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LOWER QUANTIZATION COEFFICIENT AND THE F-CONFORMAL MEASURE

BY

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Abstract. Let $F = \{f^{(i)} : 1 \le i \le N\}$ be a family of Hölder continuous functions and let $\{\varphi_i : 1 \le i \le N\}$ be a conformal iterated function system. Lindsay and Mauldin's paper [Nonlinearity 15 (2002)] left an open question whether the lower quantization coefficient for the F-conformal measure on a conformal iterated funcion system satisfying the open set condition is positive. This question was positively answered by Zhu. The goal of this paper is to present a different proof of this result.

1. Introduction. The term 'quantization' in this paper refers to the idea of estimating a given probability on $\mathbb{R}^d$ with a discrete probability, that is, a 'quantized' version of the probability supported on a finite set. Following the work of Graf and Luschgy (cf. [GL1, GL2]), we define the *quantization dimension* (or perhaps better, the *quantization dimension function*) as follows. Given a Borel probability measure μ on $\mathbb{R}^d$, a number $r \in (0, +\infty)$ and a natural number $n \in \mathbb{N}$, the nth *quantization error* of order r for μ is defined by

$$e_{n,r} = \inf\Big\{\Big(\int d(x,\alpha)^r\,d\mu(x)\Big)^{1/r} : \alpha \subset \mathbb{R}^d,\ \mathrm{card}(\alpha) \le n\Big\},$$

where $d(x,\alpha)$ denotes the distance from the point x to the set α with respect to a given norm $\|\cdot\|$ on $\mathbb{R}^d$. We note that if $\int \|x\|^r\,d\mu(x) < \infty$ then there is some set α for which the infimum is achieved (cf. [GL1]). The *upper* and *lower quantization dimensions* for μ of order r are defined by

$$\overline{D}_r(\mu) := \limsup_{n\to\infty} \frac{\log n}{-\log e_{n,r}}, \qquad \underline{D}_r(\mu) := \liminf_{n\to\infty} \frac{\log n}{-\log e_{n,r}}.$$

If $\overline{D}_r(\mu)$ and $\underline{D}_r(\mu)$ coincide, we call their common value the *quantization dimension* of μ of order r and we denote it by $D_r(\mu)$. For $s > 0$, we define the *s-dimensional upper* and *lower quantization coefficients* of μ of order r by $\limsup_{n\to\infty} n e^s_{n,r}(\mu)$ and $\liminf_{n\to\infty} n e^s_{n,r}(\mu)$ respectively.

Under the open set condition Graf and Luschgy determined the quantization dimension $D_r := D_r(\mu)$ for an arbitrary self-similar measure μ,

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and proved that the D_r-dimensional upper and lower quantization coefficients of μ are both positive and finite (cf. [GL1, GL2]). These results were extended later by Lindsay and Mauldin (cf. [LM]) to the F-conformal measure m associated with a conformal iterated function system determined by finitely many conformal mappings. They established a relationship between the quantization dimension and the multifractal spectrum of m. They also proved that the upper quantization coefficient of m is finite; however, they left it open whether the lower quantization coefficient is positive. Zhu gave an affirmative answer to this question (cf. [Z]). He did not use Hölder's inequality which appears both in Graf–Luschgy's (cf. [GL1, GL2]) and Lindsay–Mauldin's work (cf. [LM]), instead in the proof he mainly applied a class of finite maximal antichains.

From our work, it can be seen that the asymptotic behavior of $\sum_{|\omega|=n}(\|\varphi'_\omega\|^r\|\exp(S_\omega(F))\|)^{\kappa_r/(r+\kappa_r)}$, which occurs in Lindsay and Mauldin's paper, is not a hurdle in analyzing the κ_r-dimensional lower quantization coefficient. We first introduce some lemmas (Lemmas 3.5 and 3.7), and then following the techniques of Lindsay and Mauldin, using Hölder's inequality we give a different proof that the lower quantization coefficient of the F-conformal measure is positive. The method of this paper can be used in analyzing the lower quantization coefficients for many other probability measures (for example: ergodic measure with bounded distortion, Moran measure, ergodic Markov measure associated with a recurrent self-similar set, probability measure generated by a set of bi-Lipschitz mappings, Gibbs measure).

2. Basic definitions and lemmas. Let $V \subset \mathbb{R}^d$. Recall that a map $\varphi : V \to V$ is called *contracting* if there exists $0 < \gamma(\varphi) < 1$ such that $|\varphi(x) - \varphi(y)| \le \gamma(\varphi)|x - y|$. Let $\{\varphi_1, \ldots, \varphi_N\}$ be a collection of contracting maps of an open set $V \subset \mathbb{R}^d$ such that $\varphi_i(X) \subset X$ for all $1 \le i \le N$, where $X \subset V$ is a compact set such that $X = \mathrm{cl}(\mathrm{int}X)$ and $N \ge 2$. Any such collection is called an *iterated function system.* By [H], there is a unique nonempty compact set J, called the *limit set* for the iterated function system, such that

$$J = \bigcup_{j=1}^{N} \varphi_j(J). \tag{1}$$

The iterated function system is said to satisfy the *open set condition* (OSC) if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\varphi_i(U) \subset U$ for all $1 \le i \le N$ and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for every pair $i, j \in \{1, \ldots, N\}$, $i \ne j$.

A $\mathcal{C}^1$ map $\varphi : V \to \mathbb{R}^d$ is *conformal* if the differential $\varphi'(x) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies $|\varphi'(x)y| = |\varphi'(x)| \cdot |y| \ne 0$ for all $x \in V$ and $y \in \mathbb{R}^d$, $y \ne 0$, where

$|\varphi'(x)|$ represents the norm of the derivative at $x \in \mathbb{R}^d$. An iterated function system $\{\varphi_i : X \to X\}_{1\le i\le N}$ satisfying the open set condition on a compact set $X \subset \mathbb{R}^d$ with $X = \mathrm{cl}(\mathrm{int}X)$ is said to be a *conformal iterated function system* (cIFS) if each φ_i extends to an injective conformal map $\varphi_i : V \to V$ on an open connected set $V \supset X$ such that $\varphi_i : V \to \varphi_i(V) \subset V$ is a conformal $\mathcal{C}^{1+\gamma}$ diffeomorphism with $0 < \gamma < 1$ and $\|\varphi_i'\| = \sup\{|\varphi_i'(x)| : x \in V\} < 1$. In this case the unique nonempty compact set $J \subset X$ satisfying (1) is called a *self-conformal set*. Since $\{\varphi_i : 1 \le i \le N\}$ is a finite system of conformal maps, by [PRSS] the open set condition is equivalent to the *strong open set condition* (SOSC), i.e., the open set U can be chosen so that $U \cap J \neq \emptyset$.

Let $I := \{1,\ldots,N\}$ be a finite index set, $I^* := \bigcup_{n\ge0} I^n$ be the set of all finite words including the empty word $\emptyset$, and $I^\infty := \prod_{n=1}^\infty I$ be the set of all infinite words over I. Let σ be the left shift on I^∞, i.e., for $\omega = (\omega_1,\omega_2,\ldots) \in I^\infty$ we have $\sigma(\omega) = (\omega_2,\omega_3,\ldots)$. For $\omega = (\omega_1,\omega_2,\ldots,\omega_n) \in I^n$ we write $|\omega| = n$ for the length of ω, and set $\sigma(\omega) = (\omega_2,\omega_3,\ldots,\omega_n)$; moreover, $\omega|_k = (\omega_1,\omega_2,\ldots,\omega_k)$, $k \le n$, denotes the truncation of ω to length k. The length of the empty word is zero. We write $\omega\tau = \omega * \tau = (\omega_1,\ldots,\omega_{|\omega|},\tau_1,\tau_2,\ldots)$ to denote the juxtaposition of $\omega = (\omega_1,\omega_2,\ldots,\omega_{|\omega|}) \in I^*$ and $\tau = (\tau_1,\tau_2,\ldots) \in I^* \cup I^\infty$. For $\omega \in I^*$ and $\tau \in I^* \cup I^\infty$ we say that τ is an *extension* of ω if $\tau|_{|\omega|} = \omega$. For $\omega = (\omega_1,\omega_2,\ldots,\omega_{|\omega|}) \in I^*$, let us write

$$\varphi_\omega = \begin{cases} \mathrm{Id}_{\mathbb{R}^d}, & \omega = \emptyset,\\ \varphi_{\omega_1}\circ\varphi_{\omega_2}\circ\cdots\circ\varphi_{\omega_{|\omega|}}, & |\omega| \ge 1.\end{cases}$$

We call $\Gamma \subset I^*$ a *finite maximal antichain* if Γ is a finite set of words such that every element in I^∞ is an extension of some word in Γ, but no word of Γ is an extension of another word in Γ. Of course, this requires that the index set I is finite. We will make this assumption in the remainder of the paper. We denote by $|\Gamma|$ the cardinality of Γ.

Let us now state the following two well-known lemmas for conformal iterated function systems (for details of the proof see [P]).

LEMMA 2.1. *There exists a constant $K \ge 1$ such that $|\varphi_\omega'(x)| \le K|\varphi_\omega'(y)|$ for all $x, y \in V$ and all $\omega \in I^*$.*

LEMMA 2.2. *There exists a constant $\tilde{K} \ge K$ such that*

$$\tilde{K}^{-1}\|\varphi_\omega'\| d(x,y) \le d(\varphi_\omega(x),\varphi_\omega(y)) \le \tilde{K}\|\varphi_\omega'\| d(x,y)$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where d is the metric on X.

From Lemma 2.1, the following lemma easily follows.

LEMMA 2.3 (cf. [R]). *Let $K \geq 1$ be as in Lemma 2.1. Then for all $\omega, \tau \in I^*$,*

$$K^{-1}\|\varphi'_\omega\|\,\|\varphi'_\tau\| \leq \|\varphi'_{\omega\tau}\| \leq K\|\varphi'_\omega\|\,\|\varphi'_\tau\|.$$

Let $F = \{f^{(i)} : X \to \mathbb{R}\}_{i\in I}$ be a family of Hölder continuous functions (cf. [MU]), i.e., for some $\beta > 0$ we have $V_\beta(F) = \sup_{n\geq 1} V_n(F) < \infty$, where for each $n \geq 1$,

$$V_n(F) = \sup_{\omega\in I^n} \sup_{x,y\in X} |f^{(\omega_1)}(\varphi_{\sigma(\omega)}(x)) - f^{(\omega_1)}(\varphi_{\sigma(\omega)}(y))|e^{\beta(n-1)},$$

and also $\sum_{i\in I} \|e^{f^{(i)}}\| < \infty$, where $\|\cdot\|$ denotes the supremum norm taken over X.

For $n \geq 1$ and $\omega \in I^n$, set $S_\omega(F) := \sum_{j=1}^n f^{(\omega_j)} \circ \varphi_{\sigma^j(\omega)}$. Then the *topological pressure* of F is defined by

$$P(F) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \|\exp(S_\omega(F))\|.$$

As in [LM], we may assume $P(F) = 0$. By [MU], there exists a probability measure m (the F-*conformal measure*) supported on J such that for any continuous function $g : X \to \mathbb{R}$ and $n \geq 1$,

$$\int g\,dm = \sum_{|\omega|=n} \int \exp(S_\omega(F)) \cdot (g \circ \varphi_\omega)\,dm. \tag{2}$$

Let $\beta(q)$ be the temperature function for $G_{q,\beta} := \{\beta \log |\varphi'_i| + qf^{(i)}\}_{i\in I}$, i.e., $P(G_{q,\beta(q)}) = 0$. Below, we write $P(G_{q,\beta(q)})$ as $P(q, \beta(q))$. As in [LM], for each $r \in (0, +\infty)$ there exists a unique $\kappa_r \in (0, +\infty)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r \|\exp(S_\omega(F))\|\right)^{\frac{\kappa_r}{r+\kappa_r}} = 0, \tag{3}$$

which implies $P(q_r, rq_r) = 0$, i.e., $\beta(q_r) = rq_r$ where $q_r = \kappa_r/(r + \kappa_r)$. Let us now write

$$V_{n,r}(m) = \inf\Big\{\int d(x,\alpha)^r\,dm(x) : \alpha \subset \mathbb{R}^d,\ \mathrm{card}(\alpha) \leq n\Big\},$$
$$u_{n,r}(m) = \inf\Big\{\int d(x,\alpha\cup U^c)^r\,dm(x) : \alpha \subset \mathbb{R}^d,\ \mathrm{card}(\alpha) \leq n\Big\},$$

where U is the set from the strong open set condition and U^c denotes the complement of U. We see that

$$u_{n,r}^{1/r} \leq V_{n,r}^{1/r} = e_{n,r}.$$

We call sets $\alpha_n \subset \mathbb{R}^d$ for which the above infima are achieved n-*optimal sets* for $e_{n,r}, V_{n,r}$ or $u_{n,r}$ respectively. As stated before, Graf and Luschgy have shown that n-optimal sets exist when $\int \|x\|^r\,dm(x) < \infty$.

It is proved in [LM] that the quantization dimension $D_r := D_r(m)$ of order r for the probability measure m exists and equals $\beta(q_r)/(1-q_r) = \kappa_r$; furthermore, the κ_r-dimensional upper quantization coefficient is finite.

3. Main result. In this section we prove our main result given by the following theorem.

THEOREM 3.1. *Let m be the F-conformal measure associated with the family of strongly Hölder continuous functions $\{f^{(i)} : X \to X\}_{i\in I}$ and the conformal iterated function system $\{\varphi_i : X \to X\}_{i\in I}$. Let κ_r be the quantization dimension for the probability measure m. Then* $\liminf ne_{n,r}^{\kappa_r}(m) > 0$.

To prove the above theorem we need the following lemmas and corollary.

LEMMA 3.2 ([LM, Lemma 2]). *There exists a constant $C \ge 1$ such that for any $x, y \in X$ and $\omega \in I^*$,*

$$\frac{\exp(S_\omega(F)(x))}{\exp(S_\omega(F)(y))} \le C.$$

In particular, for any $x \in X$ and $\omega \in I^*$, $\exp(S_\omega(F)(x)) \ge C^{-1}\|\exp(S_\omega(F))\|$.

Using Lemma 3.2, we can deduce the following lemma.

LEMMA 3.3. *Let $C \ge 1$ be as in Lemma 3.2. Then for $\omega, \tau \in I^*$, and $x, y \in X$,*

$$C^{-2} \le \frac{\exp(S_{\omega\tau}(F)(x))}{\|\exp(S_\omega(F))\|\,\|\exp(S_\tau(F))\|} \le C^2.$$

Proof. For $x \in X$ and $\omega, \tau \in I^*$, we have

$$\begin{aligned}\exp(S_{\omega\tau}(F)(x)) &= \exp\Big(\sum_{j=1}^{|\omega|} f^{(\omega_j)} \circ \varphi_{\sigma^j(\omega)}(\varphi_\tau(x)) + \sum_{j=1}^{|\tau|} f^{(\tau_j)} \circ \varphi_{\sigma^j(\tau)}(x)\Big)\\ &\ge C^{-2}\|\exp(S_\omega(F))\|\,\|\exp(S_\tau(F))\|.\end{aligned}$$

The remaining inequality easily follows from the calculation

$$\begin{aligned}\exp(S_{\omega\tau}(F)(x)) &\le \|\exp(S_\omega(F))\|\,\|\exp(S_\tau(F))\|\\ &\le C^2\|\exp(S_\omega(F))\|\,\|\exp(S_\tau(F))\|. \ \blacksquare\end{aligned}$$

Let us now give the following lemma.

LEMMA 3.4. *Let $C \ge 1$ be as in Lemma 3.2. Then for $\tau \in I^*$,*

$$\|\exp(S_\tau(F))\| \le C.$$

Proof. By (2), for any Borel subset A of X and any $\tau \in I^n$ $(n \ge 1)$, we have

$$
\begin{aligned}
m(\varphi_\tau(A)) &= \sum_{|\omega|=n} \int \exp(S_\omega(F)(x)) \cdot (1_{\varphi_\tau(A)} \circ \varphi_\omega(x))\, dm(x) \\
&= \int \exp(S_\tau(F)(x)) \cdot (1_{\varphi_\tau(A)} \circ \varphi_\tau(x))\, dm(x) \\
&= \int_A \exp(S_\tau(F)(x))\, dm(x) \geq C^{-1} \|\exp(S_\tau(F))\| m(A).
\end{aligned}
$$

Thus

$$
\|\exp(S_\tau(F))\| \leq C \cdot \frac{m(\varphi_\tau(A))}{m(A)} \leq C. \blacksquare
$$

LEMMA 3.5. *Let* $0 < r < +\infty$ *and* κ_r *be as in* (3)*. Then for any* $n \geq 1$,

$$
(K^r C)^{-\frac{\kappa_r}{r+\kappa_r}} \leq \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r \|\exp(S_\omega(F))\| \right)^{\frac{\kappa_r}{r+\kappa_r}} \leq (K^r C)^{\frac{\kappa_r}{r+\kappa_r}}.
$$

Proof. For $\omega \in I^*$, let $s_\omega = \|\varphi'_\omega\|^r \|\exp(S_\omega(F))\|$. Then for $\omega, \tau \in I^*$ with $|\omega| = n$ and $|\tau| = p$ $(n, p \geq 1)$, by Lemmas 2.3 and 3.3, we obtain $(K^r C)^{-2} s_\omega s_\tau \leq s_{\omega\tau} \leq (K^r C)^2 s_\omega s_\tau$. Hence by the standard theory of subadditive sequences, $\lim_{n\to\infty} n^{-1} \log \sum_{|\omega|=n} s_\omega^t$ exists for any $t \in \mathbb{R}$. Let us denote this limit by $h(t)$. Then for $t \geq 0$, we have

$$
h(t) = \lim_{p\to\infty} \frac{1}{np} \log \sum_{|\omega|=np} s_\omega^t,
$$

and so

$$
\lim_{p\to\infty} \frac{1}{np} \log \Big(\sum_{|\omega|=n} s_\omega^t (K^r C)^{-t} \Big)^p \leq h(t) \leq \lim_{p\to\infty} \frac{1}{np} \log \Big(\sum_{|\omega|=n} s_\omega^t (K^r C)^t \Big)^p,
$$

which implies

$$
\frac{1}{n} \log \sum_{|\omega|=n} s_\omega^t (K^r C)^{-t} \leq h(t) \leq \frac{1}{n} \log \sum_{|\omega|=n} s_\omega^t (K^r C)^t,
$$

and therefore

$$
e^{nh(t)} (K^r C)^{-t} \leq \sum_{|\omega|=n} s_\omega^t \leq e^{nh(t)} (K^r C)^t.
$$

Now substitute $t = \frac{\kappa_r}{r+\kappa_r}$; then by (3) we have $h(t) = 0$, which yields

$$
(K^r C)^{-\frac{\kappa_r}{r+\kappa_r}} \leq \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r \|\exp(S_\omega(F))\| \right)^{\frac{\kappa_r}{r+\kappa_r}} \leq (K^r C)^{\frac{\kappa_r}{r+\kappa_r}}
$$

for any $n \geq 1$, ending the proof. $\blacksquare$

COROLLARY 3.6. *Let* m *be an* F*-conformal measure,* $0 < r < +\infty$ *and* κ_r *be as in* (3)*. Then for any* $\omega \in I^n$ *with* $n \geq 1$,

$$
(K^r C)^{-\frac{2\kappa_r}{r+\kappa_r}} \leq \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r m(\varphi_\omega(X)) \right)^{\frac{\kappa_r}{r+\kappa_r}} \leq (K^r C)^{\frac{2\kappa_r}{r+\kappa_r}}.
$$

Proof. We know, for any $\omega \in I^*$, that

$$m(\varphi_\omega(X)) = \int \exp(S_\omega(F)(x))\, dm(x),$$

and so

$$m(\varphi_\omega(X)) \le \|\exp(S_\omega(F))\| \quad \text{and} \quad m(\varphi_\omega(X)) \ge C^{-1}\|\exp(S_\omega(F))\|.$$

Hence, we have

$$\begin{aligned} C^{-1}\|\varphi'_\omega\|^r m(\varphi_\omega(X)) \le \|\varphi'_\omega\|^r m(\varphi_\omega(X)) &\le \|\varphi'_\omega\|^r \|\exp(S_\omega(X))\| \\ &\le C\|\varphi'_\omega\|^r m(\varphi_\omega(X)). \end{aligned}$$

Then

$$\begin{aligned} C^{-\frac{\kappa_r}{r+\kappa_r}} \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r m(\varphi_\omega(X))\right)^{\frac{\kappa_r}{r+\kappa_r}} &\le \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r \|\exp(S_\omega(X))\|\right)^{\frac{\kappa_r}{r+\kappa_r}} \\ &\le C^{\frac{\kappa_r}{r+\kappa_r}} \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r m(\varphi_\omega(X))\right)^{\frac{\kappa_r}{r+\kappa_r}}, \end{aligned}$$

from which, by Lemma 3.5, it follows that

$$\begin{aligned} \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r m(\varphi_\omega(X))\right)^{\frac{\kappa_r}{r+\kappa_r}} &\le C^{\frac{\kappa_r}{r+\kappa_r}} \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r \|\exp(S_\omega(X))\|\right)^{\frac{\kappa_r}{r+\kappa_r}} \\ &\le (K^r C)^{\frac{2\kappa_r}{r+\kappa_r}}, \end{aligned}$$

and

$$\begin{aligned} \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r m(\varphi_\omega(X))\right)^{\frac{\kappa_r}{r+\kappa_r}} &\ge C^{-\frac{\kappa_r}{r+\kappa_r}} \sum_{|\omega|=n} \left(\|\varphi'_\omega\|^r \|\exp(S_\omega(X))\|\right)^{\frac{\kappa_r}{r+\kappa_r}} \\ &\ge (K^r C)^{-\frac{2\kappa_r}{r+\kappa_r}}, \end{aligned}$$

and thus the corollary is obtained. ■

The following lemma plays a crucial role in this paper.

LEMMA 3.7. *Let $0 < r < +\infty$ and κ_r be as in (3). Let Γ be a finite maximal antichain. Then*

$$\sum_{\omega\in\Gamma} \left(\|\varphi'_\omega\|^r \|\exp(S_\omega(F))\|\right)^{\frac{\kappa_r}{r+\kappa_r}} \ge (K^r C)^{-\frac{6\kappa_r}{r+\kappa_r}}.$$

Proof. As Γ is a finite maximal antichain, there exists a finite sequence of positive integers $n_1 < \cdots < n_k$ such that

$$\Gamma = \Gamma_{n_1} \cup \cdots \cup \Gamma_{n_k},$$

where $\Gamma_{n_j} = \{\omega \in \Gamma : |\omega| = n_j\}$ for $1 \le j \le k$. Let M be a positive integer and $M \ge n_k$. We know that if m is an F-conformal measure, then for any $\omega, \tau \in I^*$ it follows that

$$\begin{aligned} m(\varphi_{\omega\tau}(X)) \le \|\exp(S_{\omega\tau}(F))\| &\le \|\exp(S_\omega(F))\|\, \|\exp(S_\tau(F))\| \\ &\le C^2 m(\varphi_\omega(X)) m(\varphi_\tau(X)). \end{aligned}$$

Then, using Corollary 3.6, we have

$$\begin{aligned}
\sum_{\omega\in\Gamma}\big(\|\varphi'_\omega\|^r\|\exp(S_\omega(F))\|\big)^{\frac{\kappa_r}{r+\kappa_r}} &\geq \sum_{\omega\in\Gamma}\big(\|\varphi'_\omega\|^r m(\varphi_\omega(X))\big)^{\frac{\kappa_r}{r+\kappa_r}}\\
&= \sum_{j=1}^{k}\sum_{\omega\in\Gamma_{n_j}}\big(\|\varphi'_\omega\|^r m(\varphi_\omega(X))\big)^{\frac{\kappa_r}{r+\kappa_r}}\\
&\geq (K^rC)^{-\frac{2\kappa_r}{r+\kappa_r}}\sum_{j=1}^{k}\sum_{\omega\in\Gamma_{n_j}}\big(\|\varphi'_\omega\|^r m(\varphi_\omega(X))\big)^{\frac{\kappa_r}{r+\kappa_r}}\\
&\quad\times\sum_{|\tau|=M-n_j}\big(\|\varphi'_\tau\|^r m(\varphi_\tau(X))\big)^{\frac{\kappa_r}{r+\kappa_r}}\\
&= (K^rC)^{-\frac{2\kappa_r}{r+\kappa_r}}\sum_{j=1}^{k}\sum_{\omega\in\Gamma_{n_j}}\sum_{|\tau|=M-n_j}\big(\|\varphi'_\omega\|^r\|\varphi'_\tau\|^r m(\varphi_\omega(X))m(\varphi_\tau(X))\big)^{\frac{\kappa_r}{r+\kappa_r}}\\
&\geq (K^rC)^{-\frac{2\kappa_r}{r+\kappa_r}}\sum_{j=1}^{k}\sum_{\omega\in\Gamma_{n_j}}\sum_{|\tau|=M-n_j}\big(K^{-r}\|\varphi'_{\omega\tau}\|^r C^{-2} m(\varphi_{\omega\tau}(X))\big)^{\frac{\kappa_r}{r+\kappa_r}}\\
&\geq (K^rC)^{-\frac{4\kappa_r}{r+\kappa_r}}\sum_{|\omega|=M}\big(\|\varphi'_\omega\|^r m(\varphi_\omega(X))\big)^{\frac{\kappa_r}{r+\kappa_r}} \geq (K^rC)^{-\frac{6\kappa_r}{r+\kappa_r}}. \quad\blacksquare
\end{aligned}$$

Let us now state the following well-known lemma.

LEMMA 3.8 (cf. [LM, Lemma 3]). *Let $\Gamma \subseteq I^*$ be a finite maximal antichain. Then there exists $n_0 = n_0(\Gamma)$ such that for every $n \geq n_0$, there exists a set $\{n_\omega := n_\omega(n)\}_{\omega\in\Gamma}$ of positive integers such that $\sum_{\omega\in\Gamma} n_\omega \leq n$ and*

$$u_{n,r} \geq (\tilde{K}^rC)^{-1}\sum_{\omega\in\Gamma}\|\varphi'_\omega\|^r\|\exp(S_\omega(F))\|u_{n_\omega,r}.$$

Proof of Theorem 3.1. Let $\Gamma \subseteq I^*$ be a finite maximal antichain. By Lemma 3.8, we have n_0 and for $n \geq n_0$ the numbers $\{n_\omega := n_\omega(n)\}_{\omega\in\Gamma}$ which satisfy the conclusion of that lemma. Set $c = \min\{n^{r/\kappa_r}u_{n,r} : n \leq n_0\}$. Clearly each $u_{n,r} > 0$ and hence $c > 0$. Suppose $n \geq n_0$ and $k^{r/\kappa_r}u_{k,r} \geq c$ for all $k < n$. Hence using Lemma 3.8, we have

$$\begin{aligned}
n^{r/\kappa_r}u_{n,r} &\geq n^{r/\kappa_r}(\tilde{K}^rC)^{-1}\sum_{\omega\in\Gamma}\|\varphi'_\omega\|^r\|\exp(S_\omega(F))\|u_{n_\omega,r}\\
&= n^{r/\kappa_r}(\tilde{K}^rC)^{-1}\sum_{\omega\in\Gamma}\|\varphi'_\omega\|^r\|\exp(S_\omega(F))\|(n_\omega(n))^{-r/\kappa_r}(n_\omega(n))^{r/\kappa_r}u_{n_\omega,r}\\
&\geq c(\tilde{K}^rC)^{-1}\sum_{\omega\in\Gamma}\|\varphi'_\omega\|^r\|\exp(S_\omega(F))\|\left(\frac{n_\omega(n)}{n}\right)^{-r/\kappa_r}.
\end{aligned}$$

Using Hölder's inequality (with exponents less than 1), we have

$$n^{r/\kappa_r} u_{n,r} \geq c(\tilde{K}^r C)^{-1} \Big(\sum_{\omega \in \Gamma} \big(\|\varphi'_\omega\|^r \|\exp(S_\omega(F))\| \big)^{\kappa_r/(r+\kappa_r)} \Big)^{(r+\kappa_r)/\kappa_r} \times \bigg(\sum_{\omega \in \Gamma} \Big(\frac{n_\omega(n)}{n} \Big)^{(-r/\kappa_r)(-\kappa_r/r)} \bigg)^{-r/\kappa_r}.$$

By Lemma 3.7 and the fact that $\sum_{\omega \in \Gamma} n_\omega(n) \leq n$, we see that

$$n^{r/\kappa_r} u_{n,r} \geq c(\tilde{K}^r C)^{-1} (K^r C)^{-6}.$$

Therefore, by induction,

$$\liminf_{n\to\infty} n u_{n,r}^{\kappa_r/r} \geq \big(c(\tilde{K}^r C)^{-1} (K^r C)^{-6} \big)^{\kappa_r/r} > 0, \quad \text{i.e.,} \quad \liminf n e_{n,r}^{\kappa_r} > 0.$$

Hence the proof of the theorem is complete.

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