COLLOQUIUM MATHEMATICUM

VOL. 102 2005 NO. 1

STRONG NO-LOOP CONJECTURE FOR ALGEBRAS WITH TWO SIMPLES AND RADICAL CUBE ZERO

BY

BERNT T. JENSEN (Leeds)

Abstract. Let Λ be an artinian ring and let $\mathfrak{r}$ denote its Jacobson radical. We show that a simple module of finite projective dimension has no self-extensions when Λ is graded by its radical, with at most two simple modules and $\mathfrak{r}^4 = 0$, in particular, when Λ is a finite-dimensional algebra over an algebraically closed field with at most two simple modules and $\mathfrak{r}^3 = 0$.

1. Introduction. Let Λ be an artinian ring. Many important problems remain to be solved in connection with the homological properties of Λ-modules. We mention the finitistic dimension conjecture (see [7], [3]) and the Cartan determinant conjecture (see [2]). Both these problems deal with studying homological dimensions of Λ-modules. In this paper we will consider the so called no-loop conjectures (see [5], [4]). The (*weak*) *no-loop conjecture* says that if $\mathrm{Ext}^1_\Lambda(S,S) \neq 0$ for a simple Λ-module S, then the global dimension of Λ is infinite. The *strong no-loop conjecture* says that if $\mathrm{Ext}^1_\Lambda(S,S) \neq 0$ then the projective dimension of S is infinite.

The weak no-loop conjecture was proven in [5] for a large class of finite-dimensional algebras over a field k, including all finite-dimensional algebras, if k is algebraically closed. The strong no-loop conjecture seems to be more difficult, in fact it has only been established for very special classes of algebras.

For a Λ-module U, let $\mathrm{pd}_\Lambda U$ denote the projective dimension of U. Let $\mathrm{gldim}\,\Lambda$ denote the global dimension of Λ. Let $\mathfrak{r}$ denote the Jacobson radical of Λ. The strong no-loop conjecture holds if $\mathfrak{r}^2 = 0$. For in this case if $\mathrm{Ext}^1_\Lambda(S,S) \neq 0$ then S is a summand of its own syzygy and so $\mathrm{pd}_\Lambda S = \infty$. If Λ has only one simple module up to isomorphism then all non-projective modules have infinite projective dimension. So in this case $\mathrm{pd}_\Lambda S = \infty$ if $\mathrm{Ext}^1_\Lambda(S,S) \neq 0$. For algebras with two simple modules up to isomorphism and for radical cube zero algebras the situation is much more complicated.

2000 *Mathematics Subject Classification*: 16D10, 16E10.

Key words and phrases: artinian ring, ring graded by radical, projective dimension, Cartan determinant.

The artinian ring Λ is a filtered ring using its radical filtration. Let $\operatorname{gr}\Lambda = \bigoplus_i (\operatorname{gr}\Lambda)_i$ denote the corresponding graded ring of Λ. We say that Λ is *graded by its radical* if the canonical isomorphisms $\mathfrak{r}^i/\mathfrak{r}^{i+1} \to (\operatorname{gr}\Lambda)_i$ lift to an isomorphism of rings $\Lambda \cong \operatorname{gr}\Lambda$. The following is our main result.

THEOREM. *Let Λ be an artinian ring graded by its radical with $\mathfrak{r}^4 = 0$ and with at most two simple modules up to isomorphism. If S is a simple module of finite projective dimension then $\operatorname{Ext}^1_\Lambda(S,S) = 0$.*

Now we only have to note that any basic finite-dimensional algebra over an algebraically closed field with radical cube zero is graded by its radical and we obtain the following

COROLLARY. *Let Λ be a finite-dimensional k-algebra over an algebraically closed field with $\mathfrak{r}^3 = 0$ and with at most two simple modules up to isomorphism. If S is a simple module of finite projective dimension then $\operatorname{Ext}^1_\Lambda(S,S) = 0$.*

The paper is organized as follows. In Section 2 we recall some notation and prove some basic lemmas needed to establish our main result. In Section 3 we prove our main result.

2. Definitions and some basic results. Let Λ be an artinian ring graded by its radical. That is, Λ is a graded artinian ring

$$\Lambda = \bigoplus_{i=0}^{L} \Lambda_i$$

where Λ_0 is semisimple and $\Lambda_i\Lambda_j = \Lambda_{i+j}$ for $i,j \in \{0,\dots,L\}$ with $i+j \le L$. Unless otherwise stated, from now on, all modules will be graded left Λ-modules of finite length and all homomorphisms between Λ-modules will be graded of degree 0. For a Λ-module $M = \bigoplus_i M_i$ we denote by $M[j]$ the shifted Λ-module given by $M[j]_i := M_{i-j}$.

Note that the finiteness of the projective dimension of a simple Λ-module S is independent of whether we use graded projective resolutions or not. Moreover the extension group $\operatorname{Ext}^1_\Lambda(S, T[1])$ of two simple Λ-modules S and T generated in the same degree may be identified with the group of extensions of S by T we get by forgetting the grading. So for the questions we are interested in we have not lost any generality by considering graded modules over a graded ring.

Let $S_1,\dots,S_n$ be a complete set of representatives of simple Λ-modules generated in degree 0. Let $P_1,\dots,P_n$ be a corresponding set of representatives of indecomposable projective Λ-modules. That is, $P_i/\mathfrak{r}P_i \cong S_i$ for $i = 1,\dots,n$.

Let $[S_i]$ denote the element of $\mathbb{Z}[t]^n$ given by $[S_i]_j = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta. To every P_i we associate the element $[P_i]$ in $\mathbb{Z}[t]^n$ given by

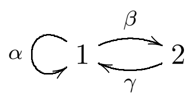
$$[P_i] = \sum_{r=0}^{L} \sum_{j=1}^{n} \mathcal{C}_{ij}^{(r)} [S_j] t^r$$

where $\mathcal{C}_{ij}^{(r)}$ is the largest integer m such that $\mathfrak{r}^r P_i / \mathfrak{r}^{r+1} P_i \cong S_j[r]^m \oplus U$. Let $\mathcal{C} = \mathcal{C}(\Lambda)$ be the graded Cartan matrix of Λ (see [6]). That is, $\mathcal{C}$ is an n by n matrix with coefficients in $\mathbb{Z}[t]$ given by

$$\mathcal{C} = \sum_{r=0}^{L} \mathcal{C}^{(r)} t^r$$

where $\mathcal{C}^{(r)}$ is a matrix with coefficients in $\mathbb{Z}$ and where $\mathcal{C}_{ij}^{(r)}$ was defined above. In other words, the ith column of $\mathcal{C}$ is $[P_i]$. Note that $\mathcal{C}^{(0)}$ is the identity matrix.

Example 1. Let Λ be the algebra with quiver



and relations $\beta\gamma$, $\alpha^2 - \gamma\beta$, $\beta\alpha\gamma$. Then Λ is graded by its radical. A basis of the projective P_1 at vertex 1 is $e_1, \alpha, \beta, \beta\alpha, \alpha^2, \alpha^3$. Thus

$$[P_1] = \begin{pmatrix} 1 + t + t^2 + t^3 \\ t + t^2 \end{pmatrix}$$

A basis of the projective P_2 at vertex 2 is $e_2, \gamma, \alpha\gamma$. Thus

$$[P_2] = \begin{pmatrix} t + t^2 \\ 1 \end{pmatrix}$$

Hence the graded Cartan matrix of Λ is given by

$$\mathcal{C} = \begin{pmatrix} 1 + t + t^2 + t^3 & t + t^2 \\ t + t^2 & 1 \end{pmatrix}$$

If N and M are two n by n matrices of integers we write $M \geq N$ if all entries of $M - N$ are non-negative.

Lemma 2. *The matrices $\mathcal{C}^{(r)}$ satisfy the following inequalities*:

(i) $\mathcal{C}^{(i)} \geq 0$ *for all* $i = 0, \dots, L$,
(ii) $\mathcal{C}^{(l)} \mathcal{C}^{(m)} \geq \mathcal{C}^{(l+m)}$ *for all* $l, m \in \{0, \dots, L\}$ *with* $l + m \leq L$.

Proof. Part (i) is obvious. By the Wedderburn–Artin theorem we have an isomorphism

$$\Lambda_0 \cong \bigotimes_{i=1}^{n} \mathrm{M}_{n_i}(D_i)$$

where $\mathrm{M}_{n_i}(D_i)$ is the full matrix ring over a division ring D_i. We view this isomorphism as an identification and let e_i denote the identity matrix of the matrix ring $\mathrm{M}_{n_i}(D_i)$. Let $l, m \in \{0, \dots, L\}$ with $l + m \leq L$. We have $\Lambda_m \Lambda_l = \Lambda_{l+m}$ and so we get a surjective Λ_0-Λ_0-homomorphism $e_j \Lambda_m \otimes_{\Lambda_0} \Lambda_l e_i \to e_j \Lambda_{l+m} e_i$ induced by multiplication. Now

$$e_j \Lambda_m \otimes_{\Lambda_0} \Lambda_l e_i \cong \bigoplus_{r=1}^{n} e_j \Lambda_m e_r \otimes_{\mathrm{M}_{n_r}(D_r)} e_r \Lambda_l e_i.$$

The number of indecomposable left summands of $e_j \Lambda_m e_r \otimes_{\mathrm{M}_{n_r}(D_r)} e_r \Lambda_l e_i$ is $\mathcal{C}_{ir}^{(l)} \mathcal{C}_{rj}^{(m)} n_i$. The number of indecomposable left summands of $e_j \Lambda_{l+m} e_i$ is $\mathcal{C}_{ij}^{(l+m)} n_i$. Therefore $\mathcal{C}_{ij}^{(l+m)} \leq (\mathcal{C}^{(l)} \mathcal{C}^{(m)})_{ij}$. This concludes the proof of the lemma. ∎

The following lemma is well known; see for example [1, Chapter III].

Lemma 3. *We have* $\mathcal{C}_{ij}^{(1)} > 0$ *if and only if* $\mathrm{Ext}_{\Lambda}^1(S_i, S_j[1]) \neq 0$.

Let $Q = Q(\Lambda)$ be the quiver given by the matrix $\mathcal{C}^{(1)}$. That is, Q is an oriented graph with vertices $1, \dots, n$ and $\mathcal{C}_{ij}^{(1)}$ arrows from vertex i to vertex j. Thus, by the previous lemma, $\mathrm{Ext}_{\Lambda}^1(S_i, S_i[1])$ is non-zero for some simple S_i if and only if Q has a loop at vertex i.

Let $\Delta = \sum_{i=0}^{n \cdot L} \Delta_i t^i =: \det \mathcal{C}$ denote the graded Cartan determinant of Λ. Let M_{ij} be the ijth cofactor of the matrix $\mathcal{C}$. That is, M_{ij} is $(-1)^{i+j}$ times the determinant of the matrix obtained by removing the ith column and the jth row from $\mathcal{C}$. Then

$$\mathcal{C}^{-1} = \frac{1}{\Delta} (M_{ji})_{ij}$$

is a matrix over the field of rational functions $\mathbb{Q}(t)$. For non-zero polynomials $a_1, \dots, a_n \in \mathbb{Z}[t]$ let $\gcd(a_1, \dots, a_n)$ denote their greatest common factor. We let the coefficient of the lowest degree term of $\gcd(a_1, \dots, a_n)$ be positive.

Lemma 4. *Let* Δ, $\mathcal{C}$ *and* Q *be as above. Then*

(i) $\Delta_0 = 1$.
(ii) Δ_1 *is the number of loops of* Q.
(iii) $\gcd(M_{1j}, \dots, M_{nj}) | \Delta$ *for all* $j = 1, \dots, n$.
(iv) *If* $\mathrm{gldim}\, \Lambda < \infty$ *then* $\Delta = 1$.
(v) *If* $\mathrm{pd}_{\Lambda} S_j < \infty$ *then* $\Delta = \gcd(M_{1j}, \dots, M_{nj})$.

Proof. We have $\Delta_0 = \det \mathcal{C}^{(0)}$. Now $\mathcal{C}^{(0)}$ is the identity matrix and so (i) follows. The constant terms in the polynomials off the diagonal of $\mathcal{C}$ are all zero. Hence Δ_1 is the trace of $\mathcal{C}^{(1)}$. This proves (ii). We have $\Delta = \sum_{i=1}^{n} \mathcal{C}_{ij} M_{ij}$, which proves (iii).

Let $\mathrm{pd}_\Lambda S_j < \infty$. We have a graded projective resolution

$$0 \to Q_m \to \cdots \to Q_1 \to Q_0 \to S_j \to 0$$

of S_j. Thus $[S_j] = \sum_{i=0}^{m} (-1)^i [Q_i] = \sum_{i=1}^{n} f_{ij}(t)[P_i]$ for polynomials $f_{ij}(t) \in \mathbb{Z}[t]$. Here

$$f_{ij} = \frac{M_{ij}}{\Delta}$$

for $i = 1, \dots, n$ and so $\Delta = \gcd(M_{1j}, \dots, M_{nj})$ by (iii). This proves (v).

If $\operatorname{gldim} \Lambda < \infty$, then again by the graded projective resolution of the simples we see that $\mathcal{C}^{-1}$ is a matrix with entries in $\mathbb{Z}[t]$ and consequently Δ is a unit in $\mathbb{Z}[t]$. Hence $\Delta = 1$ by (i). This proves (iv). ∎

Example 5. Let Λ be as in Example 1. Then

$$\mathcal{C}^{-1} = \frac{1}{\Delta} (M_{ji})_{ij} = \frac{1}{\Delta} \begin{pmatrix} 1 & -t - t^2 \\ -t - t^2 & 1 + t + t^2 + t^3 \end{pmatrix}.$$

Moreover $\gcd(M_{11}, M_{21}) = \gcd(1, -t - t^2) = 1$ and $\gcd(M_{12}, M_{22}) = \gcd(-t - t^2, 1 + t + t^2 + t^3) = 1 + t$. We see that $\Delta = 1 + t - t^3 - t^4$. Consequently $\mathrm{pd}_\Lambda S_1 = \infty$ and $\mathrm{pd}_\Lambda S_2 = \infty$ by Lemma 4(v).

3. Proof of the Theorem. Let Λ be an artinian ring graded by its radical with at most two simple modules up to isomorphism such that $\mathfrak{r}^4 = 0$. That is, $\Lambda = \bigoplus_{i=0}^{3} \Lambda_i$ where Λ_0 is semisimple and $\Lambda_i \Lambda_j = \Lambda_{i+j}$ for $i, j \in \{0, \dots, 3\}$ with $i + j \leq 3$. We may also assume that $\Lambda_2 \neq 0$ and that Λ has exactly two simple modules S_1, S_2 up to isomorphism. We assume that $\mathrm{Ext}^1_\Lambda(S_1, S_1[1]) \neq 0$ and that $\mathrm{pd}_\Lambda S_1 < \infty$. We will obtain a contradiction, which proves the theorem.

Let

$$\mathcal{C} = \sum_{i=0}^{3} \mathcal{C}^{(i)} t^i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the graded Cartan matrix of Λ. We have $\Delta = ad - bc$ and

$$\mathcal{C}^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

where Δ is the graded Cartan determinant of Λ. Thus by Lemma 4(v) we have $\gcd(M_{11}, M_{21}) = \gcd(d, -c) = ad - bc$. So there exist polynomials λ

and μ such that $d = \lambda\Delta$, $c = \mu\Delta$ and $a\lambda - b\mu = 1$. Let $a = \sum_i a_i t^i$ and similarly for b, c and d. Then $a_0 = 1 = d_0$ and $b_0 = 0 = c_0$. If $b_1 = 0$ then $b = 0$ by Lemma 2(ii). Similarly if $c_1 = 0$ then $c = 0$. In either case $\mathrm{pd}_\Lambda S_1 = \infty$ (see [3]) so we may assume that $b_1, c_1 > 0$. By Lemma 3 we see that $a_1 > 0$. Since $\mathrm{pd}_\Lambda S_1 < \infty$ at least one of the projectives has radical length less than 4. Hence we have two cases to consider, either $c_3 = 0 = a_3$ or $d_3 = 0 = b_3$.

We consider first the case $d_3 = 0 = b_3$. We see that $\deg \Delta \leq 2$, where $\deg \Delta$ denotes the degree of the polynomial Δ. If $\deg \Delta = 2$ then $d = \Delta$, which is a contradiction since the linear term of Δ is $a_1 + d_1$ and $a_1 > 0$. Thus $\deg \Delta = 1$ and $\Delta = 1 + (a_1 + d_1)t$. Consequently, $\lambda = 1 - a_1 t$ since $a\lambda - b\mu = 1$. Thus $d = \lambda\Delta = 1 + d_1 t + (-a_1^2 - a_1 d_1)t^2$. But this is a contradiction since $d_2 \geq 0$. This concludes the proof in the case where $d_3 = 0 = b_3$.

We now consider the case $c_3 = 0 = a_3$. As before, $\Delta = 1 + (a_1 + d_1)t$. Thus $\mu = c_1 t$ and $\lambda = 1 - a_1 t + \lambda_2 t^2$ for some integer λ_2. Since $d = \lambda\Delta$ we get $d_2 = \lambda_2 - a_1^2 - a_1 d_1$ and $d_3 = (a_1 + d_1)\lambda_2$. Similarly,

$$c_2 = c_1(a_1 + d_1).$$

Since $a\lambda - b\mu = 1$ we see that $\lambda_2 = b_1 c_1 + a_1^2 - a_2$. Thus

$$d_2 = b_1 c_1 - a_2 - a_1 d_1, \qquad d_3 = a_1 b_1 c_1 + a_1^3 - a_1 a_2 + b_1 c_1 d_1 + a_1^2 d_1 - a_2 d_1.$$

Moreover, again by $a\lambda - b\mu = 1$, we get

$$b_2 = a_1 b_1 + \frac{a_1^3 - 2a_1 a_2}{c_1}, \qquad b_3 = a_2 b_1 + \frac{a_1^2 a_2 - a_2^2}{c_1}.$$

By Lemma 2(ii) we have $\mathcal{C}^{(1)}\mathcal{C}^{(2)} \geq \mathcal{C}^{(3)}$ and so $d_3 \leq c_1 b_2 + d_1 d_2$. Thus we get $a_1 a_2 + a_1^2 d_1 + a_1 d_1^2 \leq 0$ and so $a_2 = 0$ and $d_1 = 0$. Again by Lemma 2(ii) we have $\mathcal{C}^{(2)}\mathcal{C}^{(1)} \geq \mathcal{C}^{(3)}$ and so $d_3 \leq c_2 b_1 + d_2 d_1$. But then $a_1^3 \leq 0$, which is a contradiction since $a_1 > 0$. This completes the proof of the theorem.

REFERENCES

[1] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, Cambridge, 1995.

[2] K. R. Fuller, *The Cartan determinant and global dimension of Artinian rings*, in: Azumaya Algebras, Actions, and Modules (Bloomington, IN, 1990), Contemp. Math. 124, Amer. Math. Soc., Providence, RI, 1992, 51–72.

[3] K. R. Fuller and M. Saorin, *On the finitistic dimension conjecture for Artinian rings*, Manuscripta Math. 74 (1992), 117–132.

[4] E. L. Green, Ø. Solberg and D. Zacharia, *Minimal projective resolutions*, Trans. Amer. Math. Soc. 353 (2001), 2915–2939.

[5] K. Igusa, *Notes on the no loops conjecture*, J. Pure Appl. Algebra 69 (1990), 161–176.

[6] K. Igusa and D. Zacharia, *On the cyclic homology of monomial relation algebras*, J. Algebra 151 (1992), 502–521.

[7] B. Zimmermann Huisgen, *The finitistic dimension conjectures—a tale of* 3.5 *decades*, in: Abelian Groups and Modules (Padova, 1994), Math. Appl. 343, Kluwer, Dordrecht, 1995, 501–517.

Department of Pure Mathematics
University of Leeds
Leeds LS2 9JT, United Kingdom
E-mail: bjensen@maths.leeds.ac.uk

Received 9 March 2004;
revised 28 September 2004 (4437)