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# OPERATIONAL CALCULUS AND FOURIER TRANSFORM ON BOEHMIANS

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**Abstract.** We define various operations on the space of ultra Boehmians like multiplication with certain analytic functions which are Fourier transforms of compactly supported distributions, polynomials, and characters ($e^{ist}$, $s, t \in \mathbb{R}$), translation, differentiation. We also prove that the Fourier transform on the space of ultra Boehmians has all the operational properties as in the classical theory.

**1. Introduction.** The problem of defining a "product" of generalized functions is an interesting one. See for example [1, 6, 7, 10]. A famous result of L. Schwartz [10] says that a product of distributions cannot be defined so as to have the usual properties. On the other hand, in [8, p. 144] the product of a smooth function $f \in C^\infty$ and a distribution $u \in \mathcal{D}'$ is defined by

$$(f \cdot u)(\phi) = u(f \cdot \phi), \quad \forall \phi \in \mathcal{D},$$

where $f \cdot \phi$ is the pointwise product of functions.

In the context of Boehmians it is proved in [6] that we cannot define multiplication between the class of real analytic functions and the Boehmian space $\mathcal{B}_\infty$ as a map continuous in the second variable. In another paper [7], the product of a continuous functions and a Boehmian is defined in a natural way, and some usual properties of this product are verified.

In [2], we have constructed the space of ultra Boehmians $\mathcal{B}(\mathcal{Z}', \mathcal{Z}, \bullet, \hat{\Delta})$ and extended the Fourier transform as a continuous linear bijection. Since the operation used in the construction of this Boehmian space is multiplication, the product of ultra Boehmians by functions from multipliers of $\mathcal{Z}$ is well-defined and has good properties as we shall see later, even though defining convolution with functions will be difficult in this context.

Since the Fourier transform provides an isomorphism between $(\mathcal{D}', \mathcal{D}, *)$ and $(\mathcal{Z}', \mathcal{Z}, \cdot)$, and delta sequences in $\mathcal{Z}$ are defined as Fourier transforms

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of delta sequences in $\mathcal{D}$, the isomorphism extends in a natural way to the corresponding Boehmian spaces, and the properties of multiplication mirror the properties of convolution for $C^\infty$-Boehmians. The objective of the present paper is to obtain an operational calculus for the Fourier transform consistent with the classical theory.

**2. Classical theory.** Throughout the paper we use the normalized Lebesgue measure [9] defined by $dm(t) = (2\pi)^{-1/2}dt$ where $dt$ is the Lebesgue measure on $\mathbb{R}$. Let $C^\infty$ be the space of smooth complex-valued functions on $\mathbb{R}$, and $\mathcal{D}$ be the subspace of $C^\infty$ consisting of functions with compact supports, with the usual locally convex topology [8].

Let $\mathcal{Z}(m)$ ($m \in \mathbb{N}$) be the collection of all entire functions on $\mathbb{C}$ satisfying

$$\gamma_{m,k}(f) = \sup\{|z^\alpha f(z)|e^{-m|\operatorname{Im} z|} : z \in \mathbb{C},\, 0 \le \alpha \le k\} < \infty,\ \ k = 0, 1, \ldots.$$

It is well known that each $\mathcal{Z}(m)$ is a multi-normed space with the collection of seminorms $\{\gamma_{m,k}\}_{k=0}^\infty$. The *Zemanian space* $\mathcal{Z}$ is defined as the union $\bigcup_{m\in\mathbb{N}} \mathcal{Z}(m)$ with the inductive limit topology [11]. The dual space $\mathcal{Z}'$ of $\mathcal{Z}$ is usually referred to as the space of *ultra distributions*. We use both notations $u(\phi)$ and $\langle u(t), \phi(t)\rangle$ to denote the value of $u \in V'$ at $\phi \in V$, where $V$ is $\mathcal{D}$, $\mathcal{S}$ or $\mathcal{Z}$.

We say that $u_n \to u$ as $n \to \infty$ in $\mathcal{Z}'$ if $\sup_{f\in B} |(u_n - u)(f)| \to 0$ as $n \to \infty$ for each bounded subset $B$ of $\mathcal{Z}$.

Definition 2.1 (Operations on generalized function spaces). Let $V$ denote $\mathcal{D}$, $\mathcal{S}$ or $\mathcal{Z}$, $u \in V'$, and $s \in \mathbb{R}$. Put

$$e_s(t) = e^{ist}, \qquad t \in \mathbb{R}.$$

(i) The derivative $u' \in V$ is defined by

$$\langle u'(t), \psi(t)\rangle = \langle u(t), -\psi'(t)\rangle, \qquad \psi \in V.$$

(ii) The product $e_s \cdot u \in V'$ of $e_s$ and $u$ is defined by

$$\langle (e_s \cdot u)(t), \psi(t)\rangle = \langle u(t), e_s(t) \cdot \psi(t)\rangle, \qquad \psi \in V.$$

(iii) The operation $Mu$ is defined by

$$\langle (Mu)(t), \psi(t)\rangle = \langle u(t), t \cdot \psi(t)\rangle, \qquad \psi \in V.$$

(iv) The translation $\tau_s u$ of $u \in V'$ is defined as a member of $V'$ by

$$\langle (\tau_s u)(t), \psi(t)\rangle = \langle u(t), \psi(t+s)\rangle, \qquad \psi \in V.$$

(v) If $u \in \mathcal{D}'$ and $\phi \in \mathcal{D}$, then $u * \phi \in C^\infty$, where

$$(u * \phi)(x) = \langle u(t), \phi(x-t)\rangle.$$

The classical Fourier transform of an integrable function $f$ on $\mathbb{R}$ is defined by

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt}\,dm(x), \qquad t \in \mathbb{R}.$$

The theory of the Fourier transform on $\mathcal{S}$, $\mathcal{D}$, $\mathcal{Z}$ and on their dual spaces is well known and can be found in [8, 12].

The following lemma is taken from [8].

LEMMA 2.2. *Let $u \in \mathcal{S}'$ and $s \in \mathbb{R}$. Then*

(i) $(-iMu)^\wedge = (\hat{u})'$;
(ii) $(u')^\wedge = iM\hat{u}$;
(iii) $(e_s \cdot u)^\wedge = \tau_s \hat{u}$;
(iv) $(\tau_s u)^\wedge = e_{-s}\hat{u}$.

LEMMA 2.3. *The conclusions of Lemma* 2.2 *also hold for $u \in \mathcal{Z}'$ and $s \in \mathbb{R}$.*

The proof is very similar to that of the previous lemma and we omit it.

**3. Boehmian spaces.** For $f \in C^\infty$ and $\phi \in \mathcal{D}$ we define the convolution by

$$(f * \phi)(x) = \int_{-\infty}^{\infty} f(x-t)\phi(t)\,dm(t).$$

We call a sequence $(\delta_n)$ in $\mathcal{D}$ a *delta sequence* if

(i) $\int_{-\infty}^{\infty} \delta_n(t)\,dm(t) = 1$ for all $n \in \mathbb{N}$;
(ii) $\int_{-\infty}^{\infty} |\delta_n(t)|\,dm(t) \le M$ for all $n \in \mathbb{N}$ for some $M > 0$;
(iii) $s(\delta_n) = \sup\{|t| : t \in \mathbb{R},\ \delta_n(t) \neq 0\} \to 0$ as $n \to \infty$.

The collection of all delta sequences is denoted by $\Delta$.

We denote by $\mathcal{A}$ the collection of all pairs $((f_n), (\delta_n))$ of sequences satisfying

$$f_n * \delta_m = f_m * \delta_n, \qquad \forall m, n \in \mathbb{N}.$$

The space $\mathcal{B}_\infty$ of $C^\infty$-*Boehmians* is defined to be the collection of all equivalence classes $\left[\dfrac{f_n}{\delta_n}\right]$ given by the equivalence relation $\sim$ defined on $\mathcal{A}$ by

$$((f_n), (\delta_n)) \sim ((g_n), (\epsilon_n)) \quad \text{if} \quad f_n * \epsilon_m = g_m * \delta_n,\ \forall m, n \in \mathbb{N}.$$

The space of Schwartz distributions can be identified with a proper subspace of $\mathcal{B}_\infty$ by the identification $u \mapsto \left[\dfrac{u * \delta_n}{\delta_n}\right]$, where $(\delta_n) \in \Delta$ is arbitrary.

In addition to the operations of addition and scalar multiplication the following operations have been defined on $\mathcal{B}_\infty$. See [3, 4, 5, 7].

Definition 3.1. *Let* $X = \left[\frac{f_n}{\delta_n}\right] \in \mathcal{B}_\infty$, $u \in \mathcal{D}'$ *with compact support*, $s \in \mathbb{R}$ *and* $\alpha \in \mathbb{C}$. *Then define*

(i) $X' = \left[\frac{f_n'}{\delta_n}\right]$.

(ii) $MX = \left[\frac{(Mf_n) * \delta_n - f_n * (M\delta_n)}{\delta_n * \delta_n}\right]$, *where* $(Mf)(t) = t \cdot f(t)$ *for* $t \in \mathbb{R}$.

(iii) $\tau_s X = \left[\frac{\tau_s f_n}{\delta_n}\right]$, *where* $\tau_s f_n(t) = f_n(t-s)$ *for* $t \in \mathbb{R}$.

(iv) $e_\alpha \cdot X = \left[\frac{\lambda_n e_\alpha f_n}{\lambda_n e_\alpha \delta_n}\right]$, *where* $e_\alpha(t) = e^{\alpha t}$ *for* $t \in \mathbb{R}$ *and* $\lambda_n = (\int_{-\infty}^{\infty} e_\alpha(t)\delta_n(t)\,dt)^{-1}$.

(v) $u * X = \left[\frac{u * f_n}{\delta_n}\right]$, *where* $(u * f_n)(x) = \langle u(t), f_n(x-t)\rangle$ *for* $x \in \mathbb{R}$.

The space $\mathcal{B}_F$ of *ultra Boehmians* is the quadruple $(\mathcal{Z}', \mathcal{Z}, \bullet, \hat{\Delta})$, where the operation $\bullet : \mathcal{Z}' \times \mathcal{Z} \to \mathcal{Z}'$ is defined by

$$(u \bullet f)(g) = u(f \cdot g), \qquad g \in \mathcal{Z},$$

and $\hat{\Delta} = \{(\hat{\delta}_n) : (\delta_n) \in \Delta\}$, where $\Delta$ is defined as above.

We say that $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathcal{B}_F$ if there exist $u_{n,k}, u_k \in \mathcal{Z}'$ and $(\hat{\delta}_k) \in \hat{\Delta}$ such that $X_n = \left[\frac{u_{n,k}}{\hat{\delta}_k}\right]$, $X = \left[\frac{u_k}{\hat{\delta}_k}\right]$ and for each $k \in \mathbb{N}$,

$$u_{n,k} \to u_k \quad \text{as } n \to \infty \text{ in } \mathcal{Z}'.$$

Definition 3.2. The *Fourier transform* of an ultra Boehmian is defined by

$$\mathcal{F}\left(\left[\frac{u_n}{\hat{\delta}_n}\right]\right) = \left[\frac{\hat{u}_n * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right] \in \mathcal{B}_\infty.$$

For more details we refer the reader to [2].

**4. Auxiliary results.** Now we state and prove a few lemmas which will be useful in what follows. The first two were proved in [2].

Lemma 4.1. *Let* $m, n \in \mathbb{N}$ *and* $k \in \mathbb{N}_0$. *If* $f \in \mathcal{Z}(m)$ *and* $g \in \mathcal{Z}(n)$ *then*

$$\gamma_{m+n,k}(f \cdot g) \le \gamma_{m,k}(f) \cdot \gamma_{n,0}(g).$$

Lemma 4.2. *If* $u \in \mathcal{Z}'$, $v \in \mathcal{D}'$, $\phi \in \mathcal{D}$ *then*

(i) $(u \bullet \hat{\phi})^\wedge = \hat{u} * \check{\phi}$ *in* $\mathcal{D}'$;

(ii) $(v * \phi)^\wedge = \hat{v} \bullet \hat{\phi}$ *in* $\mathcal{Z}'$.

LEMMA 4.3. *Let $u \in \mathcal{D}'$, $\phi \in \mathcal{D}$ and $s \in \mathbb{R}$. Then*

(i) $(u * \phi)' = u' * \phi = u * \phi'$;
(ii) $\tau_s(u * \phi) = \tau_s u * \phi$;
(iii) $M(u * \phi) = (Mu) * \phi + u * (M\phi)$;
(iv) $e_s \cdot (u * \phi) = (e_s \cdot u) * (e_s \cdot \phi)$.

LEMMA 4.4. *Let $u \in \mathcal{Z}'$, $f \in \mathcal{Z}$ and $s \in \mathbb{R}$. Then*

(i) $(u \bullet f)' = u' \bullet f + u \bullet f'$;
(ii) $\tau_s(u \bullet f) = \tau_s u \bullet \tau_s f$;
(iii) $M(u \bullet f) = (Mu) \bullet f$;
(iv) $e_s \cdot (u \bullet f) = (e_s \cdot u) \bullet f$.

LEMMA 4.5. *Let $u_n \to u$ as $n \to \infty$ in $\mathcal{Z}'$ and $f_n \to f$ as $n \to \infty$ in $\mathcal{Z}$. Then*

(i) $u_n \bullet f_n \to u \bullet f$ *as* $n \to \infty$ *in* $\mathcal{Z}'$;
(ii) $u_n' \to u'$ *as* $n \to \infty$ *in* $\mathcal{Z}'$.

*Proof.* (i) Let $B$ be a bounded subset of $\mathcal{Z}$. For $g \in B$,

$$\begin{aligned}|(u_n \bullet f_n - u \bullet f)(g)| &\le |(u_n \bullet f_n - u \bullet f_n)(g)| + |(u \bullet f_n - u \bullet f)(g)| \\ &= |(u_n - u)(f_n \cdot g)| + |u((f_n - f) \cdot g)|.\end{aligned}$$

Since $(f_n)$ is a convergent sequence, $\{f_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathcal{Z}$. Hence using Lemma 4.1, we can prove that $\{f_n \cdot g : n \in \mathbb{N},\ g \in B\}$ is also a bounded subset of $\mathcal{Z}$. Since $u_n \to u$ as $n \to \infty$ uniformly on bounded subsets of $\mathcal{Z}$ we deduce that $\sup_{g \in B} |(u_n - u)(f_n \cdot g)| \to 0$ as $n \to \infty$.

Next we choose $m, p \in \mathbb{N}$ such that $\{f_n - f : n \in \mathbb{N}\} \subset \mathcal{Z}(m)$ and $B \subset \mathcal{Z}(p)$. Now for each $k \in \mathbb{N}_0$ there exists $C_{p,k} > 0$ such that $\gamma_{p,k}(g) \le C_{p,k}$ for all $g \in B$. By using the continuity of $u$ and Lemma 4.1, we have

$$\begin{aligned}|u((f_n - f) \cdot g)| &\le M_k \gamma_{m+p,k}((f_n - f) \cdot g) \quad \text{for some } k \in \mathbb{N},\ M_k > 0 \\ &\le M_k \gamma_{m,0}(f_n - f) \cdot \gamma_{p,k}(g) \\ &\le M_k C_{p,k} \gamma_{m,0}(f_n - f) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

(ii) Let $B$ be a bounded subset of $\mathcal{Z}$. Then it is easy to verify that $\{f' : f \in B\}$ is also a bounded subset of $\mathcal{Z}$. Since $u_n \to u$ as $n \to \infty$ in $\mathcal{Z}'$,

$$\sup_{f \in B} |(u_n' - u')(f)| = \sup_{f \in B} |(u_n - u)(f')| \to 0 \quad \text{as } n \to \infty,$$

hence (ii) follows.

Let $A$ denote the collection of entire functions $\psi$ called *multipliers for* $\mathcal{Z}$ [12, p. 198] satisfying $|\psi(z)| \le Ce^{m|\operatorname{Im} z|}(1 + |z|^n)$ for all $z \in \mathbb{C}$, for some positive real $C$ and $m, n \in \mathbb{N}$. It is easy to verify that $\gamma_{m+l,k}(\psi \cdot f) \le 2C\gamma_{l,k}(f)$ for all $f \in \mathcal{Z}(l)$ and hence we can define $\psi \cdot u \in \mathcal{Z}'$ for $u \in \mathcal{Z}'$ and

$\psi \in A$ by

$$(\psi \cdot u)(f) = u(\psi \cdot f), \qquad \forall f \in \mathcal{Z}.$$

It is clear from the Paley–Wiener theorem [8, Theorem 7.23] that the Fourier transform from the space of compactly supported distributions onto $A$ is a bijection. Hence we can say that the Fourier transform of $\psi \in A$ is a compactly supported distribution $\check{v}$, where $\hat{v} = \psi$. (Recall that $\langle \check{v}(t), f(t) \rangle = \langle v(t), f(-t) \rangle$ for all $f \in C^\infty$.)

Lemma 4.6. *If $\psi \in A$ and $u \in \mathcal{Z}'$ then $(\psi \cdot u)^\wedge = \hat{\psi} * \hat{u}$.*

*Proof.* First we note that $\hat{\psi} * \hat{u}$ is meaningful since $\hat{\psi}$ is compactly supported. Now for $\phi \in \mathcal{D}$,

$$\begin{aligned}(\hat{\psi} * \hat{u})(\phi) &= ((\hat{\psi} * \hat{u}) * \check{\phi})(0) = (\hat{u} * (\hat{\psi} * \check{\phi}))(0) = (\hat{u} * (\hat{\psi} * \check{\phi})(0) = \hat{u}(\check{\hat{\psi}} * \phi) \\ &= u(\psi \cdot \hat{\phi}) = (\psi \cdot u)(\hat{\phi}) = (\psi \cdot u)^\wedge(\phi).\end{aligned}$$

## 5. Operational calculus

Definition 5.1. For $X = \left[\dfrac{u_n}{\hat{\delta}_n}\right] \in \mathcal{B}_F$ and $\psi \in A$, define

$$\psi \cdot X = \left[\frac{\psi \cdot u_n}{\hat{\delta}_n}\right] \in \mathcal{B}_F.$$

It can be easily verified that the product is well defined.

Theorem 5.2. *If $\psi \in A$ and $X \in \mathcal{B}_F$ then $\mathcal{F}(\psi \cdot X) = \hat{\psi} * \mathcal{F}(X)$.*

*Proof.* Let $X = \left[\dfrac{u_n}{\hat{\delta}_n}\right]$. Using Lemma 4.6 we get

$$\begin{aligned}\mathcal{F}(\psi \cdot X) &= \mathcal{F}\left(\left[\frac{\psi \cdot u_n}{\hat{\delta}_n}\right]\right) = \left[\frac{(\psi \cdot u_n)^\wedge * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right] = \left[\frac{\hat{\psi} * \hat{u}_n * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right] \\ &= \hat{\psi} * \left[\frac{\hat{u}_n * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right] = \hat{\psi} * \mathcal{F}(X).\end{aligned}$$

Definition 5.3 (Differentiation). Given $X = \left[\dfrac{u_n}{\phi_n}\right] \in \mathcal{B}_F$ we define the derivative of $X$ by

$$X' = \left[\frac{u_n' \bullet \phi_n - u_n \bullet \phi_n'}{\phi_n \bullet \phi_n}\right].$$

Lemma 5.4. *The derivative of a Boehmian is well defined.*

*Proof.* It is well known that if $(\phi_n) \in \hat{\Delta}$ then $(\phi_n \bullet \phi_n) \in \hat{\Delta}$. First we show that $\dfrac{u_n' \bullet \phi_n - u_n \bullet \phi_n'}{\phi_n \bullet \phi_n}$ is a quotient. Using Lemma 4.4(i) we get

$$\begin{aligned}
&(u_n' \bullet \phi_n - u_n \bullet \phi_n') \bullet \phi_m \bullet \phi_m \\
&\qquad = u_n' \bullet \phi_n \bullet \phi_m \bullet \phi_m - u_n \bullet \phi_n' \bullet \phi_m \bullet \phi_m \\
&\qquad = u_n' \bullet \phi_m \bullet \phi_n \bullet \phi_m - u_n \bullet \phi_m \bullet \phi_n' \bullet \phi_m \\
&\qquad = (u_n \bullet \phi_m)' \bullet \phi_n \bullet \phi_m - u_n \bullet \phi_m' \bullet \phi_n \bullet \phi_m - u_n \bullet \phi_m \bullet \phi_n' \bullet \phi_m \\
&\qquad = (u_m \bullet \phi_n)' \bullet \phi_n \bullet \phi_m - u_m \bullet \phi_n \bullet \phi_m' \bullet \phi_n - u_m \bullet \phi_n \bullet \phi_n' \bullet \phi_m \\
&\qquad = (u_m \bullet \phi_n)' \bullet \phi_n \bullet \phi_m - u_m \bullet \phi_n' \bullet \phi_n \bullet \phi_m - u_m \bullet \phi_m' \bullet \phi_n \bullet \phi_n \\
&\qquad = u_m' \bullet \phi_n \bullet \phi_n \bullet \phi_m - u_m \bullet \phi_m' \bullet \phi_n \bullet \phi_n \\
&\qquad = (u_m' \bullet \phi_m - u_m \bullet \phi_m') \bullet \phi_n \bullet \phi_n.
\end{aligned}$$

If $\left[\dfrac{u_n}{\phi_n}\right] = \left[\dfrac{v_n}{\psi_n}\right]$ in $\mathcal{B}_F$ then $u_n \bullet \psi_m = v_m \bullet \phi_n$ for all $m, n \in \mathbb{N}$. Again using Lemma 4.4(i) we get

$$\begin{aligned}
&(u_n' \bullet \phi_n - u_n \bullet \phi_n') \bullet \psi_m \bullet \psi_m \\
&\qquad = u_n' \bullet \phi_n \bullet \psi_m \bullet \psi_m - u_n \bullet \phi_n' \bullet \psi_m \bullet \psi_m \\
&\qquad = u_n' \bullet \psi_m \bullet \phi_n \bullet \psi_m - u_n \bullet \psi_m \bullet \phi_n' \bullet \psi_m \\
&\qquad = (u_n \bullet \psi_m)' \bullet \phi_n \bullet \psi_m - u_n \bullet \psi_m' \bullet \phi_n \bullet \psi_m - u_n \bullet \psi_m \bullet \phi_n' \bullet \psi_m \\
&\qquad = (v_m \bullet \phi_n)' \bullet \phi_n \bullet \psi_m - v_m \bullet \phi_n \bullet \psi_m' \bullet \phi_n - v_m \bullet \phi_n \bullet \phi_n' \bullet \psi_m \\
&\qquad = (v_m \bullet \phi_n)' \bullet \phi_n \bullet \psi_m - v_m \bullet \phi_n' \bullet \phi_n \bullet \psi_m - v_m \bullet \psi_m' \bullet \phi_n \bullet \phi_n \\
&\qquad = v_m' \bullet \phi_n \bullet \phi_n \bullet \psi_m - v_m \bullet \psi_m' \bullet \phi_n \bullet \phi_n \\
&\qquad = (v_m' \bullet \psi_m - v_m \bullet \psi_m') \bullet \phi_n \bullet \phi_n.
\end{aligned}$$

This shows that the derivative of a Boehmian is well defined.

Inductively we can define all higher order derivatives of $X$.

Lemma 5.5 (Consistency). *Let $u \in \mathcal{Z}'$. If $X = \left[\dfrac{u \bullet \phi_n}{\phi_n}\right]$ for any $(\phi_n) \in \hat{\Delta}$ then*

$$X' = \left[\frac{u' \bullet \phi_n}{\phi_n}\right].$$

Since

$$\begin{aligned}
(u \bullet \phi_n)' \bullet \phi_n - u \bullet \phi_n \bullet \phi_n' &= (u \bullet \phi_n)' \bullet \phi_n - u \bullet \phi_n' \bullet \phi_n \\
&= u' \bullet \phi_n \bullet \phi_n, \qquad \forall n \in \mathbb{N},
\end{aligned}$$

we get

$$\left[\frac{u' \bullet \phi_n}{\phi_n}\right] = \left[\frac{u' \bullet \phi_n \bullet \phi_n}{\phi_n \bullet \phi_n}\right] = \left[\frac{(u \bullet \phi_n)' \bullet \phi_n - u \bullet \phi_n \bullet \phi_n'}{\phi_n \bullet \phi_n}\right].$$

Theorem 5.6. *If $X \in \mathcal{B}_F$ and $f \in \mathcal{Z}$ then $(X \bullet f)' = X \bullet f' + X' \bullet f$.*

*Proof.* Let $X = \left[\dfrac{u_n}{\phi_n}\right]$. Since

$$\begin{aligned}(u_n \bullet f)' \bullet \phi_n - u_n \bullet f \bullet \phi_n' &= u_n \bullet f' \bullet \phi_n + u_n' \bullet f \bullet \phi_n - u_n \bullet f \bullet \phi_n' \\ &= (u_n' \bullet \phi_n - u_n \bullet \phi_n') \bullet f + u_n \bullet \phi_n \bullet f',\end{aligned}$$

we get

$$\left[\frac{(u_n \bullet f)' \bullet \phi_n - u_n \bullet f \bullet \phi_n'}{\phi_n \bullet \phi_n}\right] = \left[\frac{(u_n' \bullet \phi_n - u_n \bullet \phi_n') \bullet f}{\phi_n \bullet \phi_n}\right] + \left[\frac{u_n \bullet \phi_n \bullet f'}{\phi_n \bullet \phi_n}\right].$$

Hence $(X \bullet f)' = X' \bullet f + X \bullet f'$.

THEOREM 5.7. *If $X \in \mathcal{B}_F$ then $\mathcal{F}(X') = iM\mathcal{F}(X)$.*

*Proof.* Let $X = \left[\dfrac{u_n}{\phi_n}\right]$, where $(\phi_n) = (\hat{\delta}_n)$ and $(\delta_n) \in \Delta$. We note that

$$\begin{aligned}\mathcal{F}(X') &= \mathcal{F}\left(\left[\frac{u_n' \bullet \phi_n - u_n \bullet \phi_n'}{\phi_n \bullet \phi_n}\right]\right) \\ &= \left[\frac{(u_n' \bullet \phi_n - u_n \bullet \phi_n')^\wedge * \check{\delta}_n * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n * \check{\delta}_n * \check{\delta}_n}\right] \\ &= \left[\frac{(u_n')^\wedge * \check{\delta}_n * \check{\delta}_n * \check{\delta}_n - \hat{u}_n * (\phi_n')^\wedge * \check{\delta}_n * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n * \check{\delta}_n * \check{\delta}_n}\right] \\ &= \left[\frac{(iM\hat{u}_n) * \check{\delta}_n * \check{\delta}_n * \check{\delta}_n - \hat{u}_n * (iM\check{\delta}_n) * \check{\delta}_n * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n * \check{\delta}_n * \check{\delta}_n}\right]\end{aligned}$$

and

$$\begin{aligned}iM\mathcal{F}(X) &= iM\left[\frac{\hat{u}_n * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right] \\ &= i\left[\frac{(M(\hat{u}_n * \check{\delta}_n)) * \check{\delta}_n * \check{\delta}_n - (\hat{u}_n * \check{\delta}_n) * (M(\check{\delta}_n * \check{\delta}_n))}{\check{\delta}_n * \check{\delta}_n * \check{\delta}_n * \check{\delta}_n}\right].\end{aligned}$$

By using Lemma 4.3(iii), we get

$$\begin{aligned}&i[(M(\hat{u}_n * \check{\delta}_n)) * \check{\delta}_n * \check{\delta}_n - (\hat{u}_n * \check{\delta}_n) * (M(\check{\delta}_n * \check{\delta}_n))] \\ &\qquad = i((M\hat{u}_n) * \check{\delta}_n + \hat{u}_n * (M\check{\delta}_n)) * \check{\delta}_n * \check{\delta}_n \\ &\qquad\qquad - \hat{u}_n * \check{\delta}_n * ((M\check{\delta}_n) * \check{\delta}_n + \check{\delta}_n * (M\check{\delta}_n))] \\ &\qquad = (iM\hat{u}_n) * \check{\delta}_n * \check{\delta}_n * \check{\delta}_n - \hat{u}_n * \check{\delta}_n * (iM\check{\delta}_n) * \check{\delta}_n.\end{aligned}$$

Hence the theorem follows.

DEFINITION 5.8. If $X = \left[\dfrac{u_n}{\hat{\delta}_n}\right] \in \mathcal{B}_F$ then define

$$MX = \left[\frac{Mu_n}{\hat{\delta}_n}\right] \in \mathcal{B}_F.$$

It is easy to verify (by using Lemma 4.4(iii)) that $MX$ is well defined and consistent with operation (iii) of Definition 2.1 on ultra distributions.

THEOREM 5.9. *If* $X \in \mathcal{B}_F$, *then* $\mathcal{F}(-iMX) = (\mathcal{F}(X))'$.

*Proof.* Let $X = \left[\frac{u_n}{\hat{\delta}_n}\right] \in \mathcal{B}_F$. Using Lemma 4.3(i), we get

$$\mathcal{F}(-iMX) = \mathcal{F}\left(\left[\frac{-iMu_n}{\hat{\delta}_n}\right]\right) = \left[\frac{(-iMu_n)^\wedge * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right] = \left[\frac{(\hat{u}_n)' * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right]$$
$$= \left[\frac{(\hat{u}_n * \check{\delta}_n)'}{\check{\delta}_n * \check{\delta}_n}\right] = \left[\frac{\hat{u}_n * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right]' = (\mathcal{F}(X))'.$$

DEFINITION 5.10. For $s \in \mathbb{R}$, $\left[\frac{u_n}{\hat{\delta}_n}\right] \in \mathcal{B}_F$, we define

$$e_s \cdot \left[\frac{u_n}{\hat{\delta}_n}\right] = \left[\frac{e_s \cdot u_n}{\hat{\delta}_n}\right].$$

By an application of Lemma 4.4(iv) we see that this product is well defined and consistent with multiplication of $e_s$ with an ultra distribution $u \in \mathcal{Z}'$.

THEOREM 5.11. *Let* $X \in \mathcal{B}_F$ *and* $s \in \mathbb{R}$. *Then* $\mathcal{F}(e_s \cdot X) = \tau_s \mathcal{F}(X)$, *where* $\tau_s$ *is the translation operator defined on* $C^\infty$*-Boehmians.*

*Proof.* Let $X = \left[\frac{u_n}{\hat{\delta}_n}\right]$. Then by Lemma 4.3(ii) we get

$$\mathcal{F}(e_s \cdot X) = \left[\frac{(e_s \cdot u_n)^\wedge * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right] = \left[\frac{(\tau_s \hat{u}_n) * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right] = \left[\frac{\tau_s(\hat{u}_n * \check{\delta}_n)}{\check{\delta}_n * \check{\delta}_n}\right]$$
$$= \tau_s \left[\frac{\hat{u}_n * \check{\delta}_n}{\check{\delta}_n * \check{\delta}_n}\right] = \tau_s \mathcal{F}(X).$$

DEFINITION 5.12. Let $s \in \mathbb{R}$ and and $X = \left[\frac{u_n}{\hat{\delta}_n}\right]$. We define the translation $\tau_s X$ of $X$ by

$$\tau_s X = \left[\frac{\lambda_n \tau_s u_n}{\lambda_n \tau_s \hat{\delta}_n}\right]$$

where $\lambda_n = (\hat{\delta}_n(-s))^{-1}$ for $n \in \mathbb{N}$.

We show that the translation is well defined. Since $(\lambda_n e_s \cdot \delta_n) \in \Delta$ (see [7]) and $\lambda_n \tau_s \hat{\delta}_n = (\lambda_n e_s \cdot \delta_n)^\wedge$, it follows that $(\lambda_n \tau_s \hat{\delta}_n) \in \hat{\Delta}$. Since

$$\begin{aligned}(\lambda_n \tau_s u_n) \bullet (\lambda_m \tau_s \hat{\delta}_m) &= \lambda_n \lambda_m (\tau_s u_n \bullet \tau_s \hat{\delta}_m) \\ &= \lambda_n \lambda_m (\tau_s (u_n \bullet \hat{\delta}_m)) \quad \text{(by Lemma 4.4(ii))} \\ &= \lambda_m \lambda_n (\tau_s (u_m \bullet \hat{\delta}_n)) \\ &= \lambda_m \lambda_n (\tau_s u_m \bullet \tau_s \hat{\delta}_n) = (\lambda_m \tau_s u_m) \bullet (\lambda_n \tau_s \hat{\delta}_n),\end{aligned}$$

we see that $\dfrac{\lambda_n\tau_s u_n}{\lambda_n\tau_s\hat{\delta}_n}$ is a quotient and hence $\left[\dfrac{\lambda_n\tau_s u_n}{\lambda_n\tau_s\hat{\delta}_n}\right] \in \mathcal{B}_F$. By a similar argument we can show that the definition is independent of the choice of the representative.

LEMMA 5.13. *The translation operator defined on $\mathcal{B}_F$ is consistent with that on $\mathcal{Z}'$.*

*Proof.* Let $u \in \mathcal{Z}'$ and $s \in \mathbb{R}$. Using the equalities

$$\begin{aligned}\lambda_n\tau_s(u \bullet \hat{\delta}_n) \bullet \hat{\delta}_m &= \lambda_n\tau_s u \bullet \tau_s\hat{\delta}_n \bullet \hat{\delta}_m \quad \text{(by Lemma 4.4(ii)}\\ &= \tau_s u \bullet \hat{\delta}_m \bullet \lambda_n\tau_s\hat{\delta}_n, \quad m, n \in \mathbb{N},\end{aligned}$$

we get

$$\left[\frac{\lambda_n\tau_s(u \bullet \hat{\delta}_n)}{\lambda_n\tau_s\hat{\delta}_n}\right] = \left[\frac{\tau_s u \bullet \hat{\delta}_n}{\hat{\delta}_n}\right].$$

THEOREM 5.14. *If $s \in \mathbb{R}$ and $X = \left[\dfrac{u_n}{\hat{\delta}_n}\right] \in \mathcal{B}_F$ then*

$$\mathcal{F}(\tau_s X) = e_{-s} \cdot \mathcal{F}(X).$$

*Proof.* We have

$$\begin{aligned}\mathcal{F}\left(\tau_s\left[\frac{u_n}{\hat{\delta}_n}\right]\right) &= \mathcal{F}\left(\left[\frac{\lambda_n\tau_s u_n}{\lambda_n\tau_s\hat{\delta}_n}\right]\right) \quad \text{where } \lambda_n = (\hat{\delta}_n(-s))^{-1}\\ &= \left[\frac{\lambda_n(\tau_s u_n)^{\wedge} * (\lambda_n e_{-s} \cdot \delta_n)^{\vee}}{(\lambda_n e_{-s} \cdot \delta_n)^{\vee} * (\lambda_n e_{-s} \cdot \delta_n)^{\vee}}\right]\\ &= \left[\frac{(\lambda_n e_{-s} \cdot \hat{u}_n) * (\lambda_n e_{-s} \cdot \check{\delta}_n)}{(\lambda_n e_{-s} \cdot \check{\delta}_n) * (\lambda_n e_{-s} \cdot \check{\delta}_n)}\right]\\ &= \left[\frac{\lambda_n^2 e_{-s} \cdot (\hat{u}_n * \check{\delta}_n)}{\lambda_n^2 e_{-s} \cdot (\delta_n * \delta_n)^{\vee}}\right] \quad \text{(by Lemma 4.3(iv))}\\ &= e_{-s} \cdot \mathcal{F}(X)\end{aligned}$$

since $\lambda_n^2 = (\int_{-\infty}^{\infty}(\delta_n * \delta_n)^{\vee} e^{-ist}\, dt)^{-1}$.

THEOREM 5.15. *Let $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathcal{B}_F$ and $f_n \to f$ as $n \to \infty$ in $\mathcal{Z}$. Then*

(i) $X_n \bullet f_n \xrightarrow{\delta} X \bullet f$ *as* $n \to \infty$;

(ii) $X_n' \xrightarrow{\delta} X'$ *as* $n \to \infty$.

*Proof.* Choose $u_{n,k}, u_k \in \mathcal{Z}'$ and $(\phi_k) \in \hat{\Delta}$ such that $X_n = \left[\dfrac{u_{n,k}}{\phi_k}\right]$, $X = \left[\dfrac{u_k}{\phi_k}\right]$ and for each $k \in \mathbb{N}$, $u_{n,k} \to u_k$ as $n \to \infty$ in $\mathcal{Z}'$. Now by using Lemma 4.5, our theorem follows.

**6. A comparative study.** In [6], it is proved that one cannot define a product between the collection of all real analytic functions and the space of Boehmians as a continuous operator on the space of Boehmians for each fixed real analytic function. With our choice of the Boehmian space, we can multiply its elements by some entire functions, polynomials in one real variable and also by characters in a simple and natural way by taking the *top space* as a dual space and defining the product as the usual multiplication. The product of a continuous function $\phi$ and a Boehmian $X = \left[\frac{f_n}{\delta_n}\right] \in \mathcal{B}_\infty$ is defined in [7] by $\phi \cdot X = \delta\text{-}\lim_{n\to\infty} \phi \cdot f_n$ whenever the limit exists for each representative $\frac{f_n}{\delta_n}$ of $X$. Though this definition is consistent with the classical product, and all polynomials and functions like $e^{\alpha t}$ $(\alpha \in \mathbb{C},\ t \in \mathbb{R})$ can be multiplied with Boehmians in $\mathcal{B}_\infty$, it is difficult to answer the question *whether the product $\phi \cdot X$ exists for any given continuous function $\phi$ other than polynomials and characters and a Boehmian $X \in \mathcal{B}_\infty$.*

In our theory there is no such complexity. We have also proved that whenever $X_n \to X$ *as* $n \to \infty$ *in* $\mathcal{B}_F$ *and* $f_n \to f$ *as* $n \to \infty$ *in* $\mathcal{Z}$, $X_n \bullet f_n \to X \bullet f$ *as* $n \to \infty$, and these definitions together with our Fourier transform on $\mathcal{B}_F$ have all the classical properties of the Fourier transform. This shows that the space of ultra Boehmians and the usual product of elements of $\mathcal{Z}'$ and entire functions in $\mathcal{Z}$ is the right choice for developing an operational calculus for the theory of Fourier transform in the context of Boehmians.

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