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EXPLICIT CONSTRUCTION OF NORMAL LATTICE CONFIGURATIONS

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Abstract. We extend Champernowne's construction of normal numbers to base b to the $\mathbb{Z}^d$ case and obtain an explicit construction of a generic point of the $\mathbb{Z}^d$ shift transformation of the set $\{0,1,\ldots,b-1\}^{\mathbb{Z}^d}$.

1. Introduction. A number $\alpha \in (0,1)$ is said to be *normal* to base b if in a b-ary expansion of α, $\alpha = .d_1 d_2 \ldots$ ($d_i \in \{0,1,\ldots,b-1\}$, $i = 1,2,\ldots$), each fixed finite block of digits of length k appears with an asymptotic frequency of b^{-k} along the sequence $(d_i)_{i\geq 1}$. Normal numbers were introduced by Borel (1909). Champernowne (1933) gave an explicit construction of such a number, namely,

$$\theta = .1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,\ldots,$$

obtained by successively concatenating all the natural numbers.

We shall call the sequence of digits obtained from a normal number a normal sequence.

Champernowne's construction is associated with the i.i.d. process of variables having uniform distribution over b states. In [AKS], [Po], and [SW], constructions of normal sequences for various stationary stochastic processes, similar to Champernowne's, were introduced.

Our goal is to extend such constructions to $\mathbb{Z}^d$-arrays ($d > 1$) of random variables, which we shall call $\mathbb{Z}^d$-processes. We shall deal with stationary $\mathbb{Z}^d$-processes, that is, processes with distribution invariant under the $\mathbb{Z}^d$-action. We shall call a specific realization of a $\mathbb{Z}^d$-process a configuration (lattice configuration). To begin with, the very definition of a normal configuration is subject to various generalizations from the 1-dimensional case.

We begin with a very simple generalization (see also [Ci], [KT], and [LeSm1]).

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1.1. *Rectangular normality.* We denote by $\mathbb{N}$ the set of non-negative integers. Let $d, b \geq 2$ be two integers, $\mathbb{N}^d = \{(n_1, \dots, n_d) \mid n_i \in \mathbb{N}, i = 1, \dots, d\}$, $\Delta_b = \{0, 1, \dots, b-1\}$, and $\Omega = \Delta_b^{\mathbb{N}^d}$.

We shall call $\omega \in \Omega$ a *configuration* (*lattice configuration*). A configuration is a function $\omega : \mathbb{N}^d \to \Delta_b$.

Given a subset F of $\mathbb{N}^d$, ω_F will be the restriction of the function ω to F. Let $\mathbf{N} \in \mathbb{N}^d$, $\mathbf{N} = (N_1, \dots, N_d)$. We denote a *rectangular block* by

$$F_{\mathbf{N}} = \{(f_1, \dots, f_d) \in \mathbb{N}^d \mid 0 \leq f_i < N_i,\ i = 1, \dots, d\},$$

$\mathbf{h} = [0, h_1) \times \dots \times [0, h_d)$, $h_i \geq 1$, $i = 1, \dots, d$; $G = G_{\mathbf{h}}$ is a fixed block of digits $G = (g_{\mathbf{i}})_{\mathbf{i} \in F_{\mathbf{h}}}$, $g_{\mathbf{i}} \in \Delta_b$; $\chi_{\omega,G}(\mathbf{f})$ is the characteristic function of the block G shifted by the vector $\mathbf{f}$ in the configuration ω:

$$\chi_{\omega,G}(\mathbf{f}) = \begin{cases} 1 & \text{if } \omega(\mathbf{f} + \mathbf{i}) = g_{\mathbf{i}},\ \forall \mathbf{i} \in F_{\mathbf{h}}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Definition 1. $\omega \in \Omega$ is said to be *rectangular normal* if for any $\mathbf{h} \subset \mathbb{N}^d$ and block $G_{\mathbf{h}}$,

$$\#\{\mathbf{f} \in F_{\mathbf{N}} \mid \chi_{\omega,G_{\mathbf{h}}}(\mathbf{f}) = 1\} - b^{-h_1 \cdots h_d} N_1 \cdots N_d = o(N_1 \cdots N_d) \tag{2}$$

as $\max(N_1, \dots, N_d) \to \infty$.

We shall say that ω is *square normal* if we consider only square blocks, i.e., $N_1 = \dots = N_d$. For clarity, we shall carry out the proof only for the case $d = 2$. The generalization to general $d > 2$ is easy and straightforward.

Construction. The formula

$$L(f_1, f_2) = \begin{cases} f_1^2 + f_2 & \text{if } f_2 < f_1, \\ f_2^2 + 2f_2 - f_1 & \text{if } f_2 \geq f_1, \end{cases} \tag{3}$$

defines a bijection between $\mathbb{N}^2$ and $\mathbb{N}$, inducing a total order on $\mathbb{N}^2$ from the usual one on $\mathbb{N}$. We define the configuration ω_n on $F_{(2nb^{2n^2}, 2nb^{2n^2})}$ as the concatenation of b^{4n^2} $2n \times 2n$ blocks of digits with the lower left corner $(2nx, 2ny)$, $0 \leq x, y < b^{2n^2}$. To each of these blocks we assign the number $L(x, y)$. Next we use the b-expansion of the number $L(x, y)$ according to the order L to obtain the digits of the relevant $2n \times 2n$ block. It is easy to obtain the analytic expression for the digits of the configuration ω_n:

$$\omega_n(2nx + s, 2ny + t) = \begin{cases} a_{s^2+t}(u) & \text{if } t < s, \\ a_{t^2+2t-s}(u) & \text{if } t \geq s, \end{cases} \tag{4}$$

where

$$u = u(x, y) = \begin{cases} x^2 + y & \text{if } y < x, \\ y^2 + 2y - x & \text{if } y \geq x, \end{cases} \tag{5}$$

s, t, x, y are integers, $0 \le x, y < b^{2n^2}, 0 \le s, t < 2n$, and

$$n = \sum_{i \ge 0} a_i(n) b^i \qquad (a_i(n) \in \{0, 1, \ldots, b-1\}) \tag{6}$$

is the b-expansion of the integer n.

Next we define inductively a sequence of increasing configurations ω'_n on $F_{(2nb^{2n^2}, 2nb^{2n^2})}$. Put $\omega'_1 = \omega_1$, $\omega'_{n+1}(\mathbf{f}) = \omega'_n(\mathbf{f})$ for $\mathbf{f} \in F_{(2nb^{2n^2}, 2nb^{2n^2})}$ and $\omega'_{n+1}(\mathbf{f}) = \omega_{n+1}(\mathbf{f})$ otherwise. Put

$$\omega_\infty = \lim \omega'_n, \qquad (\omega_\infty)_{F_{(2nb^{2n^2}, 2nb^{2n^2})}} = \omega'_n, \quad n = 1, 2, \ldots. \tag{7}$$

Theorem. *ω_∞ is rectangular normal, and for all $\mathbf{h} = (h_1, h_2)$, $\mathbf{N} = (N_1, N_2)$ and all blocks of digits $G_{\mathbf{h}}$ we have*

$$\#\{\mathbf{f} \in F_{\mathbf{N}} \mid \chi_{\omega, G_{\mathbf{h}}}(\mathbf{f}) = 1\} = b^{-h_1 h_2} N_1 N_2 + O(N_1 N_2 / \sqrt{\log N_1 N_2}). \tag{8}$$

Remark. A more general (and more complicated) construction is given in [LeSm1] but without an estimate of the error term as in (8). The proof of the Theorem is similar to that of [LeSm1]. The essential difference is using Gauss's estimate of exponential sums instead of Weil's.

The proof of the Theorem is given in Section 3.

1.2. *Related questions*

1.2.1. *Linear and polynomial normality.* Let the tiling of the plane by unit squares be given. We label the squares of the tiles of the positive quadrant of the plane by ω_{ij}, where (i, j) are the coordinates of the lower left vertex of the tile. Consider a curve $y = \phi(x)$. It is partitioned into successive intervals of the intersections with tiles. Therefore, to each curve corresponds a sequence of digits $(u_\phi(n))_{n \ge 0}$.

Definition 2. ω is said to be *polynomial normal* if for all polynomial curves ϕ the sequence $(u_\phi(n))_{n \ge 0}$ is normal to base b.

We shall say that ω is *linear normal* if we consider only first degree polynomial curves, i.e. lines.

In [LeSm3] we proved that the configuration ω_∞ (see (7)) is polynomial normal.

Now we note that the notions of linear, polynomial, square, and rectangular normal configurations define different sets in Ω. The differences are null measure subsets, but are not empty. In [LeSm2] we gave examples of: linear normal configuration which is not square normal; rectangle normal configuration which is not linear normal; rectangle and linear normal configuration which is not polynomial normal; square and linear normal configuration which is not rectangular normal.

Problem 1. Is the intersection of ω_∞ with all increasing convex curves also normal?

1.2.2. *s-dimensional surfaces in* $\mathbb{R}^d$. Consider a function $\psi : \mathbb{R}^s \to \mathbb{R}^d$. Let $G_\psi = \{\psi(\mathbf{x}) \in \mathbb{R}^d \mid \mathbf{x} \in \mathbb{R}^s\}$, $s \le d$, and

$$G'_\psi = \{\mathbf{n} \in \mathbb{Z}^d \mid \mathbf{n} + [0,1)^d \cap G_\psi \neq \emptyset\}, \qquad H_\psi : G'_\psi \to \mathbb{Z}^s,$$

and $\Psi = \{\psi\}$ is a set of functions ψ (a set of s-dimensional surfaces) such that H_ψ is a bijection.

Definition 3. The configuration $\omega \in \{0, 1, \dots, b-1\}^{\mathbb{Z}^d}$ is said to be *Ψ-normal* if $H_\psi(G'_\psi(\omega))$ is rectangular normal in $\mathbb{Z}^s$ for all $\psi \in \Psi$.

Problem 2. Let ω be a d-dimensional configuration, constructed similarly to (3)–(7), and Ψ_p be the set of all s-dimensional polynomial surfaces in $\mathbb{R}^d$. Is ω a Ψ_p-normal configuration?

1.2.3. *Connection with uniform distribution.* Let $(\mathbf{x}_n)_{n\ge1}$ be an infinite sequence of points in an s-dimensional unit cube $[0,1)^s$; $v = [0,\gamma_1) \times \cdots \times [0,\gamma_s)$ be a box in $[0,1)^s$; and $A_v(N)$ be the number of indices $n \in [1,N]$ such that $\mathbf{x}_n$ lies in v. The quantity

$$D(N) = D((\mathbf{x}_n)_{n=1}^N) = \sup_{v\in(0,1]^s} \left| \frac{1}{N} A_v(N) - \gamma_1 \cdots \gamma_s \right| \tag{9}$$

is called the *discrepancy* of $(\mathbf{x}_n)_{n=1}^N$. The sequence $(\mathbf{x}_n)_{n\ge1}$ is said to be *uniformly distributed* in $[0,1)^s$ if $D(N) \to 0$ as $N \to \infty$.

It is known (Wall, 1949) that a number α is normal to base b if and only if the sequence $\{\alpha b^n\}_{n\ge1}$ is uniformly distributed in $[0,1)$ (see [KN, p. 70]).

Let $\omega = (a_{i,j})_{i,j\ge1}$ $(a_{i,j} \in \{0,1,\dots,b-1\})$ be a configuration,

$$\alpha_m = \sum_{i=1}^\infty a_{m,i}/b^i, \qquad m = 1, 2, \dots,$$

and $s \ge 1$ be an integer. The following statement is proved in [L1]:

The lattice configuration ω is normal to base b if and only if for all $s \ge 1$ the double sequence

$$(\{\alpha_m b^n\}, \dots, \{\alpha_{m+s-1} b^n\})_{m,n\ge1}$$

is uniformly distributed in $[0,1)^s$, i.e.,

$$D((\{\alpha_m b^n\}, \dots, \{\alpha_{m+s-1} b^n\})_{1\le n\le N,\, 0\le m<M}) = o(1)$$

as $\max(M,N) \to \infty$. Hence we have another definition of normal configuration (of *normal sequence* $\alpha = (\alpha_1, \alpha_2, \dots) \in [0,1)^\infty$ to base b). It is evident that almost all sequences α are normal to all bases $b \ge 2$ (*absolutely normal*).

Different bases. Responding to a question of Steinhaus, J. Cassels and W. Schmidt (1960) proved that for all integers $q_1, q_2 \geq 2$ (with $\log_{q_1} q_2$ irrational) there exist numbers β that are normal to base q_1 and not normal to base q_2. G. Wagner (1989, see [KS]) found a constructive proof of this result for some $q_1, q_2 \geq 2$.

PROBLEM 3. Find for some integers $q_1, q_2 \geq 2$ an example of a sequence α normal to base q_1 such that α is not normal to base q_2.

Discrepancy estimate. In [L1] we proved explicitly that there exists a normal sequence $\alpha = (\alpha_m)_{m\geq 1}$ such that for all $s, N, M \geq 1$, we have

$$D((\{\alpha_m b^n\}, \ldots, \{\alpha_{m+s-1} b^n\})_{1\leq n\leq N,\, 0\leq m<M}) = O((MN)^{-1}(\log MN)^{2s+5})$$

as $\max(M, N) \to \infty$, and the constant implied by O only depends on s.

We note that according to Roth's theorem (see [DrTi, p. 29]), this estimate cannot be improved by more than a power of the logarithmic multiplier.

1.2.4. *Connection with completely uniform distribution.* Now let $(u_n)_{n\geq 1}$ be an arbitrary sequence of real numbers. Starting with the sequence $(u_n)_{n\geq 1}$, we construct for every integer $s \geq 1$ the s-dimensional sequence $(x_n^{(s)}) = (\{u_{n+1}\}, \ldots, \{u_{n+s}\})$, where $\{x\}$ is the fractional part of x. The sequence $(u_n)_{n\geq 1}$ is said to be *completely uniformly distributed* (abbreviated c.u.d.) if for any integer $s \geq 1$ the sequence $(x_n^{(s)})$ is u.d. in $[0, 1)^s$ (Korobov, 1949, see [Ko1, Ko2]).

A c.u.d. sequence is a universal sequence for computing multidimensional integrals, modeling Markov chains, random numbers, and for other problems [DrTi, KN, Ko2].

Let $b \geq 2$ be an integer, (u_n) be a c.u.d. sequence, and $a_n = [b\{u_n\}]$, $n = 1, 2, \ldots$. Then $\alpha = .a_1 a_2 \ldots$ is normal to base b (Korobov [Ko2]).

In [L2] we constructed a c.u.d. double sequence $(u_{n,m})_{n,m\geq 1}$ such that for all integers $s, t \geq 1$,

$$MND(((u_{n+i,m+j})_{i=1,j=1}^{s,t})_{n=1,m=1}^{N,M}) = O((\log(MN + 1))^{st+4})$$

for all $M, N \geq 1$. Similarly to [Ko2], we get from this an estimate of the error term in (8) as $O((\log(N_1 N_2+1))^{st+4})$ for the configuration $(a_{n,m})_{n,m\geq 1}$, where $a_{n,m} = [b\{u_{n,m}\}]$, $n, m \geq 1$. This estimate is evidently better than (8). But the configuration ω_∞ of (7) also has the polynomial normality property [LeSm3].

2. Auxiliary notation and results. To estimate the discrepancy we use the Erdős–Turán inequality (see, for example, [DrTi, p. 18])

$$ND((\beta_n)_{n=0}^{N-1}) \le \frac{3}{2}\Bigg(\frac{2N}{H+1} + \sum_{0<|m|\le H} \frac{|\sum_{n=0}^{N-1} e(m\beta_n)|}{\overline{m}}\Bigg), \tag{10}$$

where $e(y) = e^{2\pi i y}$, $\overline{m} = \max(1, |m|)$, and $H \ge 1$ is arbitrary.

We shall use the following estimates (see, for example, [Ko2, pp. 1, 29]):

$$\begin{aligned} &\Big|\sum_{x=A}^{A+P-1} e(\theta x)\Big| \le \min\Big(P, \frac{1}{2\|\theta\|}\Big), \\ &\Big|\sum_{x=A}^{A+P-1} e((ax^2+bx+c)/q)\Big| \\ &\qquad \le \max_{1\le d\le q}\Big|\sum_{x=A}^{A+q-1} e((ax^2+(b+d)x+c)/q)\Big| \cdot (1+\ln q), \end{aligned} \tag{11}$$

where $\|x\| = \min(\{x\}, 1-\{x\})$, $1 \le P \le q$, and a, b, c, q are integers.

Let (a, q) be the greatest common divisor of a and q. Similarly to [Ko2, pp. 12, 13], we obtain the following form of Gauss's estimate of exponential sums:

$$\Big|\sum_{x=A}^{A+q-1} e((ax^2+bx+c)/q)\Big| \le \sqrt{2q} \quad \text{ if } (a, q) = 1.$$

Let $a_1 = a/(a, q))$ and $q_1 = q/(a, q)$. Then

$$\begin{aligned} \Big|\sum_{x=A}^{A+P-1} e(ax^2/q)\Big| &= \Big|\sum_{x=A}^{A+P-1} e(a_1x^2/q_1)\Big| \\ &= \Big|\sum_{x=A}^{A+q_1[P/q_1]-1} e(a_1x^2/q_1) + \sum_{x=A+q_1[P/q_1]-1}^{A+P-1} e(ax^2/q)\Big| \\ &\le [p/q_1]\Big|\sum_{x=0}^{q_1-1} e(a_1x^2/q_1)\Big| + \Big|\sum_{x=A+q_1[P/q_1]-1}^{A+P-1} e(a_1x^2/q_1)\Big| \\ &\le ([P/q_1]+1)(2q_1)^{1/2}(1+\ln q_1) \le 2(P+q_1)q_1^{-1/2}(1+\ln q_1). \end{aligned}$$

Hence, for all $P \ge 1$ and $a \ne 0$ with $|a| < q$ we have

$$\Big|\sum_{x=A}^{A+P-1} e(ax^2/q)\Big| \le 2(P+q)|a|q^{-1/2}(1+\ln q). \tag{12}$$

3. Proof of the Theorem. Consider the configuration ω_n, where n satisfies the following inequality:

$$2(n-1)^2 b^{2(n-1)^2} \le \max(N_1, N_2) < 2nb^{2n^2}.$$

Let $h_1, h_2 \geq 1$ be integers, and

$$g_{i_1,i_2} \in \{0,1,\ldots,b-1\}, \quad 0 \leq i_1 < h_1,\ 0 \leq i_2 < h_2.$$

We consider the block of digits $G = (g_{i_1,i_2})_{0\leq i_1<h_1,\,0\leq i_2<h_2}$, the configuration ω_n, and the block of digits $\alpha = (\omega_n(i,j))$ $(0 \leq i < N_1 + h_1,$ $0 \leq j < N_2 + h_2)$.

To compute the number of appearances of the block G in the configuration α, we introduce the following notations (see (1), (2)):

$$(13)\quad V_{n,G}(L_1, M_1; L_2, M_2) = \bigcup_{(i,j)\in[L_1,L_1+M_1)\times[L_2,L_2+M_2)} \{(i,j) \mid \chi_{\omega_n,G}(i,j) = 1\}$$

and

$$(14)\quad V_{n,G}(N_1, N_2) = V_{n,G}(0, N_1; 0, N_2).$$

Let

$$(15)\quad N_1 = 2nN_{11} + N_{12}, \quad N_2 = 2nN_{21} + N_{22} \quad \text{with } N_{12}, N_{22} \in [0, 2n).$$

Observe that

$$(16)\quad V_{n,G}(N_1, N_2) = V_{n,G}(2nN_{11}, 2nN_{21}) \cup V_{n,G}(0, 2nN_1; 2nN_{21}, N_{22}) \cup V_{n,G}(2nN_{11}, N_{12}; 0, N_2).$$

Next, we fix $s,t \in [0,2n)$, and compute the number of appearances of G in the configuration $\alpha_1 = (\omega_n(i,j))_{0\leq i<M_1+h_1,\,0\leq j<M_2+h_2}$ such that the shift of the block G by the vector (i,j) satisfies $i \equiv s \pmod{2n}$ and $j \equiv t \pmod{2n}$. Set

$$(17)\quad A_{s,t,G}(M_1, M_2) = \bigcup_{(i,j)\in[0,2nM_1)\times[0,2nM_2)} \{(i,j) \mid \chi_{\omega_n,G}(i,j) = 1, \text{ and } i \equiv s,\ j \equiv t \pmod{2n}\}.$$

It is easy to see that

$$(18)\quad V_{n,G}(2nN_{11}, 2nN_{21}) = \bigcup_{0\leq s<2n} \bigcup_{0\leq t<2n} A_{s,t,G}(N_{11}, N_{21}),$$

and

$$(19)\quad V_{n,G}(0, 2nN_{11}; 2nN_{21}, N_{22}) = \bigcup_{0\leq s<2n} \bigcup_{0\leq t<N_{22}} (A_{s,t,G}(N_{11}, N_{21}+1) \setminus A_{s,t,G}(N_{11}, N_{21})).$$

We will show that to complete the proof of the theorem it is sufficient to prove that for all $s,t \in [0,2n)$, $M_1, M_2 \in [1, 2nb^{2n^2}]$, $n = 1,2,\ldots,$

$$\#A_{s,t,G}(M_1, M_2) = b^{-h_1h_2} M_1M_2 + O(M_1M_2b^{-s-t}).$$

Now we find an analytic expression for $\#A_{s,t,G}(M_1,M_2)$. First from (1), (2), and (17) we have

$$A_{s,t,G}(M_1,M_2)=\{(2nx+s,2ny+t)\mid(x,y)\in[0,M_1)\times[0,M_2),\ \omega_n(2nx+s+i_1,2ny+t+i_2)=g_{i_1,i_2}\,\forall(i_1,i_2)\in[0,h_1)\times[0,h_2)\}. \tag{20}$$

Next we introduce some integer sequences:

$$v=v(i_1,i_2)=v(s,t,i_1,i_2)=\begin{cases}(s+i_1)^2+t+i_2 & \text{if } t+i_2<s+i_1,\\ (t+i_2)^2+2(t+i_2)-s-i_1 & \text{otherwise,}\end{cases} \tag{21}$$

and $k_1,\dots,k_h$ $(h=h_1h_2)$ is an increasing sequence of integers from the set

$$v(s,t,i_1,i_2)+1,\qquad i_1=0,1,\dots,h_1-1,\ i_2=0,1,\dots,h_2-1. \tag{22}$$

We enumerate the set $(v(s,t,i_1,i_2))_{i_1=0,\,i_2=0}^{h_1-1,\,h_2-1}$ in increasing order with the integer sequence $\mu(i_1,i_2)\in[1,\ h_1h_2]$:

$$\mu(i_1,i_2)>\mu(j_1,j_2)\ \Leftrightarrow\ v(s,t,i_1,i_2)>v(s,t,j_1,j_2), \tag{23}$$

where $i_\nu,j_\nu\in[0,h_\nu)$, $\nu=1,2$, and we obtain

$$k_{\mu(i_1,i_2)}=v(s,t,i_1,i_2)+1,\qquad i_\nu=0,1,\dots,h_\nu-1,\ \nu=1,2. \tag{24}$$

Put

$$d_{\mu(i_1,i_2)}=g_{i_1,i_2}\qquad i_\nu=0,1,\dots,h_\nu-1,\ \nu=1,2. \tag{25}$$

Using (4)–(6), and (23)–(25), we find that the condition

$$\omega_n(2nx+s+i_1,2ny+t+i_2)=g_{i_1,i_2}\quad\forall(i_1,i_2)\in[0,h_1)\times[0,h_2) \tag{26}$$

is equivalent to

$$a_{v(s,t,i_1,i_2)}(u(x,y))=g_{i_1,i_2}\qquad\forall(i_1,i_2)\in[0,h_1)\times[0,h_2),$$

or by (24) and (25) to

$$a_{k_i-1}(u(x,y))=d_i\qquad\forall i\in[0,h_1h_2), \tag{27}$$

where

$$u(x,y)=\begin{cases}x^2+y & \text{for } x\ge y,\\ y^2+2y-x & \text{otherwise.}\end{cases} \tag{28}$$

In other words, (26) is equivalent to

$$a_{k_i-1}(u(x,y))=d_i\qquad\forall i\in[0,h_1h_2). \tag{29}$$

Now from (20), (26), and (29) we deduce that

(30) $$A_{s,t,G}(M_1,M_2)=\{(2nx+s,2ny+t)\mid(x,y)\in[0,M_1)\times[0,M_2),\ a_{k_i-1}(u(x,y))=d_i\ \forall i\in[1,h_1h_2]\}.$$

Lemma 1. *Let* $M_1,M_2\in[0,b^{2n^2})$, $s,t\in[0,2n-15h]$, *and* $h=h_1h_2$. *Then*

(31) $$\#A_{s,t,G}(M_1,M_2)=\sum_{x_2=0}^{b^{k_2-k_1-1}-1}\dots\sum_{x_h=0}^{b^{k_h-k_{h-1}-1}-1}B_{st}(M_1,M_2,d(x_2,\dots,x_h)),$$

where

(32) $$B_{st}(M_1,M_2,d)=\#\left\{(x,y)\in[0,M_1)\times[0,M_2)\ \middle|\ \{u(x,y)b^{-k_h}\}\in\left[\frac{d(x_2,\dots,x_h)}{b^{k_h-k_1+1}},\frac{d(x_2,\dots,x_h)+1}{b^{k_h-k_1+1}}\right)\right\},$$

and

(33) $$d(x_2,\dots,x_h)=d_1+x_2b+d_2b^{k_2-k_1}+\dots+x_hb^{k_{h-1}-k_1+1}+d_hb^{k_h-k_1}.$$

Proof. From (6), we infer that the condition $a_{k_i-1}(u(x,y))=d_i$ for all $i\in[1,h]$ is equivalent to the following statement:

$$u(x,y)=x_1+d_1b^{k_1-1}+x_2b^{k_1}+d_2b^{k_2-1}+\dots+x_hb^{k_{h-1}}+d_hb^{k_h-1}+x_{h+1}b^{k_h},$$

with integers $x_i\in[0,b^{k_i-k_{i-1}-1})$, $k_0=0$, $i=1,\dots,h$, and $x_{h+1}\geq 0$. Using (30) and (33), we get

(34) $$\begin{aligned}A_{s,t,G}(M_1,M_2)&=\{(2nx+s,2ny+t)\mid(x,y)\in[0,M_1)\times[0,M_2),\\ &\qquad u(x,y)=x_1+d(x_2,\dots,x_h)b^{k_1-1}+x_{h+1}b^{k_h},\\ &\qquad x_i\in[0,b^{k_i-k_{i-1}-1}),\ k_0=0,\ i=1,\dots,h,\ x_{h+1}\geq 0\}\\ &=\bigcup_{x_2=0}^{b^{k_2-k_1-1}-1}\dots\bigcup_{x_h=0}^{b^{k_h-k_{h-1}-1}-1}\{(2nx+s,2ny+t)\mid(x,y)\in[0,M_1)\times[0,M_2),\\ &\qquad u(x,y)=x_1+d(x_2,\dots,x_h)b^{k_1-1}+x_{h+1}b^{k_h}\},\end{aligned}$$

for arbitrary integers $x_1\in[0,b^{k_1-1})$, $x_{h+1}\geq 0$. Bearing in mind that the condition

$$u(x,y)=x_1+d(x_2,\dots,x_h)b^{k_1-1}+x_{h+1}b^{k_h}$$

is equivalent to

$$\{u(x,y)b^{-k_h}\}\in\left[\frac{d(x_2,\dots,x_h)}{b^{k_h-k_1+1}},\frac{d(x_2,\dots,x_h)+1}{b^{k_h-k_1+1}}\right),$$

we deduce from (34) that

$$A_{s,t,G}(M_1,M_2) = \bigcup_{x_2=0}^{b^{k_2-k_1-1}-1} \cdots \bigcup_{x_h=0}^{b^{k_h-k_{h-1}-1}-1} \Big\{ (2nx+s, 2ny+t) \mid (x,y) \in [0,M_1) \times \times [0,M_2), \{u(x,y)b^{-k_h}\} \in \Big[\frac{d(x_2,\dots,x_h)}{b^{k_h-k_1+1}}, \frac{d(x_2,\dots,x_h)+1}{b^{k_h-k_1+1}} \Big) \Big\}.$$

Now by (32) and (33) we obtain the assertion of the lemma. ∎

LEMMA 2. *Let* $1 \le M_2 \le M_1 \in [b^{2n^2-5n}, b^{2n^2})$, $s,t \in [0, 2n-15h]$, $h = h_1h_2$, $n \ge h$, *and* $0 < |m| \le H = b^{k_h-k_1+s+t}$. *Then*

$$S(m) = \sum_{y=0}^{M_2-1} \sum_{x=0}^{M_1-1} e(mu(x,y)b^{-k_h}) = O(M_1M_2H^{-1}/(s+t+1)). \tag{35}$$

Proof. Let

$$\sigma_1 = \sum_{x=0}^{M_2^2-1} e(mxb^{-k_h}), \tag{36}$$

$$\sigma_2 = \sum_{y=0}^{M_2-1} \sum_{x=0}^{M_1-1} e(m(x^2+y)b^{-k_h}), \tag{37}$$

$$\sigma_3 = \sum_{x,y=0}^{M_2-1} e(m(x^2+y)b^{-k_h}). \tag{38}$$

From (5) and (36)–(38), we obtain

$$\begin{aligned} S(m) &= \sum_{y,x=0}^{M_2-1} e(mu(x,y)b^{-k_h}) + \sum_{y=0}^{M_2-1} \sum_{x=M_2}^{M_1-1} e(mu(x,y)b^{-k_h}) \\ &= \sum_{x=0}^{M_2^2-1} e(mxb^{-k_h}) + \sum_{y=0}^{M_2-1} \sum_{x=M_2}^{M_1-1} e(m(x^2+y)b^{-k_h}) \\ &= \sigma_1 + \sum_{y=0}^{M_2-1} \sum_{x=0}^{M_1-1} e(m(x^2+y)b^{-k_h}) \\ &\quad - \sum_{x,y=0}^{M_2-1} e(m(x^2+y)b^{-k_h}) = \sigma_1 + \sigma_2 - \sigma_3. \end{aligned} \tag{39}$$

First we estimate $|\sigma_2| + |\sigma_3|$. Let

$$\sigma(y,M) = \Big| \sum_{x=0}^{M-1} e(m(x^2+y)b^{-k_h}) \Big|.$$

Using (12) we obtain

$$\sigma(y, M) \le 2(M + b^{k_h})|m|b^{-k_h/2}(1 + k_h \ln b). \tag{40}$$

By (37) and (38) we have

$$|\sigma_2| + |\sigma_3| \le 4M_2(M_1 + b^{k_h})|m|b^{-k_h/2}(1 + k_h \ln b). \tag{41}$$

Bearing in mind (22), (21) and the assumptions of the lemma, we get

$$0 \le k_h - k_1 \le 2sh_1 + 2th_2 + 2h_1^2 + 2h_2^2 \le 8nh + 4h^2, \tag{42}$$

$$\begin{aligned} (s^2 + t^2)/2 \le k_1 < k_h &\le (2n - 14h)^2 + 2n \\ &\le 4n^2 - 10n - 44nh + 200h^2. \end{aligned} \tag{43}$$

Hence there exist constants $c_1(h_1, h_2), c_2(h_1, h_2)$ such that

$$2 \log_b k_h + k_h - k_1 + s + t < k_h/4 + c_1(h_1, h_2), \tag{44}$$

$$|m|(1 + k_h \ln b)b^{-k_h/2} < c_2(h_1, h_2)H^{-1}/(s + t + 1), \tag{45}$$

where $|m| \le H = b^{k_h - k_1 + s + t}$. Therefore,

$$M_1M_2|m|b^{-k_h/2}(1 + k_h \ln b) = O(M_1M_2H^{-1}/(s + t + 1)). \tag{46}$$

We also deduce from (42) and (43) that

$$\begin{aligned} H(1 + k_h \ln b)b^{k_h/2} &\le H(1 + k_h \ln b)b^{2n^2 - 5n - 22nh + 100h^2} \\ &\le M_1 b^{k_h - k_1 + s + t - 22nh + 100h^2}(1 + k_h \ln b) \\ &\le c_2(h_1, h_2)M_1 b^{-k_h + k_1 - s - t}/(s + t + 1) \\ &= c_2(h_1, h_2)M_1 H^{-1}/(s + t + 1). \end{aligned} \tag{47}$$

Hence

$$M_2|m|b^{k_h/2}(1 + k_h \ln b) = O(M_1M_2H^{-1}/(s + t + 1)). \tag{48}$$

From (41), (46), and (48), we get

$$|\sigma_2| + |\sigma_3| = O(M_1M_2H^{-1}/(s + t + 1)). \tag{49}$$

Now we consider the sum σ_1 (see (36)). If $M_2 \le M_1H^{-1}/(s+t+1)$ then we get a trivial estimate:

$$|\sigma_1| = O(M_1M_2H^{-1}/(s + t + 1)). \tag{50}$$

Now let $M_2 > M_1H^{-1}/(s + t + 1)$. From the assumptions of the lemma and (42), we have

$$\begin{aligned} \log_b(M_1M_2H^{-1}/(s + t + 1)) &\ge \log_b(M_1^2H^{-2}/(s + t + 1)^2) \\ &\ge 4n^2 - 10n - 2(k_h - k_1 + s + t + 1) - 2 \log_b(s + t + 1) \\ &\ge 4n^2 - 10n - 2(8nh + 4h^2 + 4n) - 2 \log_b(4n + 1). \end{aligned}$$

By (43) and (44), there exists an integer $n_0 > 0$ such that

$$\begin{aligned} k_h &\le 4n^2 - 10n - 44nh + 200h^2 \\ &\le 4n^2 - 10n - 24nh - 8h^2 - 2\log_b(4n+1) \le \log_b(M_2 M_1 H^{-1}/(s+t+1)) \end{aligned}$$

for $n \ge n_0$, and

$$H = b^{k_h - k_1 + s + t} < b^{k_h}/2 \quad \text{for } n \ge n_0.$$

Hence,

$$0 < |m| b^{-k_h} \le H b^{-k_h} < 1/2 \quad \text{and} \quad b^{k_h} \le M_1 M_2 H^{-1}/(s+t+1)$$

for $n \ge n_0$. We apply (11) to estimate the sum σ_1:

$$|\sigma_1| \le b^{k_h} \le M_1 M_2 H^{-1}/(s+t+1) \quad \text{for } n \ge n_0.$$

Now by (39), (35), (49), and (50), the assertion of the lemma follows. ∎

LEMMA 3. *Under the assumptions of Lemma* 2,

$$D = D((\{u(x,y) b^{-k_h}\})_{x=0,\, y=0}^{M_1-1,\, M_2-1}) = O(b^{k_1 - k_h - s - t}). \tag{51}$$

Proof. We apply Lemma 2, (42) and the Erdős–Turán inequality, with $N = M_1 M_2$, $H = b^{k_h - k_1 + s + t}$ and $\beta_{x + M_1 y} = u(x,y) b^{-k_h}$ $(0 \le x < M_1$, $0 \le y < M_2)$:

$$\begin{aligned} D &= O\biggl(H^{-1} + (M_1 M_2)^{-1} \sum_{0<|m|\le H} \frac{|S(m)|}{\overline{m}}\biggr) \\ &= O\biggl(H^{-1}\biggl(1 + \frac{1}{s+t+1} \sum_{0<|m|\le H} \frac{1}{\overline{m}}\biggr)\biggr) \\ &= O(H^{-1}(1 + (s+t+1)^{-1} \log H)) \\ &= O(H^{-1}(1 + (s+t+1)^{-1}(k_h - k_1 + s + t))) = O(H^{-1}). \ \blacksquare \end{aligned}$$

Using the definition of discrepancy (9), from (32) we get:

COROLLARY 1. *Under the assumptions of Lemma* 2,

$$B_{st}(M_1, M_2, d(x_2, \ldots, x_h)) = M_1 M_2 b^{k_1 - k_h - 1}(1 + O(b^{-s-t})) \tag{52}$$

for all integers $x_i \in [0, b^{k_i - k_{i-1} - 1})$, $i = 1, \ldots, h$.

From Lemma 1, (32), (33), Corollary 1, and (22), we get

COROLLARY 2. *Under the assumptions of Lemma* 2,

$$\# A_{s,t,G}(M_1, M_2) = b^{-h} M_1 M_2 + O(M_1 M_2 b^{-s-t}). \tag{53}$$

LEMMA 4. *Let* $0 \le N_2 \le N_1 \in [b^{2n^2 - 5n}, b^{2n^2})$. *Then*

$$\# V_{n,G}(N_1, N_2) = b^{-h} N_1 N_2 + O(N_1 N_2 / n).$$

Proof. We use (18):

$$V_{n,G}(2nN_{11}, 2nN_{21}) = \bigcup_{0\le s,t<2n-15h} \bigcup_{2n-15h\le \max(s,t)<2n} A_{s,t,G}(N_{11}, N_{21}). \tag{54}$$

We apply (53) for the first union and the trivial estimates for the second union:

$$\begin{aligned} \#V_{n,G}(2nN_{11}, 2nN_{21}) &= \sum_{0\le s,t<2n-15h} (b^{-h}N_{11}N_{21} + O(N_{11}N_{21}b^{-s-t})) + O(N_{11}N_{21}n) \\ &= b^{-h}4n^2N_{11}N_{21} + O(N_{11}N_{21}n), \quad N_{21}\ge 1. \end{aligned} \tag{55}$$

Similarly, from (19) we obtain

$$\begin{aligned} &\#V_{n,G}(0, 2nN_{11}; 2nN_{21}, N_{22}) \\ &= \sum_{0\le s<2n-15h} \sum_{0\le t<\min(N_{22},2n-15h)} \#(A_{s,t,G}(N_{11}, N_{21}+1) \setminus A_{s,t,G}(N_{11}, N_{21})) \\ &\quad + \varepsilon_1 \sum_{s\in[2n-15h,2n),\, t\in[0,N_{22})} N_{11} + \varepsilon_2 \sum_{0\le s<2n,\, t\in[2n-15h,N_{22})} N_{11}, \end{aligned}$$

where $0 \le \varepsilon_1, \varepsilon_2 \le 1$. It is easy to see that the first sum is not empty only for $N_{22} \ge 2n - 15h$. Hence by (53) we have

$$\begin{aligned} \#V_{n,G}(0, 2nN_{11}; 2nN_{21}, N_{22}) &= \sum_{0\le s<2n-15h} \sum_{0\le t<\min(N_{22},2n-15h)} (b^{-h}N_{11} + O(N_{11}b^{-s-t})) + O(N_{11}N_{22}) \\ &= \sum_{0\le s<2n} \sum_{0\le t<N_{22}} b^{-h}N_{11} + O(N_{11}N_{22}) = b^{-h}2nN_{11}N_{22} + O(N_{11}N_{22}). \end{aligned} \tag{56}$$

We get a trivial estimate from (13)–(15):

$$\#V_{n,G}(2nN_{11}, N_{12}; 0, N_2) \le N_2N_{12} \le 2nN_2 < N_1N_2/n.$$

Now the assertion of the lemma follows from (15), (16), and (55)–(56). ∎

We introduce similar notation for the configuration ω_∞ (instead of ω_n):

$$\begin{aligned} V_G(P_1, P_2) = \{&(v_1, v_2) \in [0, P_1) \times [0, P_2) \mid \\ &\omega_\infty(v_1 + i_1, v_2 + i_2) = g_{i_1,i_2} \ \forall (i_1, i_2) \in [0, h_1) \times [0, h_2)\}. \end{aligned} \tag{57}$$

We prove the Theorem for the case $N_1 \ge N_2$. The other case is similar.

Completion of the proof of the Theorem. Let $1 \le N_2 \le N_1$ and $N_1 \ge 4b^8$. There exists $n \ge 3$ so that

$$N_1 \in [2(n-1)^2 b^{2(n-1)^2} - h, 2nb^{2n^2} - h). \tag{58}$$

Now let

$$N_1' = 2(n-1)^2 b^{2(n-1)^2} - h, \qquad N_2' = \min(N_2, N_1'). \tag{59}$$

From (57) and the definition of the configurations ω_∞, ω_n we get

$$\begin{aligned} \#V_G(N_1; N_2) = \#V_{n,G}(N_1, N_2) - \#V_{n,G}(N_1', N_2') + \#V_G(N_1', N_2') \\ + 2\varepsilon_1 h N_2' + 2\varepsilon_2 N_1 \min(h, N_2 - N_2'), \end{aligned} \tag{60}$$

with $|\varepsilon_i| \le 1$, $i = 1, 2$. It is easy to see that if $N_2 \le n$, then $N_2 = N_2'$, otherwise $h \le hN_2/n$ and

$$\begin{aligned} \#V_G(N_1, N_2) - \#V_{n,G}(N_1, N_2) = \#V_G(N_1', N_2') - \#V_{n,G}(N_1', N_2') \\ + 4\varepsilon_3 h N_1 N_2/n \quad \text{with } |\varepsilon_3| \le 1. \end{aligned} \tag{61}$$

Analogously,

$$\begin{aligned} \#V_G(N_1', N_2') - \#V_{n,G}(N_1', N_2') = \#V_G(N_1'', N_2'') - \#V_{n-1,G}(N_1'', N_2'') \\ + 4\varepsilon_4 h N_1 N_2/n \quad \text{with } |\varepsilon_4| \le 1, \end{aligned} \tag{62}$$

and

$$N_1'' = 2(n-2)^2 b^{2(n-2)^2} - h, \qquad N_2'' = \min(N_2, N_1''). \tag{63}$$

It is evident that

$$\#V_G(N_1'', N_2'') + \#V_{n,G}(N_1'', N_2'') \le 2N_1''N_2'' < 2N_1N_2/n. \tag{64}$$

From (58)–(64), we obtain

$$\begin{aligned} \#V_G(N_1, N_2) = \#V_{n,G}(N_1, N_2) - \#V_{n,G}(N_1', N_2') + \#V_{n-1,G}(N_1', N_2') \\ + O(N_1N_2/n). \end{aligned}$$

Using Lemma 4, we have

$$\begin{aligned} \#V_G(N_1, N_2) &= b^{-h}N_1N_2 - b^{-h}N_1'N_2' + O(N_1N_2/n) + b^{-h}N_1'N_2' \\ &= b^{-h}N_1N_2 + O(N_1N_2/n) = b^{-h}N_1N_2 + O(N_1N_2/\sqrt{\log N_1N_2}). \end{aligned}$$

From (57), (1) and (2) we obtain the assertion of the Theorem. ■

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