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ON UNIVERSALITY OF FINITE POWERS OF LOCALLY PATH-CONNECTED MEAGER SPACES

BY

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Abstract. It is shown that for every integer n the $(2n+1)$th power of any locally path-connected metrizable space of the first Baire category is $\mathcal{A}_1[n]$-universal, i.e., contains a closed topological copy of each at most n-dimensional metrizable σ-compact space. Also a one-dimensional σ-compact absolute retract X is found such that the power X^{n+1} is $\mathcal{A}_1[n]$-universal for every n.

A topological space X is called $\mathcal{C}$*-universal*, where $\mathcal{C}$ is a class of spaces, if X contains a closed topological copy of each space $C \in \mathcal{C}$. We denote by $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{A}_1$ the classes of metrizable compacta, Polish spaces, and metrizable σ-compact spaces, respectively. For a class $\mathcal{C}$ of spaces we denote by $\mathcal{C}[n]$ the subclass of $\mathcal{C}$ consisting of all spaces $C \in \mathcal{C}$ with $\dim C \le n$.

In terms of universality, the classical Menger–Nöbeling–Pontryagin–Lefschetz Theorem states that the cube $[0,1]^{2n+1}$ is $\mathcal{M}_0[n]$-universal for every $n \ge 0$. It is well known that the exponent $2n+1$ in this theorem is the best possible: the Menger universal compactum μ_n cannot be embedded into $[0,1]^{2n}$. Nonetheless, P. Bowers [Bo] has proved that the $(n+1)$th power D^{n+1} of any dendrite D with dense set of end-points is $\mathcal{M}_0[n]$-universal for every non-negative integer n. Moreover, every such dendrite D contains a connected G_δ-subset G whose $(n+1)$th power G^{n+1} is $\mathcal{M}_1[n]$-universal for every n (see [Bo]). Actually, these results of Bowers' are particular cases of a more general fact proved in [BCTZ]: for any locally connected Polish space X without free arcs the power X^{n+1} is $\mathcal{M}_0[n]$-universal; moreover, if the space X is nowhere locally compact, then the power X^{n+1} is $\mathcal{M}_1[n]$-universal.

Taking into account that $\mathcal{M}_0$ and $\mathcal{M}_1$ are the first classes in the Borel hierarchy it is natural to ask the following

QUESTION. *Suppose $\mathcal{C}$ is a Borel class. Is there a one-dimensional absolute retract $X \in \mathcal{C}$ whose $(n+1)$th power X^{n+1} is $\mathcal{C}[n]$-universal for every integer $n \ge 0$?*

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According to the above-mentioned results of [Bo] or [BCTZ] the answer to this question is affirmative for the multiplicative Borel classes $\mathcal{M}_0$ and $\mathcal{M}_1$. In this paper we show that the same is true for the additive Borel class $\mathcal{A}_1$.

THEOREM 1. *If X is a metrizable locally path-connected space of first Baire category, then the space X^{2n+1} is $\mathcal{A}_1[n]$-universal for every integer $n \geq 0$.*

THEOREM 2. *There exists a one-dimensional σ-compact absolute retract A whose power A^{n+1} is $\mathcal{A}_1[n]$-universal for every integer $n \geq 0$. Moreover, such a space A can be found in every dendrite with dense set of end-points.*

The exponents $2n+1$ and $n+1$ in Theorems 1 and 2 are the best possible: the Menger universal compactum μ_n admits no embedding into X^{2n} if X is a countable union of arcs, while the n-sphere S^n admits no embedding into the nth power of a one-dimensional space.

Observe also a difference between our results and Bowers'. While Bowers' results have infinite-dimensional counterparts (there exists a Polish (resp. compact) one-dimensional absolute retract whose countable power is $\mathcal{M}_1$-universal (resp. $\mathcal{M}_0$-universal)), that is not true for Theorems 1 and 2: no finite-dimensional space has $\mathcal{A}_1$-universal countable power [BC].

To prove Theorems 1 and 2 we shall apply some well known infinite-dimensional techniques adapted to our finite-dimensional needs. First we recall some definitions and notations. *All spaces considered in this paper are metrizable and separable, all maps are continuous.* By I we denote the closed interval $[0,1]$; the letters n, m, k, i, j denote non-negative integer numbers. For a space X let $\mathrm{cov}(X)$ denote the set of all open covers of X. We write $\mathcal{V} \prec \mathcal{U}$ for $\mathcal{V},\mathcal{U} \in \mathrm{cov}(X)$ if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ with $V \subset U$. For a cover $\mathcal{U} \in \mathrm{cov}(X)$ we set $\mathcal{S}t(\mathcal{U}) = \{\mathcal{S}t(U,\mathcal{U}) : U \in \mathcal{U}\}$, where $\mathcal{S}t(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ for a subset $A \subset X$. Also $\mathcal{S}t^2(\mathcal{U}) = \mathcal{S}t(\mathcal{S}t(\mathcal{U}))$. We say that two maps $f, g : Y \to X$ are *$\mathcal{U}$-near* (denoted by $(f,g) \prec \mathcal{U}$) if for every $y \in Y$ there is $U \in \mathcal{U}$ with $\{f(y), g(y)\} \subset U$.

A subset A of a space X is called a *Z_n-set in X*, n being a non-negative integer, if A is closed in X and for every map $f : I^n \to X$ and every cover $\mathcal{U} \in \mathrm{cov}(X)$ there exists a map $g : I^n \to X$ such that $(g,f) \prec \mathcal{U}$ and $g(I^n) \cap A = \emptyset$. A subset $A \subset X$ is called a *Z_∞-set in X* if A is a Z_n-set in X for every $n \in \mathbb{N}$. A space X is defined to be a *σZ_n-space* if X can be written as a countable union $X = \bigcup_{i=1}^\infty A_i$, where each A_i is a Z_n-set in X. Observe that a subset $A \subset X$ is a Z_0-set in X if and only if A is closed and nowhere dense in X, and a space X is a σZ_0-space if and only if X is of first Baire category. The property of Z_n-sets described in the subsequent lemma is well known for $n = \infty$ (see [Mi, §7.2]) and can be proved by analogy.

LEMMA 1. *If A is a Z_n-set in an absolute retract X, then for any map $f : K \to X$ of a compactum K with* $\dim K \leq n$, *any closed subset $K_0 \subset K$, and any cover $\mathcal{U} \in$* $\mathrm{cov}(X)$ *there exists a map $g : K \to X$ such that $g|K_0 = f|K_0$, $(g, f) \prec \mathcal{U}$, and $g(K \setminus K_0) \subset X \setminus A$.*

The following lemma was proved in [BT].

LEMMA 2. *If X is an absolute retract of first Baire category, then X^{n+1} is a σZ_n-space for every integer $n \geq 0$.*

We recall that a space X has the *disjoint n-cells property* if for every cover $\mathcal{U} \in \mathrm{cov}(X)$ and every map $f : I^n \times \{0,1\} \to X$ there exists a map $g : I^n \times \{0,1\} \to X$ such that $(g, f) \prec \mathcal{U}$ and $g(I^n \times \{0\}) \cap g(I^n \times \{1\}) = \emptyset$. The following lemmas are proved in [BT] and [Bo], respectively.

LEMMA 3. *If X is a non-degenerate absolute retract, then X^{2n+1} has the disjoint n-cells property for every $n \geq 0$.*

LEMMA 4. *If X is a dendrite with dense set of end-points, then X^{n+1} satisfies the disjoint n-cells property for every $n \geq 0$.*

Our next lemma is well known and can be proven by standard methods (see [Mi, §7.3]).

LEMMA 5. *If a Polish ANR X has the disjoint n-cells property for some integer* $n \geq 0$, *then it has the following stronger property*:

($\mathcal{SU}_n$) *for any open set $U \subset X$, any open cover $\mathcal{U} \in$* $\mathrm{cov}(U)$, *and any perfect map $f : K \to U$ from an at most n-dimensional locally compact space K there exists a closed embedding $g : K \to U$ such that $(f, g) \prec \mathcal{U}$.*

Recall that a map $f : X \to Y$ is called *perfect* if f is closed and $f^{-1}(y)$ is compact for every $y \in Y$.

We shall need the following easy modification of Lemma 5.4 of [DMM].

LEMMA 6. *An absolute retract X is $\mathcal{A}_1[n]$-universal for some integer* $n \geq 0$ *provided X is a σZ_n-space with property ($\mathcal{SU}_n$).*

Next, we consider the question of when a countable union of spaces with ($\mathcal{SU}_n$) satisfies that property. We say that a tower $X_1 \subset X_2 \subset \cdots$ of subsets of a space X has the *mapping absorption property for n-dimensional compacta* if for any cover $\mathcal{U} \in \mathrm{cov}(X)$, any closed subset K_0 of a compactum K with $\dim(K) \leq n$, and any map $f : K \to X$ with $f(K_0) \subset X_i$ for some i, there exists a map $g : K \to X_j$ for some $j \geq i$ such that $(g, f) \prec \mathcal{U}$ and $g|K_0 = f|K_0$.

LEMMA 7. *A tower $X_1 \subset X_2 \subset \cdots$ of subsets of a space X has the mapping absorption property for n-dimensional compacta, n being a non-*

negative integer, provided there exists a sequence of retractions $r_i : X \to X_i$, $i \in \mathbb{N}$, *converging to the identity map of* X *uniformly on compacta.*

The proof of this lemma is easy and left to the reader.

LEMMA 8. *Suppose* $X_1 \subset X_2 \subset \cdots$ *is a tower of subsets of an absolute retract* X. *If for some integer* $n \geq 0$ *this tower has the mapping absorption property for* n*-dimensional compacta and each space* X_i *has property* $(\mathcal{SU}_n)$, *then the space* X *has this property as well.*

Proof. To show that the space X has $(\mathcal{SU}_n)$, fix an open set $U \subset X$, a cover $\mathcal{U} \in \operatorname{cov}(U)$, and a perfect map $f : K \to U$ of an at most n-dimensional locally compact space K. Observe that the tower $(X_i \cap U)_{i=1}^{\infty}$ in U has the mapping absorption property for n-dimensional compacta and each space $X_i \cap U$ has property $(\mathcal{SU}_n)$. So, without loss of generality, we may assume that $U = X$.

We may also assume $\mathcal{U}$ to be so fine that every map $g : K \to U$ that is $\mathcal{U}$-near to f is perfect (see [Ch, 4.1]). Thus to prove Lemma 8, it suffices to construct an injective map $g : K \to X$ with $(g, f) \prec \mathcal{U}$. By the paracompactness of X, there is a cover $\mathcal{V} \in \operatorname{cov}(X)$ such that $\mathcal{S}t^2(\mathcal{V}) \prec \mathcal{U}$.

Write $K = \bigcup_{i=1}^{\infty} K_i$, where each K_i is a compactum lying in the interior K_{n+1}° of K_{n+1} in K. Using the mapping absorption property of the tower (X_i), by the standard approximation procedure (see, e.g., [BP, IV.§2]), construct a map $f_0 : K \to X$ such that $(f_0, f) \prec \mathcal{V}$ and for every $i \in \mathbb{N}$ there is $j \in \mathbb{N}$ with $f_0(K_i) \subset X_j$. Since $(f_0, f) \prec \mathcal{V} \prec \mathcal{U}$, the map f_0 is perfect and thus $f_0(K)$ is a closed locally compact subset in X (see [En, 3.7.21]). Observe that the subset $f_0(K) \subset X$ has the following property: every point $x \in f_0(K)$ has a neighborhood $W \subset f_0(K)$ such that $W \subset X_j$ for some j. Indeed, since f_0 is a perfect map, the preimage $f_0^{-1}(x) \subset K$ is compact and as such, lies in some K_i. Since the map f_0 is closed, $W = f_0(K) \setminus f_0(K \setminus K_{i+1}^{\circ})$ is an open neighborhood of x in $f_0(K)$. Clearly, $W \subset f_0(K_{i+1}) \subset X_j$ for some j.

Consequently, $f_0(K) = \bigcup_{i=0}^{\infty} W_i$, where

$$W_i = \{x \in f_0(K) : X_i \cap f_0(K) \text{ is a neighborhood of } x \text{ in } f_0(K)\}.$$

Evidently, each set W_i is open in the locally compact space $f_0(K)$. Hence, we may select a tower $\emptyset = L_0 \subset L_1 \subset L_2 \subset \cdots$ of compact subsets of $f_0(K)$ such that $f_0(K) = \bigcup_{i=1}^{\infty} L_i$ and for every $i \in \mathbb{N}$ the set L_i lies in W_i as well as in the interior L_{i+1}° of L_{i+1} in $f_0(K)$. Let $M_i = f_0^{-1}(L_i^{\circ})$ and $\widetilde{M}_i = f_0^{-1}(L_i)$ for every i. Clearly, M_i are open and $\widetilde{M}_i$ are compact sets in K. To produce the required injective map $g : K \to X$, we shall inductively construct maps $f_i : K \to X$, $i \in \mathbb{N}$, satisfying the following conditions:

(1) $\quad f_i | \widetilde{M}_{i-1} \cup (K \setminus M_{i+1}) = f_{i-1} | \widetilde{M}_{i-1} \cup (K \setminus M_{i+1})$;

(2) $f_i(\widetilde{M}_{i+1}) \subset X_{i+1}$;
(3) f_i is injective on $\widetilde{M}_i$;
(4) $f_i(M_{i+1}) \cap f_i(K \setminus M_{i+1}) = \emptyset$;
(5) $(f_i, f_{i-1}) \prec \mathcal{V}$.

Assume that for some $k \geq 1$ the maps f_i, $i < k$, have been constructed. By (1), $f_{k-1}|K \setminus M_k = f_0|K \setminus M_k$ and thus $f_{k-1}(M_{k+1} \setminus M_k) = f_0(M_{k+1} \setminus M_k) \subset L^\circ_{k+1} \subset X_{k+1}$. Together with (2) this yields $f_{k-1}(M_{k+1}) \subset X_{k+1}$. Let $F = f_{k-1}(\widetilde{M}_{k-1} \cup (K \setminus M_{k+1}))$. By (1), $F = f_{k-1}(\widetilde{M}_{k-1}) \cup f_0(K \setminus M_{k+1}) = f_{k-1}(\widetilde{M}_{k-1}) \cup (f_0(K) \setminus L^\circ_{k+1})$, i.e., F is a closed set in X. Consequently, $X_{k+1} \setminus F$ is an open set in X_{k+1}. It follows from (2)–(4) that $f_{k-1}(M_{k+1} \setminus \widetilde{M}_{k-1}) \subset X_{k+1} \setminus F$. Clearly, the map $f_{k-1}|M_{k+1} \setminus \widetilde{M}_{k-1} : M_{k+1} \setminus \widetilde{M}_{k-1} \to X_{k+1} \setminus F$ is perfect. Since the space X_{k+1} has property $(\mathcal{SU}_n)$, we may select a closed embedding $e : M_{k+1} \setminus \widetilde{M}_{k-1} \to X_{k+1} \setminus F$ so near to $f_{k-1}|M_{k+1} \setminus \widetilde{M}_{k-1}$ that the map $f_k : K \to X$ defined by

$$f_k(x) = \begin{cases} e(x) & \text{if } x \in M_{k+1} \setminus \widetilde{M}_{k-1}, \\ f_{k-1}(x) & \text{if } x \in \widetilde{M}_{k-1} \cup (K \setminus M_{k+1}), \end{cases}$$

is continuous and $\mathcal{V}$-near to f_{k-1}. It is easy to verify that the map f_k so defined satisfies conditions (1)–(5).

Letting $g = \lim_{i\to\infty} f_i : K \to X$ we see that g is an injective continuous map with $(g, f_0) \prec \mathcal{S}t(\mathcal{V})$. Since $(f_0, f) \prec \mathcal{V}$, we get $(g, f) \prec \mathcal{S}t^2(\mathcal{V}) \prec \mathcal{U}$. ■

Lemmas 5, 6, and 8 immediately imply

Lemma 9. *An absolute retract X is $\mathcal{A}_1[n]$-universal for some integer $n \geq 0$ provided X is a σZ_n-space containing a tower $X_1 \subset X_2 \subset \cdots$ having the mapping absorption property for n-dimensional compacta and consisting of Polish ANR's X_i with the disjoint n-cells property.*

We shall apply this lemma to establish the $\mathcal{A}_1[n]$-universality of finite powers of certain subsets of dendrites. Let D be a *dendrite*, i.e., a non-degenerate uniquely arcwise connected Peano continuum (equivalently, a compact one-dimensional absolute retract). By the *order* of a point $x \in D$ we understand the number of connected components of $D \setminus \{x\}$. Points of order 1 in D are called *end-points* of D. For points $x, y \in D$ we denote by $[x, y]$ the unique arc in D with end-points x, y. Also we set $(x, y) = [x, y] \setminus \{x, y\}$. We remark that each subcontinuum A of D is a retract of D; moreover, there is a canonical retraction r_A of D onto A such that for every $x \in D$, $[x, r_A(x)]$ is an irreducible arc between x and A. If $A_1 \subset A_2 \subset \cdots$ is a tower of subcontinua in D such that $\bigcup_{i=1}^\infty A_i$ is dense in D, then the function sequence $(r_{A_i})_{i=1}^\infty$ converges uniformly to the identity map of D.

LEMMA 10. *If D is a dendrite with dense set E of end-points, then the space $(D \setminus E)^{2n+1}$ is $\mathcal{A}_1[n]$-universal for every integer $n \geq 0$.*

Proof. Fix any integer $n \geq 0$. It is easy to see that the space $X = D \setminus E$ is a σ-compact absolute retract of first Baire category. By Lemma 2, the power X^{2n+1} is a σZ_n-space. Let $(A_i)_{i=1}^{\infty} \subset X$ be an increasing sequence of non-degenerate subcontinua in D such that $\bigcup_{i=1}^{\infty} A_i$ is dense in D. Each A_i, being a retract of D, is an absolute retract. As we said, the sequence $(r_{A_i})_{i=1}^{\infty}$ of retractions converges uniformly to the identity map of D. This implies that the sequence $\{r_{A_i}^{2n+1} : D^{2n+1} \to A_i^{2n+1}\}_{i=1}^{\infty}$ of retractions converges uniformly to the identity map of D^{2n+1}. By Lemma 7, the tower $(A_i^{2n+1})_{i=1}^{\infty}$ in X^{2n+1} has the mapping absorption property for n-dimensional compacta. By Lemma 3, each A_i^{2n+1} is a compact absolute retract with the disjoint n-cells property. Applying Lemma 9, we deduce that the space $X^{2n+1} = (D \setminus E)^{2n+1}$ is $\mathcal{A}_1[n]$-universal. ∎

Proof of Theorem 1. Let D be a dendrite such that the set E of end-points of D is dense in D and each point $x \in D$ has order ≤ 3. Let $X = D \setminus E$. Theorem 1 trivially follows from Lemma 10 and

LEMMA 11. *Every locally path-connected space Y of first Baire category contains a closed topological copy of the space $X = D \setminus E$.*

Proof. Let d be any metric on Y and let Z be the completion of Y with respect to this metric. It suffices to construct a continuous function $\varphi : D \to Z$ such that $\varphi^{-1}(Y) = X$ and $\varphi|X$ is injective.

Write $Y = \bigcup_{n=1}^{\infty} Y_n$, where $(Y_n)_{n=1}^{\infty}$ is an increasing sequence of closed nowhere dense subsets of Y. Fix any point $p \in X$ of order 2 in D and choose a sequence $(x_n)_{n=1}^{\infty}$ of points of X such that $X = \bigcup_{n=1}^{\infty} [p, x_n]$. Inductively, we shall construct two sequences $(X_n)_{n=1}^{\infty}$ and $(X'_n)_{n=1}^{\infty}$ of trees in X as follows. Let $X_1 = [p_0, p_1]$, $X'_1 = [p'_0, p'_1]$, where the points p_0, p_1, p'_0, p'_1 are chosen so that $[p, x_1] \subset (p'_0, p'_1) \subset [p'_0, p'_1] \subset (p_0, p_1)$. Assuming that X_n and X'_n have been constructed, choose points p_{n+1}, p'_{n+1} in $X \setminus X_n$ so that $[p, x_{n+1}] \subset [p, p'_{n+1}) \subset [p, p'_{n+1}] \subset [p, p_{n+1})$. Let $X'_{n+1} = X'_n \cup [p, p'_{n+1}]$ and $X_{n+1} = X_n \cup [p, p_{n+1}]$. Because the dendrite D contains at most countably many points of order > 2, we may suppose that all points p'_n have order 2 in D.

Let M_n denote the set of points $y \in D$ such that $p_n \in [p, y)$. Clearly, M_n is open in D and $\overline{M}_n = M_n \cup \{p_n\}$.

Let T_n be the (finite) set of all points of X'_n of order 3 in X_n. We have $T_n \subset T_{n+1}$ for every n. For every n fix a subset $S_n \subset X'_n$ consisting of points of order 2 in D such that $S_n \supset \{p'_0, \ldots, p'_n\} \cup S_{n-1}$ and the following condition is satisfied for the set $R_n = T_n \cup S_n$:

(1) $\operatorname{diam}(L) < 1/n$ for every connected component L of $X'_n \setminus R_n$.

We are going to inductively construct continuous functions $\varphi_n : D \to Y$ and real numbers $c_n > 0$ so that the following conditions are satisfied for every $n \in \mathbb{N}$:

(2) $d(\varphi_n, \varphi_{n+1}) < 2^{-n}$;
(3) $\varphi_n|X_n$ is an embedding;
(4) $\varphi_n = \varphi_n \circ r_{X_n}$;
(5) $\varphi_{n+1}(\bar{L}) = \varphi_n(\bar{L})$ for every $k \leq n$ and every component L of $X'_k \setminus R_{n+1}$;
(6) $d(\varphi_n(\overline{M}_k), \overline{Y}_k) \geq (1 + 1/n)c_k$ for every $k \leq n$.

Since Y_1 is nowhere dense in Y and Y is locally path-connected, Y contains an arc J_1 with an end-point $y_1 \notin Y_1$. Let $\varphi_1|X_1$ be any homeomorphism of X_1 onto J_1 such that $\varphi_1(p_1) = y_1$. Define φ_1 by letting $\varphi_1 = (\varphi_1|X_1) \circ r_{X_1}$ and put $c_1 = \frac{1}{2}d(y_1, \overline{Y}_1) > 0$. Evidently, conditions (3), (4) and (6) are satisfied.

Suppose we have constructed φ_k, c_k, $1 \leq k \leq n$, for some $n \geq 1$. Consider the point $q_{n+1} \in X_n$ such that $[q_{n+1}, p_{n+1}]$ is the irreducible arc between X_n and p_{n+1}. Since $p'_{n+1} \notin X_n$, we have $q_{n+1} \in [p, p'_{n+1})$. We claim that $q_{n+1} \notin R_n = T_n \cup S_n$. Indeed, notice first that q_{n+1} is not of order 3 in X_n (otherwise it would be of order ≥ 4 in D). Consequently, $q_{n+1} \notin T_n$. If q_{n+1} is of order 2 in X_n, then it is of order 3 in D and thus $q_{n+1} \notin S_n$. Finally, if q_{n+1} is of order 1 in X_n, then $q_{n+1} = p_i$ for some $i \leq n$. We claim that $q_{n+1} = p_i \notin X'_n$. Otherwise, by the construction of X'_n, we would have $i < n$ and $[p, p_i] \subset [p, p'_j) \subset X_n$ for some $j \in \{i + 1, \ldots, n\}$, which would imply that $p_i = q_{n+1}$ is not of order 1 in X_n. Since $S_n \subset X'_n$, we have $q_{n+1} \notin S_n$.

If $q_{n+1} \in X'_n$, denote by L_0 the component of $X'_n \setminus R_n$ containing q_{n+1}; if $q_{n+1} \notin X'_n$ let L_0 be the component of $X_n \setminus X'_n$ containing q_{n+1}. We distinguish between two cases:

(a) q_{n+1} is of order two in X_n. Then L_0 contains an arc $A = [u, r]$ such that $A^\circ = (u, r)$ is an open neighborhood of q_{n+1} in X_n.
(b) There exists $i \in \{0, \ldots, n\}$ such that $q_{n+1} = p_i$ is of order one in X_n. Then L_0 contains an arc $A = [u, q_{n+1}]$ such that $A^\circ = (u, q_{n+1}]$ is an open neighborhood of q_{n+1} in X_n.

Let $\alpha = \min\{2^{-n}, (1/n - 1/(n + 1)) \min_{1 \leq k \leq n} c_k\} > 0$. Without loss of generality, we may assume that

(7) $\operatorname{diam} \varphi_n(A) < \frac{1}{2}\alpha$.

Since Y is of first Baire category, $\varphi_n(X_n)$ is nowhere dense in Y. Then the local path-connectedness of Y allows us to find an arc $B \subset Y$ with end-points $\varphi_n(q_{n+1})$ and $y_{n+1} \notin Y_{n+1} \cup \varphi_n(X_n)$ such that

(8) $\operatorname{diam} B < \frac{1}{2} \min\{\alpha, d(\varphi_n(q_{n+1}), \varphi_n(X_n \setminus A^\circ))\}$.

Let $B' = [y_{n+1}, z_{n+1}]$ be an irreducible subarc of B between y_{n+1} and $\varphi_n(X_n)$. Define an embedding φ'_{n+1} of X_{n+1} into $\varphi_n(X_n) \cup B' \subset Y$ as follows. Let $\varphi'_{n+1}|X_n \setminus A^\circ = \varphi_n|X_n \setminus A^\circ$. In case (a), $A \cup [q_{n+1}, p_{n+1}]$ and $\varphi_n(A) \cup B'$ are triodes and we can extend φ'_{n+1} to an embedding of X_{n+1} so that $\varphi'_{n+1}(A) = \varphi_n(A)$ and $\varphi'_{n+1}([q_{n+1}, p_{n+1}]) = B'$. In case (b), let A' be the subarc of $\varphi_n(A)$ with end-points $\varphi_n(u)$ and z_{n+1}. We extend φ'_{n+1} onto X_{n+1} so that $\varphi'_{n+1}|(A \cup [q_{n+1}, p_{n+1}]) = A' \cup B'$.

Let $\varphi_{n+1} = \varphi'_{n+1} \circ r_{X_{n+1}}$. Since $X_n \subset X_{n+1}$, we have $r_{X_n} = r_{X_n} \circ r_{X_{n+1}}$ and if $\varphi_{n+1}(x) \neq \varphi_n(x)$, then $r_{X_{n+1}}(x) \in A \cup [q_{n+1}, p_{n+1}]$, and consequently, both points $\varphi_n(x)$ and $\varphi_{n+1}(x)$ belong to the set $\varphi_n(A) \cup B'$ which has diameter $< \alpha \leq 2^{-n}$ according to (7) and (8). Thus (2) follows.

Let $c_{n+1} = \frac{1}{2} d(y_{n+1}, \overline{Y}_{n+1}) > 0$. If $x \in \overline{M}_{n+1}$, then $r_{X_{n+1}}(x) = p_{n+1}$ and hence $\varphi_{n+1}(x) = y_{n+1}$ satisfies $d(\varphi_{n+1}(x), \overline{Y}_{n+1}) > (1+1/(n+1))c_{n+1}$. Let $k \leq n$ and let $x \in \overline{M}_k$. If $\varphi_{n+1}(x) = \varphi_n(x)$, then $d(\varphi_{n+1}(x), \overline{Y}_k) \geq (1 + 1/n)c_k$ and if $\varphi_{n+1}(x) \neq \varphi_n(x)$, then

$$\begin{aligned} d(\varphi_{n+1}(x), \overline{Y}_k) &\geq d(\varphi_n(x), \overline{Y}_k) - d(\varphi_n(x), \varphi_{n+1}(x)) \\ &\geq \left(1 + \frac{1}{n}\right)c_k - \left(\frac{1}{n} - \frac{1}{n+1}\right)c_k = \left(1 + \frac{1}{n+1}\right)c_k. \end{aligned}$$

Let $k \leq n$ and L be a component of $X'_k \setminus R_k$. Using the fact that R_k contains the points p'_j, $0 \leq j \leq k$, and is contained in R_n, it is easy to show that either $L \cap L_0 = \emptyset$ or $L_0 \subset L$. In both cases, the construction of the map φ_{n+1} guarantees that $\varphi_{n+1}(\overline{L}) = \varphi_k(\overline{L})$.

According to (2) the sequence (φ_n) converges uniformly to a continuous map $\varphi : D \to Z$. Let x, x' be two distinct points of X. Since $X = \bigcup_{n=1}^\infty X'_n$, condition (1) allows us to find an integer m and components L and L' of $X'_m \setminus R_m$ such that $x \in \overline{L}$, $x' \in \overline{L'}$ and $\overline{L} \cap \overline{L'} = \emptyset$. It follows from (5) that $\varphi(x) \in \varphi_m(\overline{L})$ and $\varphi(x') \in \varphi_m(\overline{L'})$. By (3), the sets $\varphi_m(\overline{L})$ and $\varphi_m(\overline{L'})$ are disjoint and hence $\varphi(x) \neq \varphi(x')$ and $\varphi|X$ is injective. The preceding arguments also give $\varphi(X) \subset Y$.

Let $x \in E$. The equality $[p, x) = \bigcup_{n=1}^\infty (X_n \cap [p, x))$ implies the existence of infinitely many indices n_k such that $x \in M_{n_k}$. For every such n_k, (6) implies $d(\varphi(x), \overline{Y}_{n_k}) \geq c_{n_k} > 0$. Since the sequence (Y_n) is increasing, $\varphi(x) \notin \bigcup_{n=1} \overline{Y}_n \supset Y$. This yields $\varphi^{-1}(Y) = X$. ∎

Proof of Theorem 2. Let D be a dendrite with dense set E of end-points. It is not difficult to construct an increasing sequence $(D_i)_{i=1}^\infty$ of nowhere dense subdendrites in D such that the union $A = \bigcup_{i=1}^\infty D_i$ is dense in D and each dendrite D_i has dense set of end-points. The space A, being a connected subspace of D, is an absolute retract. Since each D_i is nowhere dense in D, A is an absolute retract of first Baire category. We claim that the power A^{n+1} is $\mathcal{A}_1[n]$-universal for every integer $n \geq 0$.

Fix any integer $n \geq 0$. By Lemma 2, the power A^{n+1} is a σZ_n-space and by Lemma 4, each D_i^{n+1} is a compact absolute retract with the disjoint n-cells property. Similarly to the proof of Lemma 10, it can be verified that the tower $D_1^{n+1} \subset D_2^{n+1} \subset \ldots \subset A^{n+1}$ has the mapping absorption property for n-dimensional compacta. Therefore it is legitimate to apply Lemma 9 to conclude that the space A^{n+1} is $\mathcal{A}_1[n]$-universal. ∎

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