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A DISJOINTNESS TYPE PROPERTY OF CONDITIONAL EXPECTATION OPERATORS

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Abstract. We give a characterization of conditional expectation operators through a disjointness type property similar to band-preserving operators. We say that the operator $T : X \to X$ on a Banach lattice X is semi-band-preserving if and only if for all $f, g \in X$, $f \perp Tg$ implies that $Tf \perp Tg$. We prove that when X is a purely atomic Banach lattice, then an operator T on X is a weighted conditional expectation operator if and only if T is semi-band-preserving.

1. Introduction. In this note we study two abstract disjointness type conditions which are satisfied by all conditional expectation operators on Banach lattices. There is an extensive literature devoted to finding conditions which characterize conditional expectation operators and an extensive literature studying disjointness-preserving and band-preserving operators. However, as far as we know, to date there have been no attempts to characterize conditional expectation operators through a property related to disjointness.

Of course, conditional expectation operators are never disjointness-preserving let alone band-preserving. However they do preserve some bands, namely they satisfy the following disjointness type condition:

$$\text{(SBP)} \qquad f \perp Tg \;\Rightarrow\; Tf \perp Tg \quad \forall f, g \in X$$

(here X is a Banach lattice and T is a linear operator on X).

Note that condition (SBP) is a weakening of the condition which defines band-preserving operators. Recall that a linear operator T on a Banach lattice X is called *band-preserving* if $TB \subset B$ for every band $B \subset X$. Thus T is band-preserving if and only if one of the following two equivalent conditions is satisfied:

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$$\text{(BP1)} \qquad f \perp g \;\Rightarrow\; Tf \perp g \quad \forall f,g \in X,$$

$$\text{(BP2)} \qquad f \lhd g \;\Rightarrow\; Tf \lhd g \quad \forall f,g \in X.$$

(We use the notation $f \lhd g$ to mean that f belongs to a band generated by $\{g\}$.)

Thus condition (SBP) is the same as (BP1) with the additional constraint that g belongs to the range of T. Hence, clearly (BP1) implies (SBP), and (BP1) and (SBP) are equivalent if T is surjective. Conditional expectation operators are our principal examples of non-band-preserving operators which do satisfy (SBP).

We will say that an operator T is *semi-band-preserving* if T satisfies (SBP). Our main result (Theorem 4.7 and Corollary 4.11) asserts that when X is a purely atomic Banach lattice, then an operator T on X is a weighted conditional expectation operator if and only if T is semi-s band-preserving.

Further, we study a condition which arises from the weakening of (BP2) by adding the constraint that g belongs to the range of T, similarly to the definition of semi-band-preserving operators. Namely we consider

$$\text{(SCP)} \qquad f \lhd Tg \;\Rightarrow\; Tf \lhd Tg \quad \forall f,g \in X.$$

We will say that an operator T is *semi-containment-preserving* if T satisfies (SCP). It is clear that all surjective semi-containment-preserving operators are band-preserving. It is also easy to see that all conditional expectation operators are semi-containment-preserving but not band-preserving.

In contrast to the fact that (BP1) and (BP2) are equivalent, conditions (SBP) and (SCP) are independent in general (see Examples 3.1 and 3.2). However if the Banach lattice X is purely atomic then it follows from our characterization of semi-band-preserving operators that all semi-band-preserving operators are semi-containment-preserving (see Corollary 4.10). It is easy to construct on almost all Banach lattices a semi-containment-preserving operator T so that T is not semi-band-preserving; one can even find projections with this property (see Example 3.2). However we prove (Theorem 5.1 and Corollary 5.3) that if X is a strictly monotone purely atomic Banach lattice and P is a projection of norm one on X then P is a weighted conditional expectation operator if and only if P is semi-containment-preserving. (Thus, in particular, semi-containment-preserving projections of norm one on strictly monotone purely atomic Banach lattices are semi-band-preserving.)

We finish these general remarks about semi-band-preserving and semi-containment-preserving operators by recalling a pair of conditions which are very similar to (SBP) and (SCP). Let X denote a vector lattice and T be a linear operator on X. Consider:

(DP) $$f \perp g \;\Rightarrow\; Tf \perp Tg \quad \forall f,g \in X,$$

(β) $$f \lhd g \;\Rightarrow\; Tf \lhd Tg \quad \forall f,g \in X.$$

Condition (DP) is the well known condition defining disjointness-preserving operators, and condition (β) has recently been identified by Abramovich and Kitover [2] as being equivalent to the fact that T^{-1} is disjointness-preserving (provided that T is bijective and X has sufficiently many components). Abramovich and Kitover [2] showed that in general conditions (DP) and (β) are independent, but if T is a continuous (or just regular) linear operator between normed vector lattices then (DP) implies (β), and if X is a Banach lattice and T is bijective then (DP) is equivalent to (β).

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2. Preliminaries. We use standard lattice and Banach space notations as may be found e.g. in [5, 6, 7]. Below we recall basic definitions that we use.

A closed subspace Y of a Banach lattice X is called a *band* in X if

(1) $|y| \le |x|$ for some $x \in Y$ implies that $y \in Y$,
(2) $\sup A \in Y$ for every subset of $A \subseteq Y$ which has a supremum in X.

An element u in a Banach lattice X is called an *atom* if it follows from $0 \ne v \le u$ that $v = u$. X is called a *purely atomic Banach lattice* if it coincides with the band generated by its atoms. Examples of purely atomic Banach lattices include c_0, c, ℓ_p $(1 \le p \le \infty)$ and Banach spaces with 1-unconditional bases. A Banach lattice X is called *nonatomic* if it contains no atoms.

For an element u in a Banach lattice X, an element $v \in X$ is said to be a *component* of u if $|v| \wedge |u - v| = 0$. A lattice X is called *essentially one-dimensional* if for any two nondisjoint elements $x_1, x_2 \in X$ there exist nonzero components u_1 of x_1 and u_2 of x_2 such that u_1 and u_2 are proportional. This class of lattices is strictly larger than purely atomic lattices and does include some nonatomic lattices (see [3, Chapter 11]).

A Banach lattice X is called *strictly monotone* if for all elements x, y in X with $x, y > 0$ we have $\|x + y\| > \|x\|$.

In this note we mainly consider Banach lattices of (equivalence classes of) functions on a σ-finite measure space (Ω, Σ, μ) which are subspaces of $L_1(\Omega, \Sigma, \mu) + L_\infty(\Omega, \Sigma, \mu)$.

By the Radon–Nikodym Theorem for each $f \in L_1(\Omega, \Sigma, \mu) + L_\infty(\Omega, \Sigma, \mu)$ and for every σ-subalgebra $\mathcal{A}$ of Σ so that μ restricted to $\mathcal{A}$ is σ-finite (i.e.

so that $\mathcal{A}$ does not have atoms of infinite measure) there exists a unique, up to equality a.e., $\mathcal{A}$-measurable locally integrable function h so that

$$\int_{\Omega} gh\,d\mu = \int_{\Omega} gf\,d\mu$$

for every bounded, integrable and $\mathcal{A}$-measurable function g on Ω. The function h is called the *conditional expectation of f with respect to $\mathcal{A}$* and it is usually denoted by $\mathcal{E}(f\,|\,\mathcal{A})$. The operator $\mathcal{E}(\cdot\,|\,\mathcal{A})$ is called the *conditional expectation operator generated by $\mathcal{A}$*. Sometimes, particularly when (Ω,Σ,μ) is purely atomic, $\mathcal{E}(\cdot\,|\,\mathcal{A})$ is also called an *averaging operator*. When X is a purely atomic Banach lattice with a basis $\{e_i\}_{i\in\mathbb{N}}$ then averaging operators on X have the following form:

The σ-finite σ-subalgebra $\mathcal{A}$ is generated by a family $\{A_j\}_{j=1}^{\infty}$ of mutually disjoint finite subsets of $\mathbb{N}$, and for all $x=\sum_{i=1}^{\infty} x_i e_i$ the conditional expectation $\mathcal{E}(x\,|\,\mathcal{A})$ is defined by

$$\mathcal{E}(x\,|\,\mathcal{A}) = \sum_{j=1}^{\infty}\Big(\frac{1}{\operatorname{card}(A_j)}\sum_{n\in A_j} x_n\Big)\Big(\sum_{n\in A_j} e_n\Big).$$

Conditional expectation operators have been extensively studied by many authors since 1930s; for one of the most recent presentations of the subject see [1]. One of the main directions in the research concerning conditional expectation operators is to identify a property or properties of an operator T that guarantee that T is a conditional expectation operator (see [4]).

Let X be a Banach lattice of functions on a measure space (Ω,Σ,μ) and let $k\in L_1(\Omega,\Sigma,\mu)+L_\infty(\Omega,\Sigma,\mu)$, $w\in X'$. Then $\mathcal{E}(wf\,|\,\mathcal{A})$ is well defined for all $f\in X$. Assume in addition that $k\mathcal{E}(wf\,|\,\mathcal{A})\in X$ for all $f\in X$ and put

$$Tf = k\mathcal{E}(wf\,|\,\mathcal{A}).$$

The operator T thus defined is called a *weighted conditional expectation operator*. Note that when X is a purely atomic Banach lattice or when $\mathcal{A}$ is a σ-subalgebra of Σ generated by a family $\{A_j\}_{j=1}^{\infty}$ of mutually disjoint sets of finite measure on Σ then weighted conditional expectation operators on X have the following form:

$$Tf = \sum_{j=1}^{\infty}\langle\psi_j, f\rangle u_j, \tag{1}$$

where $\{\psi_j\}_{j=1}^{\infty}\subset X'$ and $\{u_j\}_{j=1}^{\infty}\subset X$ are so that for all j, $\operatorname{supp}\psi_j\subset A_j$ and $\operatorname{supp} u_j\subset A_j$.

Recall that when X is a space of (equivalence classes of) functions on (Ω,Σ,μ) then $\operatorname{supp} f$ is the minimal closed subset of Ω so that $f(t)=0$ for a.e. $t\in\Omega\setminus\operatorname{supp} f$.

Note that a weighted conditional expectation operator is a projection if and only if $\mathcal{E}(kw \mid \mathcal{A})$ is the function constantly equal to 1 in the case when μ is a finite measure, or if and only if $\langle \psi_j, u_j \rangle = 1$ for all j in the case when $\mathcal{A}$ is a σ-subalgebra of Σ generated by a family of mutually disjoint sets $\{A_j\}_{j=1}^{\infty}$ (i.e. when T has the form (1)).

3. Definitions of semi-band-preserving and semi-containment-preserving operators. Let X be a Banach lattice and T be a linear operator on X. As discussed in the Introduction we are interested in the following two conditions:

(SBP) $$f \perp Tg \;\Rightarrow\; Tf \perp Tg \quad \forall f, g \in X,$$

(SCP) $$f \lhd Tg \;\Rightarrow\; Tf \lhd Tg \quad \forall f, g \in X.$$

We will say that an operator T is *semi-band-preserving* if T satisfies (SBP), and that T is *semi-containment-preserving* if T satisfies (SCP).

It is easy to see that all conditional expectation operators and weighted conditional expectation operators are both semi-band-preserving and semi-containment-preserving.

Conditions (SBP) and (SCP) are weakenings of conditions (BP1) and (BP2) (respectively) which define band-preserving operators, but in contrast to the fact that conditions (BP1) and (BP2) are always equivalent, in general conditions (SBP) and (SCP) are independent of each other, as the following two simple examples demonstrate.

EXAMPLE 3.1. *Let X be a Banach lattice of functions on $[0,1]$ such that the constant function $\varphi_1 = \mathbf{1} = \chi_{[0,1]}$, and the function φ_2 defined by $\varphi_2(t) = t$ if $t \in [0,1/2]$ and $\varphi_2(t) = 0$ if $t \in (1/2,1]$, belong to X and there exist functionals $\psi_1, \psi_2 \in X'$ with $\operatorname{supp}\psi_1 \cup \operatorname{supp}\psi_2 \subseteq [0,1/2]$. Then there exists a linear operator T on X which is semi-band-preserving but not semi-containment-preserving.*

Construction. Define, for all $f \in X$,

$$Tf = \langle \psi_1, f\rangle \varphi_1 + \langle \psi_2, f\rangle \varphi_2.$$

Then the operator T is semi-band-preserving. Indeed, $f \perp Tg$ implies that either $f = 0$ or $\operatorname{supp} Tg \subset [0,1/2]$ and $\operatorname{supp} f \subset [1/2,1]$. But then $Tf = 0$ so $Tf \perp Tg$.

However T is not semi-containment-preserving. Indeed, let $f, g \in X$ be such that $\langle \psi_1, f\rangle = 0$, $\langle \psi_1, g\rangle \neq 0$ and $\operatorname{supp} g \subset [0,1/2]$. Then $Tf = \langle \psi_2, f\rangle \varphi_2$ and so $\operatorname{supp} Tf = [0,1/2]$. On the other hand, $\operatorname{supp} Tg = [0,1]$ since $\langle \psi_1, g\rangle \neq 0$. Thus $g \lhd Tf$ but $Tg \not\lhd Tf$. ∎

EXAMPLE 3.2. *Let X be any Banach lattice with $\dim X \geq 2$. Then there exists a semi-containment-preserving operator Q on X which is not* *semi-band-preserving. Moreover Q can be chosen to be a projection, and if X is not strictly monotone then Q can be chosen to be a projection of norm arbitrarily close to one.*

Construction. Let f_1, f_2 be nonzero elements in X with $f_1 \perp f_2$, and let ψ be a functional on X so that $\langle \psi, f_1 \rangle \neq 0$ and $\langle \psi, f_2 \rangle \neq 0$. Define, for all $f \in X$,

$$Qf = \langle \psi, f \rangle f_1.$$

Then Q is trivially semi-containment-preserving since the range of Q is one-dimensional. However Q is not semi-band-preserving since $f_2 \perp Qf_1$, but $Qf_2 \not\perp Qf_1$.

Moreover if $\langle \psi, f_1 \rangle = 1$ then Q is a projection. Further if X is not strictly monotone, then for any $\varepsilon > 0$, it is possible to choose $f_1 \perp f_2$, $f_2 \neq 0$, with $\|f_1 + f_2\| = \|f_1\| = 1$ and $\psi \in X'$ so that $\langle \psi, f_1 \rangle = 1$, $\langle \psi, f_2 \rangle \neq 0$ and $\|\psi\| < 1 + \varepsilon$, which will result in Q being a projection of norm smaller than $1 + \varepsilon$. ∎

4. Semi-band-preserving operators. Our next goal is to characterize weighted conditional expectation operators on purely atomic lattices as semi-band-preserving operators.

In the following X will be a Banach lattice of (equivalence classes of) real-valued functions on a measure space (Ω, Σ, μ). For any linear operator $T : X \to X$ define

$$\Sigma_T = \{A \subset \Omega : \exists f \in X \text{ with } \operatorname{supp}(Tf) = A\}.$$

Here and in the following, all set relations are considered modulo sets of measure zero.

We start with a simple lemma, which we formulate here for easy reference.

LEMMA 4.1. (1) *If $A, B \in \Sigma_T$, then $A \cup B \in \Sigma_T$.*

(2) *If $\{A_j\}_{j\in\mathbb{N}} \subset \Sigma_T$ is a family of mutually disjoint sets, then $\bigcup_{j=1}^{\infty} A_j \in \Sigma_T$.*

Proof. These facts are immediate. For (1), let f, g be concrete representations of functions in X so that $\operatorname{supp}(Tf) = A$ and $\operatorname{supp}(Tg) = B$. Define

$$h(t) = \begin{cases} \dfrac{Tf(t)}{Tg(t)} & \text{if } Tg(t) \neq 0, \\ 0 & \text{if } Tg(t) = 0, \end{cases}$$

and

$$V(h) = \{a \in \mathbb{R} : \mu(h^{-1}\{a\}) > 0\}.$$

Clearly $\operatorname{card}(V(h)) \le \aleph_0$ and thus there exists $\alpha \in \mathbb{R}$ so that $-\alpha \notin V(h)$. It is easy to see that this implies that $\operatorname{supp}(T(f+\alpha g)) = A \cup B$ (recall that all set relations are considered modulo sets of measure zero).

Part (2) is even quicker. Indeed let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of elements of X such that $\|f_j\| = 1$ and $\operatorname{supp}(Tf_j) = A_j$ for all $j \in \mathbb{N}$. Then $\sum_{j=1}^{\infty} 2^{-j} f_j$ belongs to X and, since the sets $\{A_j\}_{j\in\mathbb{N}}$ are mutually disjoint,

$$\operatorname{supp}\Big(T\Big(\sum_{j=1}^{\infty} 2^{-j} f_j\Big)\Big) = \bigcup_{j=1}^{\infty} A_j,$$

as desired. ■

Define $S_T = \bigcup_{A\in\Sigma_T} A \subset \Omega$. Then, for each $f \in X$,

$$\operatorname{supp}(Tf) \subseteq S_T. \tag{2}$$

Now we immediately obtain:

PROPOSITION 4.2. *If T is a semi-band-preserving operator on X then for every $f \in X$ with* $\operatorname{supp} f \subseteq \Omega \setminus S_T$ *we have $Tf = 0$.*

Proof. Indeed, by (2), $\operatorname{supp}(Tf) \subseteq S_T$ so $f \perp Tf$. By (SBP) we get $Tf \perp Tf$. Thus $Tf = 0$. ■

When the space X is essentially one-dimensional we can deduce a further important property of semi-band-preserving operators. We have:

PROPOSITION 4.3. *Suppose that X is essentially one-dimensional and T is a semi-band-preserving operator on X. If $A, B \in \Sigma_T$ and $A \subset B$, then $B \setminus A \in \Sigma_T$.*

Proof. Let $h, g \in X$ be such that $\operatorname{supp}(Th) = B$ and $\operatorname{supp}(Tg) = A$. Since $A \subset B$ and X is essentially one-dimensional there exists $C \subset A$ so that the components $(Th)\chi_C$ and $(Tg)\chi_C$ of Th and Tg, respectively, are proportional. Let $\{A_i\}_{i\in I}$ denote the family of subsets of A maximal with respect to the property that $(Th)\chi_{A_i}$ and $(Tg)\chi_{A_i}$ are proportional. Then $\{A_i\}_{i\in I}$ are mutually disjoint and, by the essential one-dimensionality of X,

$$A = \bigcup_{i\in I} A_i.$$

Moreover, for each $i \in I$ there exists a scalar $a_i \neq 0$ so that

$$(Th)\chi_{A_i} = a_i (Tg)\chi_{A_i}. \tag{3}$$

Consider $g_i = h - a_i g$ for $i \in I$. Then $\operatorname{supp}(Tg_i) = B \setminus A_i$ by the maximality of A_i's. Thus $B \setminus A_i \in \Sigma_T$ for all $i \in I$. Moreover $g\chi_{A_i} \perp Tg_i$. Hence, by (SBP),

$$T(g\chi_{A_i}) \perp Tg_i.$$

That is, for all $i \in I$,

$$\operatorname{supp}(T(g\chi_{A_i})) \subset A_i. \tag{4}$$

But, since $\{A_i\}_{i\in I}$ are mutually disjoint,

$$Tg = \sum_{i\in I} T(g\chi_{A_i}),$$

and, by (4),

$$(Tg)\chi_{A_i} = T(g\chi_{A_i}). \tag{5}$$

Thus, by (3) and (5), we get

$$\begin{aligned}(Th)\chi_A &= \sum_{i\in I} (Th)\chi_{A_i} = \sum_{i\in I} a_i(Tg)\chi_{A_i} \\ &= \sum_{i\in I} a_i T(g\chi_{A_i}) = T\Big(\sum_{i\in I} a_i g\chi_{A_i}\Big) = T(h\chi_A).\end{aligned}$$

So $(Th)\chi_A \in T(X)$. Hence

$$(Th)\chi_{B\setminus A} = Th - (Th)\chi_A \in T(X).$$

Thus $B \setminus A \in \varSigma_T$. ■

Remark 4.4. Note that the above proof also shows that if X is essentially one-dimensional and T is a semi-band-preserving operator on X then the subspace $T(X)$ is essentially one-dimensional. We will prove a stronger result in Theorem 4.7.

Remark 4.5. Proposition 4.3 fails in general nonatomic Banach lattices. Indeed, let T be the semi-band-preserving operator defined in Example 3.1. It is easy to see that $[0,1], [0,1/2] \in \varSigma_T$ and $[1/2,1] = [0,1] \setminus [0,1/2]$ does not belong to $\varSigma_T$.

Note that when ψ_1 and ψ_2 are positive then T is positive, and when $\psi_i(\varphi_j) = \delta_{ij}$ for $i,j = 1,2$, then T is a projection. However it follows from [4, Theorem 3.10] that when T is an order-continuous positive semi-band-preserving projection on a Banach lattice of functions on $[0,1]$ then T satisfies the assertion of Proposition 4.3.

By de Morgan Laws, as a corollary of Lemma 4.1 and Proposition 4.3 we immediately obtain:

Corollary 4.6. *Suppose that X is an essentially one-dimensional Banach lattice and T is a semi-band-preserving operator on X. Then T satisfies the following two properties*:

$$A, B \in \varSigma_T \ \Rightarrow\ A \cap B \in \varSigma_T, \tag{I1}$$

$$\{A_j\}_{j\in\mathbb{N}} \subset \varSigma_T \text{ and } A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \ \Rightarrow\ \bigcap_{j\in\mathbb{N}} A_j \in \varSigma_T. \tag{I2}$$

These properties allow us to give the full characterization of semi-band-preserving operators on essentially one-dimensional Banach lattices. Namely we have:

THEOREM 4.7. *Let X be an essentially one-dimensional Banach lattice. Then an operator $T : X \to X$ is semi-band-preserving if and only if the range of T is the linear span of a collection $\{u_j\}_{j\in J}$ of mutually disjoint elements in $T(X)$ and T is a weighted conditional expectation operator*, *i.e. T has the following form for all f in X*:

$$Tf = \sum_{j\in J} \langle \psi_j, f\rangle u_j, \tag{6}$$

where $\{\psi_j\}_{j\in J}$ are nonzero functionals on X so that for all $j \in J$ if $f \perp u_j$ then $\langle \psi_j, f\rangle = 0$ (see (1)).

Proof. It is not difficult to see that all weighted conditional expectation operators are semi-band-preserving.

For the other direction, let $\omega_0 \in S_T = \bigcup_{A\in\Sigma_T} A \subset \Omega$. Then, by (I2) and Zorn's Lemma, among all $A \in \Sigma_T$ such that $\omega_0 \in A$, there exists a set $A_0 \in \Sigma_T$ minimal with respect to inclusion.

Next, we claim that the subspace of $T(X)$ consisting of those elements in $T(X)$ whose support is contained in A_0 is one-dimensional.

Suppose for contradiction that there exist $f, g \in X$ such that $\mathrm{supp}(Tf) = A_0$, $\mathrm{supp}(Tg) \subseteq A_0$ and Tf, Tg are linearly independent. Since X is essentially one-dimensional there exist nonzero components u_1, u_2 of Tf, Tg respectively so that

$$u_1 = ku_2$$

for some scalar k. Clearly $\mathrm{supp}\, u_1 = \mathrm{supp}\, u_2$ and since Tf, Tg are linearly independent,

$$\mathrm{supp}\, u_1 = B \subsetneq A_0.$$

Consider $h = f - kg$. Then $Th = Tf - kTg$ and $C = \mathrm{supp}(Th)$ belongs to Σ_T and

$$\emptyset \neq C \subseteq A_0 \setminus B \subsetneq A_0.$$

By Proposition 4.3 we also see that $A_0 \setminus C$ belongs to Σ_T.

Now ω_0 belongs to one of the sets C, or $A_0 \setminus C$, which contradicts the minimality of the set A_0.

It now follows immediately that there exist mutually disjoint elements $\{u_j\}_{j\in J}$ in $T(X)$ with minimal supports in Σ_T. Thus $T(X) = \overline{\mathrm{span}}\{u_j\}_{j\in J}$ and T has the form (6) since T is a linear operator. Condition (SBP) implies that for all $j \in J$, if $f \perp u_j$, then since $u_j \in T(X)$, also $Tf \perp u_j$ and thus $\langle \psi_j, f\rangle = 0$, as required in (6). ∎

REMARK 4.8. The above proof is very similar in spirit to that of the characterization of the form of norm one projections in ℓ_p, $1 < p < \infty$ ([5, Theorem 2.a.4]).

REMARK 4.9. Theorem 4.7 is not valid in general nonatomic lattices. The counterexample is very similar to Example 3.1. Indeed, let X be any Banach lattice of functions on $[0,1]$ such that the constant function $\mathbf{1}$ and the function $\psi : [0,1] \to [0,1]$ defined by $\psi(t) = t$ belong to X. Then $\mathrm{span}\{\mathbf{1}, \psi\} \subset X$ is 2-dimensional in X and therefore it is complemented in X, i.e. there exists a projection $T : X \to X$ with $T(X) = \mathrm{span}\{\mathbf{1}, \psi\}$. But for every $g \in X$ we have $\mathrm{supp}(Tg) = [0,1]$. Thus $f \perp Tg$ implies $f = 0$ and thus T is trivially semi-band-preserving. Clearly T is not a weighted conditional expectation operator. Further, note that every function in the range of T has full support and hence T is also trivially semi-containment-preserving.

We finish this section with two immediate corollaries of Theorem 4.7.

COROLLARY 4.10. *Let X be an essentially one-dimensional Banach lattice. Then every semi-band-preserving operator T on X is semi-containment-preserving.*

COROLLARY 4.11. *Let X be a purely atomic Banach lattice. Then an operator T on X is a weighted conditional expectation operator if and only if T is semi-band-preserving.*

5. Semi-containment-preserving projections. In this section we obtain an analogue of our main result, Theorem 4.7, for semi-containment-preserving operators. However, as Example 3.2 demonstrates, on any Banach lattice which contains nonzero elements f_1, f_2 with $f_1 \perp f_2$ there exists a semi-containment-preserving projection Q which is not semi-band-preserving and thus is not a weighted conditional expectation operator. Moreover if X is not strictly monotone then such a Q can be chosen to be a projection of norm one.

Also an example described in Remark 4.9 demonstrates that in general nonatomic Banach lattices there may exist a semi-containment-preserving projection which is not a weighted conditional expectation operator. Thus our characterization below has natural restrictions. We prove:

THEOREM 5.1. *Let X be an essentially one-dimensional strictly monotone Banach lattice and let $P : X \to X$ be a projection of norm one. Then P is semi-containment-preserving if and only if the range of P is the linear span of a collection $\{u_j\}_{j\in J}$ of mutually disjoint elements of $P(X)$ and P is a weighted conditional expectation operator, i.e. P has the following form* *for all f in X:*

$$Pf = \sum_{j\in J} \langle \psi_j, f\rangle u_j, \tag{7}$$

where $\{\psi_j\}_{j\in J}$ are nonzero functionals on X so that for all $j \in J$, $\operatorname{supp}\psi_j \subseteq \operatorname{supp} u_j$, $\langle\psi_j, u_j\rangle = 1 = \|\psi_j\| = \|u_j\|$ and $\langle\psi_j, u_i\rangle = 0$ if $i \neq j$ (see (1)).

Proof. As before, we note that all weighted conditional expectation operators are semi-containment-preserving, so we just need to prove one implication in Theorem 5.1.

Our method of proof depends on the following lemma:

LEMMA 5.2. *Suppose that X is a strictly monotone (not necessarily essentially one-dimensional) Banach lattice and $P : X \to X$ is a semi-containment-preserving projection of norm one. Let $\{A_j\}_{j\in\mathbb{N}} \subset \Sigma_P$ with $A_1 \supseteq A_2 \supseteq \cdots$. Then*

$$\bigcap_{j\in\mathbb{N}} A_j \in \Sigma_P.$$

Using this lemma the proof of Theorem 5.1 is the same as that of Theorem 4.7. Indeed, Lemma 5.2 states that when X and P satisfy the assumptions of Theorem 5.1 then P has property (I2) from Corollary 4.6. Thus, following the proof of Theorem 4.7 word for word, we deduce that there exist mutually disjoint elements $\{u_j\}_{j\in J}$ in $P(X)$ so that $P(X) = \overline{\operatorname{span}}\{u_j\}_{j\in J}$ and P has the form (7) since P is a linear operator. Condition (SCP) implies that $\operatorname{supp}\psi_j \subseteq \operatorname{supp} u_j$ for all $j \in J$, and since P is a projection of norm one we have $\langle\psi_j, u_j\rangle = 1 = \|\psi_j\| = \|u_j\|$ and $\langle\psi_j, u_i\rangle = 0$ if $i \neq j$, as required in (7). ∎

Proof of Lemma 5.2. Since $\{A_j\}_{j\in\mathbb{N}} \subset \Sigma_P$, there exist $\{f_j\}_{j\in\mathbb{N}} \subset X$ so that $\operatorname{supp}(Pf_j) = A_j$. Define $A = \bigcap_{j\in\mathbb{N}} A_j$ and set $g = (Pf_1)\cdot\chi_A$. Then $\operatorname{supp} g = A \subset \operatorname{supp}(Pf_j)$ for all $j \in \mathbb{N}$. Thus, by (SCP),

$$\operatorname{supp}(Pg) \subset \operatorname{supp}(Pf_j)$$

for all $j \in \mathbb{N}$. Hence

$$\operatorname{supp}(Pg) \subset A.$$

Denote $\operatorname{supp}(Pg)$ by B. Then

$$(Pg)\cdot\chi_{A_1\setminus B} = 0.$$

Further

$$\begin{aligned} Pf_1 &= (Pf_1)\cdot\chi_{A_1\setminus A} + (Pf_1)\cdot\chi_A = (Pf_1)\cdot\chi_{A_1\setminus A} + g, \\ Pf_1 &= P(Pf_1) = P((Pf_1)\cdot\chi_{A_1\setminus A}) + Pg, \\ (Pf_1)\cdot\chi_{A_1\setminus B} &= P((Pf_1)\cdot\chi_{A_1\setminus A})\cdot\chi_{A_1\setminus B} + (Pg)\cdot\chi_{A_1\setminus B} \\ &= P((Pf_1)\cdot\chi_{A_1\setminus A})\cdot\chi_{A_1\setminus B}. \end{aligned}$$

Since P has norm one we get

$$\|(Pf_1)\cdot\chi_{A_1\setminus B}\| = \|P((Pf_1)\cdot\chi_{A_1\setminus A})\cdot\chi_{A_1\setminus B}\| \le \|P((Pf_1)\cdot\chi_{A_1\setminus A})\| \le \|(Pf_1)\cdot\chi_{A_1\setminus A}\|.$$

Since X is strictly monotone and $\operatorname{supp}(Pf_1) = A_1$ we conclude that

$$A_1 \setminus B \subseteq A_1 \setminus A.$$

Since $B \subset A$, we get

$$A = B = \operatorname{supp}(Pg).$$

Thus $A \in \Sigma_P$, as desired. ∎

We finish this section with an immediate corollary of Theorem 5.1 similar to Corollary 4.11.

COROLLARY 5.3. *Let X be a purely atomic strictly monotone Banach lattice and let $P : X \to X$ be a projection of norm one. Then P is a weighted conditional expectation operator if and only if P is semi-containment-preserving.*

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