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ON BAILEY PAIRS AND CERTAIN q-SERIES RELATED TO QUADRATIC AND TERNARY QUADRATIC FORMS

BY

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Abstract. We provide a new approach to establishing certain q-series identities that were proved by Andrews, and show how to prove further identities using conjugate Bailey pairs. Some relations between some q-series and ternary quadratic forms are established.

1. Introduction and main results. In an effort to expand on the theory of Bailey chains, and the umbral methods that L. J. Rogers employed in proving the Rogers–Ramanujan identities (the second proof offered by Rogers) [14], Andrews [4] provided a new method for establishing certain new q-series identities using an umbral calculus approach. A particularly nice example that was established in [4], and bears a close resemblance to Euler's Pentagonal Number Theorem [10], is given by

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q)_{2n}}=\frac{\sum_{n,m\in\mathbb{Z}}(-1)^{n+m}q^{n(3n-1)/2+m(3m-1)/2+nm}}{(q)_\infty^2}, \tag{1.1}$$

where

$$(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$$

(see [10]); for convenience we also put $(a)_n=(a;q)_n$. The q-series in (1.1) is, in fact, of the Rogers–Ramanujan type, and has the equivalent form

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q)_{2n}}=\frac{1}{(q^2,q^3,q^4,q^5,q^{11},q^{12},q^{13},q^{14};q^{16})_\infty},$$

which follows from the limiting case $\rho_1,\rho_2\to\infty$ of Bailey's lemma (see (2.3) below), with the Bailey pair [17, A(5)] with $a=1$. Further, it should be mentioned that the q-series in (1.1) is indeed related to the Virasoro character [9, p. 20, (A.4), A(5), $k=1$] (see also [18, Theorem 4.1, $(p,p')=(1,3)$, $r=0$, $s=1$]).

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The motivation of this study is two-fold. We establish another approach to proving identities like (1.1), and go a step further by finding identities that are related to certain ternary quadratic forms. The ternary quadratic form expansions we find in this study are of the form

$$\sum_{(i,j,k)\in E} (-1)^{L(i,j,k)} q^{A(i,j,k)+B(i,j,k)} \tag{1.2}$$

for some set $E \subset \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$, where $A(i,j,k)$ is some quadratic form, $B(i,j,k)$ is some indefinite quadratic form, and $L(i,j,k)$ is a linear form.

2. A proof of the identity (1.1). The purpose of this section is to provide a proof of (1.1), which will put the new identities we establish in the next section into some more meaningful perspective. And because our proofs rely on new Bailey pairs, we believe our proofs can be viewed as novel. However, we should mention that the methods used in this section are in fact closely related to those in [4], the difference being that we are not relying on the s-fold extension (our interest here would be the 2-fold extension) of Bailey's lemma [4, Theorem 1], but instead using new Bailey pairs. We mention that Berkovich [7] has found an interesting proof for identities contained in [4], relying on certain q-polynomial identities, and which should also be compared with the methods in this paper as well as [4].

For more material on Bailey's lemma, and related material, we recommend the papers [1, 5, 6, 9, 11, 15, 17].

Recall that (α_n, β_n) is said to be a *Bailey pair relative to a* if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(aq)_{n+r}(q)_{n-r}}, \tag{2.1}$$

or (see [1])

$$\alpha_n = \frac{(1-aq^{2n})(a)_n(-1)^n q^{n(n-1)/2}}{(1-a)(q)_n} \sum_{j=0}^{n} (q^{-n})_j (aq^n)_j q^j \beta_j. \tag{2.2}$$

Equation (2.2) will prove to be the most useful in this study. Now we know from [6] or [17, eq. (1.3)] that given a Bailey pair relative to a, we have

$$\sum_{n=0}^{\infty} (\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n \beta_n = \frac{(aq/\rho_1)_\infty (aq/\rho_2)_\infty}{(aq)_\infty (aq/\rho_1\rho_2)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n \alpha_n}{(aq/\rho_1)_n(aq/\rho_2)_n}. \tag{2.3}$$

We define a pair of sequences (A_n, B_n) to be a *Bailey pair in its symmetric form* if

$$B_n = \sum_{r=-n}^{n} \frac{A_r}{(aq)_{n+r}(q)_{n-r}}. \tag{2.4}$$

Then letting $\rho_1, \rho_2 \to \infty$ in (2.3) with $a = 1$ gives

$$\sum_{n=0}^{\infty} q^{n^2} B_n = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2} A_n. \tag{2.5}$$

With the above tools, we are now ready to prove (1.1).

PROPOSITION 2.1. *Identity* (1.1) *is valid.*

Proof. Recall [3, 5] that relative to $a = 1$, we have the pair of sequences (A_n, B_n) where

$$A_n = (-z)^n q^{n(n+1)/2}, \tag{2.6}$$

$$B_n = \frac{(zq; z^{-1})_n}{(q)_{2n}}. \tag{2.7}$$

The variable z in this pair turns out to be a key part of our proof, because inserting the pair (2.6)–(2.7) into (2.5), and setting $z = q^N$, gives

$$\sum_{n=0}^{N} \frac{(q^{N+1}; q^{-N})_n q^{n^2}}{(q)_{2n}} = \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n q^{n(3n+1)/2+Nn}. \tag{2.8}$$

Comparing equation (2.8) with (2.2) (multiplying through by $(1-q)^{-1}(-1)^N(1 - q^{2N+1})q^{N(N-1)/2}$), it is not hard to see that we have the Bailey pair (α_N, β_N) with respect to $a = q$, where

$$\beta_N = \frac{q^{N(N-1)}}{(q)_{2N}}, \tag{2.9}$$

$$\alpha_N = (q)_\infty^{-1}(1-q)^{-1}(-1)^N(1-q^{2N+1})q^{N(N-1)/2} \times \sum_{n\in\mathbb{Z}} (-1)^n q^{n(3n+1)/2+Nn}. \tag{2.10}$$

Inserting this new Bailey pair into (2.3) and letting $\rho_1, \rho_2 \to \infty$ and $a = q$, we find that equation (1.1) follows after making the observation that $\sum_{n\in\mathbb{Z}}(-1)^n q^{n(3n+1)/2+Nn}$ is unchanged by replacing N by $-N - 1$. ∎

We mention a few more identities, somewhat less elegant than (1.1). Rogers [14, p. 341] (see also Slater [17, eq. (46)]) established that the left hand side of (2.11) is indeed related to an infinite product with modulus 10.

The series on the left side of equation (2.12) is in Slaters' list (see [17, eq. (100)]). See also [12] for a good list of identities of these types.

THEOREM 2.2. *We have*

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q;q^2)_n (q)_n} = \frac{(-q)_\infty}{(q)_\infty^2} \sum_{n,m\in\mathbb{Z}} (-1)^{n+m} q^{n^2+m(3m+1)/2+nm}, \tag{2.11}$$

$$\sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q;q^2)_n (q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty^2} \times \sum_{n,m\in\mathbb{Z}} (-1)^{n+m} q^{n(3n+1)+m(2m+1)+2nm}. \tag{2.12}$$

Proof. In the case of (2.11), we take the Bailey pair (2.9)–(2.10) and insert it in (2.3) with $\rho_1 = -q$, and $\rho_2 \to \infty$. For (2.12), we take the Bailey pair (2.6)–(2.7) and insert it into the $\rho_1 = -q$, $\rho_2 \to \infty$ (with q replaced by q^2) case of (2.3) to get

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(zq^2;q^2)_n (z^{-1};q^2)_n}{(q;q^2)_n (q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n\in\mathbb{Z}} q^{n(2n+1)}(-z)^n. \tag{2.13}$$

Now proceeding as before, with $z = q^{2N}$, we get the Bailey pair (relative to $a = q^2$)

$$\beta_N = \frac{q^{N^2-2N}}{(q;q^2)_N (q^4;q^4)_N}, \tag{2.14}$$

$$\alpha_N = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} (1-q^2)^{-1} (-1)^N (1-q^{4N+2}) q^{N(N-1)} \times \sum_{n\in\mathbb{Z}} (-1)^n q^{n(2n+1)+2Nn}. \tag{2.15}$$

Now insert the pair (2.14)–(2.15) into (2.3), let $\rho_1 \to \infty$ and $\rho_2 \to \infty$, and use the fact that $\sum_{n\in\mathbb{Z}} (-1)^n q^{n(2n+1)+2Nn}$ is unchanged when N is replaced by $-N-1$. ∎

3. Ternary quadratic forms. We first recall some information about conjugate Bailey pairs. The pair (γ_N, δ_N) is said to be a *conjugate Bailey pair with respect to a* if the two sequences satisfy (see [6])

$$\gamma_N = \sum_{j=N}^{\infty} \frac{\delta_j}{(q)_{j-N} (aq)_{j+N}}.$$

Andrews and Warnaar [5] have taken a very interesting approach to constructing conjugate Bailey pairs, which, upon obtaining the desired conjugate pair, leads to an easy method toward obtaining new q-series by inserting known Bailey pairs. Rowell [15] has also built on this topic, and gave some more identities and applications of conjugate Bailey pairs. This will make our task of finding new q-series identities related to ternary quadratic forms considerably easier: we will simply find applications of some of the conjugate Bailey pairs found in [5, 15]. Therefore, we need the following proposition, which contains equivalent forms of conjugate Bailey pairs proved by Andrews and Warnaar [5, Theorem 2], as well as two conjugate pairs proved by Rowell [15, p. 375, eq. (13), and p. 374, eq. (9)]. However, no previous knowledge of the material contained in [5, 15] will be required to understand our results. Lastly, we mention that, in these expansions, one must be careful in the order of summation with the ternary quadratic forms to ensure the sum over $N \in \mathbb{Z}$ does not sum to 0 by the Jacobi triple product identity. Therefore, it will be taken that the sum over $N \in \mathbb{Z}$ is to be done last.

PROPOSITION 3.1. *If (A_n, B_n) is a Bailey pair relative to $a = 1$ (in the symmetric transform [5] or [14]), that is, the pair of sequences satisfy*

$$B_n = \sum_{r=-n}^{n} \frac{A_r}{(q)_{n+r}(q)_{n-r}},$$

(with q replaced by q^2 for (3.1) and (3.2)) then

$$\sum_{n=0}^{\infty}(-q)_{2n}q^n B_n = \frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sum_{2|j|\le n} A_j q^{-2j^2}, \tag{3.1}$$

$$\sum_{n=0}^{\infty}(q)_{2n}q^n B_n = \sum_{n=0}^{\infty} q^{n(n+1)/2}\sum_{2|j|\le n} A_j q^{-2j^2}. \tag{3.2}$$

Moreover,

$$\sum_{n=0}^{\infty}(q)_n B_n (-1)^n q^{n(n+1)/2} = \sum_{n=0}^{\infty}(-1)^n q^{n(3n+1)/2}(1-q^{2n+1})\sum_{|j|\le n} A_j q^{-j^2}. \tag{3.3}$$

From (3.1)–(3.3) we can obtain the following new Bailey pairs.

LEMMA 3.2. *Relative to $a = q^2$, the pair of sequences (A_N, B_N) is a Bailey pair where (with q replaced by q^2 in (2.4))*

$$(3.4)\qquad \begin{aligned} B_N &= \frac{q^{-N}}{(q)_{2N}},\\ A_N &= \frac{(-q)_\infty}{(q)_\infty}(1-q^2)^{-1}(-1)^N q^{N(N-1)}\\ &\quad\times\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sum_{2|j|\le n}(-1)^j q^{-j(j-1)+2Nj},\end{aligned}$$

and also

$$(3.5)\qquad \begin{aligned} B_N &= \frac{q^{-N}}{(-q)_{2N}},\\ A_N &= (1-q^2)^{-1}(-1)^N q^{N(N-1)}\\ &\quad\times\sum_{n=0}^{\infty} q^{n(n+1)/2}\sum_{2|j|\le n}(-1)^j q^{-j(j-1)+2Nj}.\end{aligned}$$

Further, we also have, relative to $a=q$, the Bailey pair

$$(3.6)\qquad \begin{aligned} B_N &= \frac{(q)_N(-1)^N q^{N(N-1)/2}}{(q)_{2N}},\\ A_N &= \frac{(-1)^N q^{N(N-1)/2}}{1-q}\\ &\quad\times\sum_{n=0}^{\infty}(-1)^n q^{n(3n+1)/2}(1-q^{2n+1})\sum_{|j|\le n}(-1)^j q^{-(j^2-j)/2+Nj}.\end{aligned}$$

Proof. In each case we are inserting the Bailey pair (2.6)–(2.7) into (3.1), (3.2), and (3.3) to obtain the pairs (3.4), (3.5), and (3.6) respectively. We only give the details for (3.5), and leave (3.4) and (3.6) to the reader. In the case of (3.5), inserting (2.6)–(2.7) into (3.2) and setting $z=q^{2N}$ gives

$$(3.7)\qquad \sum_{n=0}^{N}\frac{(q)_{2n}(q^{2N+2},q^{-2N};q^2)_n q^n}{(q^2;q^2)_{2n}} = \sum_{n=0}^{\infty} q^{n(n+1)/2}\sum_{2|j|\le n}(-1)^j q^{-j(j-1)+2Nj}.$$

The A_N in (3.5) follows after multiplying both sides of (3.7) by $(-1)^N q^{N(N-1)} \times (1-q^{4N+2})$, and observing that

$$\sum_{n=0}^{\infty} q^{n(n+1)/2}\sum_{2|j|\le n}(-1)^j q^{-j(j-1)+2Nj}$$

is unchanged by replacing N by $-N-1$, coupled with (2.2). ∎

It is a direct consequence of (2.3) that these Bailey pairs lead us to the following identities.

THEOREM 3.3. *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{(q^{n+1})_n} = \frac{1}{(q)_\infty} \sum_{\substack{N\in\mathbb{Z}\\ n\ge 0\\ |j|\le n}} (-1)^{N+n+j} q^{N(3N+1)/2+n(3n+1)/2-j(j-1)/2+Nj}(1-q^{2n+1}), \tag{3.8}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q)_{2n}} = \frac{1}{(q^2;q^2)_\infty} \sum_{\substack{N\in\mathbb{Z}\\ n\ge 0\\ 2|j|\le n}} (-1)^{N+j} q^{N(3N+1)+n(n+1)/2-j(j-1)+2Nj}. \tag{3.9}$$

Proof. The proof of (3.8) is simply (3.6) inserted into (2.3) with $\rho_1, \rho_2 \to \infty$ and $a = q$, and (3.9) is also this special case of (2.3), but with (3.5). ∎

The q-series in (3.8) and (3.9) are indeed related to indefinite quadratic forms. However, even though a Hecke-type series for (3.9) has been given in [13], one for (3.8) has not been offered in the literature.

Next we consider a lemma that is equivalent to a special case (namely $r = q$) of a conjugate Bailey pair due to Bressoud [8] and Singh [16]:

$$\delta_n = q^n, \qquad \gamma_n = \frac{q^n}{(q)_\infty^2} \sum_{i=0}^{\infty} (-1)^i q^{\binom{i+1}{2}+2ni}.$$

LEMMA 3.4. *Assuming the hypothesis of Proposition* 3.1, *we have*

$$\sum_{n=0}^{\infty} B_n q^n = \frac{1}{(q)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sum_{2|j|\le n} A_j q^{-2j^2}. \tag{3.10}$$

LEMMA 3.5. *Relative to $a = q$, the pair of sequences (A_N, B_N) is a Bailey pair where*

$$\begin{aligned} B_N &= \frac{1}{(q)_{2N}}, \\ A_N &= (q)_\infty^{-2}(1-q)^{-1}(-1)^N q^{N(N-1)/2} \\ &\quad \times \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sum_{2|j|\le n} (-1)^j q^{-j(3j-1)/2+Nj}. \end{aligned} \tag{3.11}$$

Proof. The proof is identical to previous proofs, using the Bailey pair (2.6)–(2.7) and Lemma 3.4. The details are omitted. ∎

We can now obtain a new expansion for $(q)_\infty^3$ from the above tools.

THEOREM 3.6. *We have, provided the sum over $N \in \mathbb{Z}$ is to be done last in the order of summation, the identity*

$$(q)_\infty^3 = \sum_{\substack{N\in\mathbb{Z}\\ n\geq 0\\ 2|j|\leq n}} (-1)^{N+n+j} q^{N(N-1)/2+n(n+1)/2-j(3j-1)/2+Nj}. \tag{3.12}$$

Proof. Insert the pair in Lemma 3.5 into (2.4) with $a = q$ and let $n \to \infty$. ∎

4. Conclusions. The emphasis here has been placed on converting conjugate Bailey pairs into Bailey pairs. We have shown how we take a known general Bailey pair, apply it to some conjugate Bailey pairs, and obtain some new Bailey pairs, and the implication of this seems to be something new. With an infinite number of conjugate Bailey pairs and Bailey pairs, it is clear that there are plenty of interesting results like those in Section 3 that are within reach. Moreover, it would be nice to find some way of producing results for the $(s + 1)$-fold extension from the s-fold extension of Bailey's lemma, and vice versa.

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