseteq \bigcup \mathcal{I}_0$ and $|g_0| \geq 2\tau$ on I'_x. Let $\mathcal{I}_1$ be the collection of all such intervals I'_x. Observe that since $g_0 \in C(X)$, the set $A_1 := \bigcup_{I \in \mathcal{I}_1} \operatorname{bd} I$ has no accumulation point in X.

For each $x \in X \setminus \operatorname{int} \bigcup \mathcal{I}_0$ with $|g_0|(x) < 4\tau$, there is a maximal interval $I''_x \ni x$ such that $I''_x \subseteq X \setminus \operatorname{int} \bigcup \mathcal{I}_0$ and $|g_0| \leq 5\tau$ on I''_x. Let $\mathcal{I}_2$ be the collection of all such intervals I''_x. Observe that since $g_0 \in C(X)$, the set $A_2 := \bigcup_{I \in \mathcal{I}_2} \operatorname{bd} I$ has no accumulation point in X.

We have constructed a discrete set $A := A_0 \cup A_1 \cup A_2$ such that the family of all components J of $X \setminus A$ can be divided into four pairwise disjoint classes:

- $J \in \mathcal{J}_1$ if $|f_0| \geq 3\tau$ and $|g_0| \geq 2\tau$ on J,
- $J \in \mathcal{J}_2$ if $J \notin \mathcal{J}_1$ and $|f_0| \geq 3\tau \geq |g_0|$ on J,
- $J \in \mathcal{J}_3$ if $J \notin \mathcal{J}_1$ and $|f_0| \leq 4\tau \leq |g_0|$ on J,
- $J \in \mathcal{J}_4$ if $J \notin \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$ and $|f_0| \leq 4\tau$ and $|g_0| \leq 5\tau$ on J.

Now we define a function $h_0 \colon X \to \mathbb{R}$. Fix an $x \in X$. If $x \in X \setminus \bigcup \mathcal{J}_4$, then we put $h_0(x) := (f_0 g_0)(x)$. Extend h_0 continuously to the whole X so that for each $J \in \mathcal{J}_4$, we have

$$\inf h_0[J] = -20\tau^2, \quad \sup h_0[J] = 20\tau^2, \tag{1}$$

and observe that

$$|h_0 - f_0 g_0| \leq 20\tau^2 + |f_0| \cdot |g_0| \leq 40\tau^2 < \varepsilon - \tau^2 \quad \text{on } J. \tag{2}$$

We shall prove that $B(h_0, \tau^2) \subseteq U \cdot V$. Since $B(h_0, \tau^2) \subseteq B(f_0 g_0, \varepsilon)$, this will complete the proof.

Fix an $h \in B(h_0, \tau^2)$. We need to define functions $f \in U$ and $g \in V$ such that $h = fg$. We will define these functions in four steps.

STEP 1. Let $J \in \mathcal{J}_1$. For each $x \in \operatorname{cl} J$, define

$$f(x) := \sqrt{\frac{|hf_0|(x)}{|g_0|(x)}} \cdot \operatorname{sgn} f_0(x), \qquad g(x) := \sqrt{\frac{|hg_0|(x)}{|f_0|(x)}} \cdot \operatorname{sgn} g_0(x).$$

Then clearly f and g are continuous and $h = fg$ on $\operatorname{cl} J$. (Notice that the function $\operatorname{sgn} \circ h = \operatorname{sgn} \circ h_0 = \operatorname{sgn} \circ (f_0 g_0)$ is constant on $\operatorname{cl} J$.) Moreover

$$(3) \quad |f - f_0| = \left| \sqrt{\frac{|hf_0|}{|g_0|}} - |f_0| \right| = |h - f_0 g_0| \cdot \frac{|f_0|}{\sqrt{\frac{|hf_0|}{|g_0|}} + |f_0|} \cdot \frac{1}{|g_0|} \le \tau^2 \cdot 1 \cdot \frac{1}{2\tau} < \tau,$$

and similarly $|g - g_0| < \tau$ on $\operatorname{cl} J$. Hence $|f| > 2\tau$ and $|g| > \tau$ on $\operatorname{cl} J$.

STEP 2. Let $J \in \mathcal{J}_2$. Let $\psi_J \colon \operatorname{cl} J \to (-\tau, \tau)$ be a linear function such that

$$\psi_J(x) = \sqrt{\frac{|hf_0|(x)}{|g_0|(x)}} \cdot \operatorname{sgn} f_0(x) - f_0(x) \quad \text{if } x \in \operatorname{bd} J \cap \operatorname{bd} J' \text{ for some } J' \in \mathcal{J}_1$$

(cf. (3)). For each $x \in \operatorname{cl} J$, define

$$f(x) := (f_0 + \psi_J)(x), \qquad g(x) := \frac{h(x)}{f(x)}.$$

Then clearly f and g are continuous, $h = fg$ on $\operatorname{cl} J$, and $|f - f_0| < \tau$ on $\operatorname{cl} J$ (whence $|f| > 2\tau$ on $\operatorname{cl} J$). Moreover

$$\begin{aligned} |g - g_0| = \left| \frac{h}{f_0 + \psi_J} - g_0 \right| &\le \frac{|h - f_0 g_0|}{|f_0 + \psi_J|} + |\psi_J| \cdot \frac{|g_0|}{|f_0 + \psi_J|} \\ &< \frac{\tau^2}{3\tau - \tau} + \tau \cdot \frac{3\tau}{3\tau - \tau} = 2\tau \end{aligned}$$

on $\operatorname{cl} J$.

STEP 3. Let $J \in \mathcal{J}_3$. We proceed as in Step 2. Let $\psi_J \colon \operatorname{cl} J \to (-\tau, \tau)$ be a linear function such that

$$\psi_J(x) = \sqrt{\frac{|hg_0|(x)}{|f_0|(x)}} \cdot \operatorname{sgn} g_0(x) - g_0(x) \quad \text{if } x \in \operatorname{bd} J \cap \operatorname{bd} J' \text{ for some } J' \in \mathcal{J}_1,$$

and for each $x \in \operatorname{cl} J$, define

$$g(x) := (g_0 + \psi_J)(x), \qquad f(x) := \frac{h(x)}{g(x)}.$$

(Notice that $\operatorname{bd} J \cap \operatorname{bd} J' = \emptyset$ whenever $J' \in \mathcal{J}_2$.) Then $|f - f_0| < 2\tau$ and $|g - g_0| < \tau$ on $\operatorname{cl} J$, whence $|g| > 3\tau$ on $\operatorname{cl} J$.

STEP 4. Finally let $J \in \mathcal{J}_4$. Put $a := \inf J$ and $b := \sup J$. Notice that we have already defined functions f and g on $\operatorname{bd}_X J = A \cap \operatorname{cl} J$, and recall that $|f| > \tau$ or $|g| > \tau$ on $\operatorname{bd}_X J$. We consider four cases.

Assume that none of a and b belongs to A; i.e., $J = X$. Define $f := \tau$ and $g := h/f$ on $\operatorname{cl} J$. Then clearly $h = fg$,

$$|f - f_0| \le \tau + |f_0| \le 5\tau,$$

and

$$|g - g_0| \le \frac{|h|}{\tau} + |g_0| < \frac{|h_0| + \tau^2}{\tau} + 5\tau \le 26\tau \tag{4}$$

on $\operatorname{cl} J$.

Now let $a \in A$ and $b \notin A$. If $|f|(a) > \tau$, then we define $f := f(a)$ and $g := h/f$ on $\operatorname{cl} J$. Then clearly $h = fg$,

$$|f - f_0| \le |f - f_0|(a) + |f_0|(a) + |f_0| < 2\tau + 4\tau + 4\tau = 10\tau,$$

and

$$|g - g_0| \le \frac{|h|}{\tau} + |g_0| < 26\tau$$

on $\operatorname{cl} J$ (cf. (4)). In the opposite case $a \in \operatorname{bd} J'$ for some $J' \in \mathcal{J}_3$. Define $g := g(a)$ and $f := h/g$ on $\operatorname{cl} J$. Then clearly $h = fg$,

$$|g - g_0| \le |g - g_0|(a) + |g_0|(a) + |g_0| < \tau + 5\tau + 5\tau = 11\tau,$$

and

$$|f - f_0| \le \frac{|h|}{|g|(a)} + |f_0| < \frac{|h_0| + \tau^2}{3\tau} + 4\tau \le 11\tau$$

on $\operatorname{cl} J$.

The case where $a \notin A$ and $b \in A$ is analogous.

Finally assume that $a, b \in A$. Notice that by (1), $h(x_0) = 0$ for some $x_0 \in J$. Let, e.g., $f(a) \ge |g|(a)$ and $-|f|(b) \ge g(b)$. (The other cases are analogous.) Then $f(a) \ge \sqrt{|h|(a)}$ (recall that $f(a)g(a) = h(a)$) and similarly $g(b) \le -\sqrt{|h|(b)}$. Define

$$\begin{aligned}
f(x) &:= \max\left\{\sqrt{|h|(x)}, f(a) \cdot \frac{x - x_0}{a - x_0}\right\}, & g(x) &:= \frac{h(x)}{f(x)} & &\text{if } x \in [a, x_0),\\
f(x_0) &:= 0, & g(x_0) &:= 0,\\
g(x) &:= \min\left\{-\sqrt{|h|(x)}, g(b) \cdot \frac{x - x_0}{b - x_0}\right\}, & f(x) &:= \frac{h(x)}{g(x)} & &\text{if } x \in (x_0, b].
\end{aligned}$$

Since $|f| \le \sqrt{|h|}$ on $(x_0, b]$, f is continuous at x_0. Similarly g is continuous at x_0. Clearly $h = fg$ on $\operatorname{cl} J$. Observe that:

- for each $x \in [a, x_0)$,

$$\begin{aligned}
|f - f_0|(x) &\le \max\{\sqrt{|h|(x)}, f(a)\} + |f_0|(x)\\
&\le \max\{\sqrt{21\tau^2}, f_0(a) + 2\tau\} + 4\tau \le 10\tau,\\
|g - g_0|(x) &\le \sqrt{|h|(x)} + |g_0|(x) \le \sqrt{21\tau^2} + 5\tau < 10\tau,
\end{aligned}$$

- for $x = x_0$,

$$|f - f_0|(x) = |f_0|(x) \le 4\tau,$$
$$|g - g_0|(x) = |g_0|(x) \le 5\tau,$$

- for each $x \in (x_0, b]$,

$$\begin{aligned} |f - f_0|(x) &\le \sqrt{|h|(x)} + |f_0|(x) \le \sqrt{21\tau^2} + 4\tau < 9\tau, \\ |g - g_0|(x) &\le \max\{\sqrt{|h|(x)}, |g|(b)\} + |g_0|(x) \\ &\le \max\{\sqrt{21\tau^2}, |g_0|(b) + 2\tau\} + 5\tau \le 12\tau. \end{aligned}$$

We constructed functions $f, g\colon X \to \mathbb{R}$ such that $h = fg$. Since the set A is discrete in X, the functions f and g are continuous. Moreover by construction, $f \in B(f_0, 11\tau) \subseteq U$ and $g \in B(g_0, 26\tau) \subseteq V$. So, $h \in U \cdot V$.

We have proved that $B(h_0, \tau^2) \subseteq U \cdot V$. By (2), $B(h_0, \tau^2) \subseteq B(f_0 g_0, \varepsilon)$. This completes the proof. ∎

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