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JORDAN ∗-DERIVATION PAIRS ON STANDARD OPERATOR ALGEBRAS AND RELATED RESULTS

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Abstract. Motivated by Problem 2 in [2], Jordan $*$-derivation pairs and n-Jordan $*$-mappings are studied. From the results on these mappings, an affirmative answer to Problem 2 in [2] is given when $E = F$ in (1) or when $\mathcal{A}$ is unital. For the general case, we prove that every Jordan $*$-derivation pair is automatically real-linear. Furthermore, a characterization of a non-normal prime $*$-ring under some mild assumptions and a representation theorem for quasi-quadratic functionals are provided.

1. Introduction. Let R be a $*$-ring. An additive mapping $D : R \to R$ is called a *Jordan $*$-derivation* if $D(x^2) = D(x)x^* + xD(x)$ $(x \in R)$. A Jordan $*$-derivation of the form $D_a(x) = ax^* - xa$ for some $a \in R$ is called *inner.* The study of Jordan $*$-derivations has been motivated by the problem of the representability of quasi-quadratic functionals by sesquilinear ones (see, for instance, [4], [5], [8] and the references there). It turns out that the solvability of the latter problem is intimately connected with the structure of Jordan $*$-derivations [6], [7].

Later, Zalar introduced a more general notion of Jordan $*$-derivation pairs. In [2], Molnár generalized it further. Let $\mathcal{M}$ be an R-bimodule. He calls the additive pair (E, F) a *Jordan $*$-derivation pair* if $E, F : R \to \mathcal{M}$ satisfy the system of equations

$$(1)\qquad \begin{cases} E(x^3) = E(x)x^{*2} + xF(x)x^* + x^2E(x), \\ F(x^3) = F(x)x^{*2} + xE(x)x^* + x^2F(x) \end{cases}$$

for all $x \in R$. A Jordan $*$-derivation pair in this note is in the sense of (1). We also call Jordan $*$-derivation pairs of the form $E_{a,b}(x) = ax^* - xb$, $F_{a,b}(x) = bx^* - xa$ for some $a, b \in \mathcal{M}$ *inner*. Here we should mention that if in the above $\mathcal{M}$ is a $*$-ring and R is a subring of $\mathcal{M}$, then R need not be *self-adjoint*, i.e., $x \in R$ implies $x^* \in R$. This convention is also applicable to other $*$-mappings in subsequent sections. We use this without any further explanations.

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For a given real or complex Hilbert space $\mathcal{H}$, throughout this note, by $\mathcal{B}(\mathcal{H})$ we mean the algebra of all bounded linear operators on $\mathcal{H}$. We denote by $\mathcal{F}(\mathcal{H})$ the subalgebra of all bounded finite-rank operators. A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is called *standard* provided that $\mathcal{A}$ contains $\mathcal{F}(\mathcal{H})$. In [2], Molnár gave a class of *complex* $*$-algebras $\mathcal{A}$ such that every Jordan $*$-derivation pair from $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$ can be represented by some two double centralizers (refer to [2] for the definition). As a result, he proved that every Jordan $*$-derivation pair from a standard operator algebra $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ is inner, where $\mathcal{H}$ is a *complex* Hilbert space. Furthermore, he proposed two open problems. One of them is whether the above result holds for real Hilbert spaces of dimension greater than 1. (The necessity of $\dim \mathcal{H} > 1$ can be seen from [5].) Let us state this problem itself more precisely:

Problem (Problem 2 of [2]). Let $\mathcal{H}$ be a real Hilbert space of dimension greater than 1. Suppose (E, F) is a Jordan $*$-derivation pair from a standard operator algebra $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Are there $S, T \in \mathcal{B}(\mathcal{H})$ such that $E(A) = SA^* - AT$ and $F(A) = TA^* - AS$ for all $A \in \mathcal{A}$?

As said in [2], answering the above question would be interesting and may turn out to be rather difficult.

In this note, motivated by the results in [1], [2], [8] and inspired by the idea of [3], we make some contributions to solving the above problem and give some related results. More precisely, we first study n-Jordan $*$-mappings ($n \geq 3$), another natural generalization of Jordan $*$-derivations, and prove that an n-Jordan $*$-mapping is a Jordan $*$-derivation in some cases (Proposition 2.1 and Theorem 2.3).

As a result, an affirmative answer to Problem 2 of [2] is given when $E = F$ in (1). Under some suitable conditions, we also show that if Jordan $*$-derivations are inner then so are Jordan $*$-derivation pairs (Proposition 3.1). As an application, we solve Problem 2 in [2] when $\mathcal{A}$ is unital (Corollary 3.2). For the general case, we prove that every Jordan $*$-derivation pair is automatically real-linear (Theorem 3.3). As two more applications of Proposition 3.1, a characterization of a non-normal prime $*$-ring under some mild assumptions and a representation theorem for quasi-quadratic functionals are provided (Proposition 4.1 and Corollary 4.2).

2. n-Jordan $*$-mappings. We start this section with a simple observation. Follow the notation of the Problem and set $G := E+F$ and $H := E-F$. It is easy to check that they satisfy the operator equations

$$G(A^3) = G(A)A^{*2} + AG(A)A^* + A^2G(A), \tag{2}$$

$$H(A^3) = H(A)A^{*2} - AH(A)A^* + A^2H(A), \tag{3}$$

for all $A \in \mathcal{A}$. So, to solve Problem 2 of [2], it is enough to find the solutions to equations (2) and (3). In this section, we take care of equation (2). Actually, we consider a natural generalization of (2).

Let R be a $*$-ring, $\mathcal{M}$ an R-bimodule, and $n \geq 3$. Consider additive mappings $J : R \to \mathcal{M}$ satisfying

$$J(x^n) = \sum_{i=0}^{n-1} x^i J(x) x^{*(n-1-i)} \qquad (x \in R). \tag{4}$$

We call a mapping satisfying (4) an *n-Jordan ∗-mapping*. Clearly, the mapping G in (2) is nothing but a 3-Jordan $*$-mapping. Furthermore, a Jordan $*$-derivation is an n-Jordan $*$-mapping for any $n \geq 3$. This, in fact, can be proved by simple induction once one notices the identity $J(x^n) = J(x \cdot x^{n-2} \cdot x)$ and the fact that a Jordan $*$-derivation is a 3-Jordan $*$-mapping (see, e.g., [5]). In this section, we show that the converse is also true in some cases. First of all, the following result says this is the case when R is a unital real or complex $*$-algebra.

PROPOSITION 2.1. *If $\mathcal{A}$ is a unital real or complex ∗-algebra and $\mathcal{M}$ is a unitary $\mathcal{A}$-bimodule, then every n-Jordan ∗-mapping $J : \mathcal{A} \to \mathcal{M}$ is a Jordan ∗-derivation.*

Proof. It follows from (4) that

$$J((x+my)^n) = \sum_{i=0}^{n-1} (x+my)^i J(x+my)(x+my)^{*(n-1-i)}$$

for all $m \in \mathbb{Z}$. This can be written in the form $\sum_{i=0}^{n} c_i m^i$ with coefficients $c_i \in \mathcal{M}$. Since it holds for all integers m, each c_i must be 0. In particular, $c_1 = 0$ gives

$$\begin{aligned} \sum_{k=0}^{n-1} J(x^k y x^{n-k-1}) \\ &= \sum_{k=0}^{n-1} \Big\{ \sum_{l=0}^{k-1} x^l J(x) x^{*(k-l-1)} y^* x^{*(n-k-1)} \\ &\quad + x^k J(y) x^{*(n-k-1)} + \sum_{l=0}^{n-k-2} x^k y x^l J(x) x^{*(n-k-l-2)} \Big\}. \end{aligned} \tag{5}$$

Let $x = y = 1$ in (5) to get $nJ(1) = n^2 J(1)$, which implies that $J(1) = 0$. Then putting $y = 1$ in (5) and using $J(1) = 0$, we have

$$nJ(x^{n-1}) = \sum_{k=0}^{n-1} \Big\{ \sum_{l=0}^{k-1} x^l J(x) x^{*(n-l-2)} + \sum_{l=0}^{n-k-2} x^{(k+l)} J(x) x^{*(n-k-l-2)} \Big\},$$

that is,

$$nJ(x^{n-1}) = n\sum_{l=0}^{n-2} x^l J(x) x^{*(n-2-l)}.$$

Thus the above J is just an $(n-1)$-Jordan $*$-mapping. The proof is now completed by applying the above arguments successively. ∎

It follows from Proposition 2.1 and Theorem in [2] that every n-Jordan $*$-mapping is inner when $\mathcal{A}$ is a unital complex $*$-algebra. That is, we have

Corollary 2.2. *Suppose that $\mathcal{A}$ is a unital complex $*$-algebra and that $\mathcal{M}$ is a unitary $\mathcal{A}$-bimodule. Then every n-Jordan $*$-mapping is of the form $J(x) = ax^* - xa$ for some $a \in \mathcal{M}$.*

Proof. Indeed, Proposition 2.1 tells us that J is a Jordan $*$-derivation. So, applying Theorem in [2], we get $J(x) = T(x^*) - S(x)$ for some double centralizer (T, S), where $T, S : \mathcal{A} \to \mathcal{M}$. Notice that $\mathcal{A}$ is unital. It is easy to see that T and S satisfy $T(x) = T(1)x$, $S(x) = xS(1)$ for all $x \in R$, and $T(1) = S(1) := a$. Therefore, $J(x) = ax^* - xa$, as required. ∎

If the $\mathcal{A}$ in the last proposition is non-unital, the situation is more complicated. However, we have the following

Theorem 2.3. *Let $\mathcal{H}$ be a real or complex Hilbert space of dimension greater than* 1*. Assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a standard operator algebra. Then every n-Jordan $*$-mapping $J : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is inner*: $J(A) = TA^* - AT$ $(A \in \mathcal{A})$ *for some $T \in \mathcal{B}(\mathcal{H})$.*

Proof. If $\dim \mathcal{H} < \infty$, then we have $\mathcal{A} = \mathcal{B}(\mathcal{H})$ for any standard operator algebra $\mathcal{A}$. Thus $\mathcal{A}$ is unital. Theorem 2.3 easily follows from Proposition 2.1 and Theorem in [4] (or Theorem 2.3 in [5]).

Now there is no loss of generality in assuming that $\dim \mathcal{H} = \infty$. Suppose first that $\mathcal{A} = \mathcal{F}(\mathcal{H})$. The following idea is very much inspired by a result of Šemrl in [3]. Let $A \in \mathcal{F}(\mathcal{H})$. Suppose that $\operatorname{Im} A$ is spanned by a set of orthonormal vectors $v_1, \ldots, v_t$, $t\ (\in \mathbb{N}) < \infty$. It is well known that, by Zorn's lemma, the orthonormal set $\{v_1, \ldots, v_t\}$ can be extended to an orthonormal basis $\{v_1, \ldots, v_t\} \cup \{v_\alpha : \alpha \in \Lambda\}$ of $\mathcal{H}$. We now pick an arbitrary pair $\{\beta, \gamma\} \subset \{1, \ldots, t\} \cup \Lambda$. Let us choose a countable set

$$\{v_m : m \in \mathbb{N}\} \subset \{v_1, \ldots, v_t\} \cup \{v_\alpha : \alpha \in \Lambda\}$$

so that $\{v_\beta, v_\gamma\} \subset \{v_m : m \in \mathbb{N}\}$. Let P_m be the orthogonal projection onto $\operatorname{span}\{v_1, \ldots, v_m\}$ $(m \in \mathbb{N})$.

By M_m we mean the algebra of $m \times m$ matrices. Define a mapping $\Phi_m : M_m \to \mathcal{B}(\mathcal{H})$ by

$$\begin{cases} \Phi_m((a_{ij}))\Big(\sum_{k=1}^{\infty} t_k v_k\Big) = \sum_{i=1}^{m} \Big(\sum_{k=1}^{m} a_{ik} t_k\Big) v_i, \\ \Phi_m((a_{ij}))|_V = 0 \end{cases}$$

for all $(a_{ij}) \in M_m$, where V is the orthogonal complement of the subspace spanned by $\{v_m : m \in \mathbb{N}\}$. It is not hard to verify that $J_m : M_m \to M_m$ given by

$$J_m((a_{ij})) = \Phi_m^{-1}(P_m J(\Phi_m((a_{ij})))P_m) \tag{6}$$

is an n-Jordan $*$-mapping. Since M_m is a unital $*$-algebra, by Proposition 2.1 and Theorem in [4] (also refer to the beginning of the proof), one can find matrices $(c_{ij}^m) \in M_m$ such that

$$J_m((a_{ij})) = (c_{ij}^m)(a_{ij})^* - (a_{ij})(c_{ij}^m) \tag{7}$$

for all $(a_{ij}) \in M_m$. Moreover, the above matrices (c_{ij}^m) can be uniquely chosen so that

$$c_{ij}^m = c_{ij}^k \quad \text{when } \max\{i, j\} \le \min\{m, k\}. \tag{8}$$

Indeed, for any $(a_{ij}) \in M_m$, let us pick $(b_{ij}) \in M_{m+1}$ as follows:

$$(b_{ij}) = \begin{pmatrix} (a_{ij}) & 0 \\ 0 & 0 \end{pmatrix}.$$

For convenience, let

$$(c_{ij}^{m+1})_m := \begin{pmatrix} c_{11}^{m+1} & \cdots & c_{1m}^{m+1} \\ \cdots & \cdots & \cdots \\ c_{m1}^{m+1} & \cdots & c_{mm}^{m+1} \end{pmatrix}.$$

Comparing $J_{m+1}((b_{ij}))$ and $J_m((a_{ij}))$ we get

$$(c_{ij}^{m+1})_m(a_{ij})^* - (a_{ij})(c_{ij}^{m+1})_m = (c_{ij}^m)(a_{ij})^* - (a_{ij})(c_{ij}^m).$$

Thus we can let $(c_{ij}^{m+1})_m = (c_{ij}^m)$. To get (8) it is enough to apply the above procedure successively.

In view of (6)–(8), we get $P_m(J(A^2) - J(A)A^* - AJ(A))P_m = 0$ for all $m \ge t$. Thus

$$P_\beta(J(A^2) - J(A)A^* - AJ(A))P_\gamma = 0.$$

Therefore $J(A^2) = J(A)A^* + AJ(A)$ according to the arbitrary choice of β, γ. Invoking Theorem in [4], we can find $T \in \mathcal{B}(\mathcal{H})$ such that $J(A) = TA^* - AT$.

It remains to prove the case where $\mathcal{A}$ is an arbitrary standard operator algebra. To this end, first notice that $J(A) = TA^* - AT$ for some $T \in \mathcal{B}(\mathcal{H})$ also defines an n-Jordan $*$-mapping from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. Applying the approach in the proof of Corollary 2 in [2], it suffices to prove that any n-Jordan $*$-derivation vanishing on $\mathcal{F}(\mathcal{H})$ is zero on $\mathcal{A}$. Letting $y = A \in \mathcal{A}$ and $x = P$ in (5) where $P \in \mathcal{F}(\mathcal{H})$ is an arbitrary projection, one can easily get

$$0 = J(A)P + (n-2)PJ(A)P + PJ(A).$$

Multiplying the above equation by P from both the left and the right, we get $nPJ(A)P = 0$. It follows that $J(A) = 0$ since P is an arbitrary finite-rank projection. This completes the proof. ∎

REMARK 2.4. Applying Theorem 2.3 for $n = 3$, we have solved the Problem when $E = F$.

3. Jordan $*$-derivation pairs. Let us first state one useful proposition which, basically, says that under some conditions if Jordan $*$-derivations are inner then so are Jordan $*$-derivation pairs (cf. Proposition 2.4 in [8]).

PROPOSITION 3.1. *Let R be a $*$-ring with identity 1 and elements $1/2$, $1/3$. Suppose that every Jordan $*$-derivation from R to a unitary R-bimodule $\mathcal{M}$ is inner. If (E, F), where $E, F : R \to \mathcal{M}$, is a Jordan $*$-derivation pair, then (E, F) is also inner.*

Proof. As before, let $G := E + F$ and $H := E - F$. So G and H satisfy equations (2) and (3), respectively. Since $1/2 \in R$, to obtain E and F it suffices to solve (2) and (3) for G and H, respectively. Using the same procedure as in Proposition 2.1, we deduce that G is a Jordan $*$-derivation. By our assumption, there is a constant $c \in \mathcal{M}$ such that

$$G(x) = cx^* - xc. \tag{9}$$

Similarly to getting (5), one can show that

$$\begin{aligned} H(x^2y + xyx + yx^2) = {} & H(x)x^*y^* - xH(x)y^* + x^2H(y) \\ & + H(x)y^*x^* - xH(y)x^* + xyH(x) \\ & + H(y)x^{*2} - yH(x)x^* + yxH(x). \end{aligned} \tag{10}$$

Setting $x = 1$ in (10) results in

$$2H(y) = H(1)y^* + yH(1). \tag{11}$$

Combining (9) and (11) yields

$$\begin{cases} E(x) = (c + E(1))x^*/2 - x(c - E(1))/2, \\ F(x) = (c - E(1))x^*/2 - x(c + E(1))/2. \end{cases} \tag{12}$$

Thus the pair (E, F) is inner, which completes our proof. ∎

We are now ready to give some applications of Proposition 3.1. First of all, an affirmative answer to Problem 2 of [2] will be given when $\mathcal{A}$ is unital. This is done in the following

COROLLARY 3.2. *Let $\mathcal{H}$ be a real Hilbert space of dimension greater than 1 and suppose that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a unital standard operator algebra. Then every Jordan $*$-derivation pair from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ is inner.*

Proof. This follows from Proposition 3.1 and Theorem in [4] directly. ∎

By Corollary 3.2, to solve Problem 2 of [2], it remains to study the case where $\mathcal{A}$ is not unital. As we have seen, it is sufficient to find G and H in (2) and (3). It follows from Theorem 2.3 that we have obtained G. From the proof of Theorem 2.3, we see that we have eventually converted the study of n-Jordan ∗-mappings (in particular $n = 3$) to that of well studied Jordan ∗-derivations. But it seems that this approach cannot be applicable to studying those mappings satisfying equation (3). Hence, to solve (3) in the non-unital case is still open. However, the next theorem says that Jordan ∗-derivation pairs are automatically real-linear in the general case.

THEOREM 3.3. *Let $\mathcal{H}$ be a real Hilbert space with $\dim \mathcal{H} > 1$ and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a standard operator algebra. Then, for every Jordan ∗-derivation pair $(E, F) : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, E and F are both real-linear.*

Proof. If $\dim \mathcal{H} < \infty$, then clearly $\mathcal{A} = \mathcal{B}(\mathcal{H})$ as before. By Corollary 3.2 every Jordan ∗-derivation pair on $\mathcal{A}$ is inner. Thus the assertion follows naturally. Below we can assume $\dim \mathcal{H} = \infty$.

Suppose first that $\mathcal{A} = \mathcal{F}(\mathcal{H})$. Following the notation in the proof of Theorem 2.3, it is routine to verify that (E_m, F_m) defined by

$$E_m((a_{ij})) = \Phi_m^{-1}(P_m E(\Phi_m((a_{ij})))P_m), \tag{13}$$

$$F_m((a_{ij})) = \Phi_m^{-1}(P_m F(\Phi_m((a_{ij})))P_m) \tag{14}$$

is a Jordan ∗-derivation pair from M_m to M_m. It follows from Proposition 3.1 that

$$E_m((a_{ij})) = (c_{ij}^m)(a_{ij})^* - (a_{ij})(d_{ij}^m), \tag{15}$$

$$F_m((a_{ij})) = (d_{ij}^m)(a_{ij})^* - (a_{ij})(c_{ij}^m) \tag{16}$$

for some (c_{ij}^m) and (d_{ij}^m) in M_m. Using an argument completely similar to that in the proof of Theorem 2.3, we can choose unique matrices (c_{ij}^m) and (d_{ij}^m) such that

$$c_{ij}^m = c_{ij}^k, \quad d_{ij}^m = d_{ij}^k \quad \text{when } \max\{i, j\} \le \min\{m, k\}. \tag{17}$$

According to (13)–(17), we obtain

$$P_\beta(E(\lambda A) - \lambda E(A))P_\gamma = 0, \qquad P_\beta(F(\lambda A) - \lambda F(A))P_\gamma = 0$$

for all $\lambda \in \mathbb{R}$. Therefore E and F are both real-linear:

$$E(\lambda A) = \lambda E(A), \quad F(\lambda A) = \lambda F(A) \quad (\lambda \in \mathbb{R}, A \in \mathcal{F}(\mathcal{H})). \tag{18}$$

Now let $\mathcal{A}$ be an arbitrary standard operator algebra. In view of (18), both $G := E + F$ and $H := E - F$ are real-linear on $\mathcal{F}(\mathcal{H})$. Now replacing x by $P \in \mathcal{F}(\mathcal{H})$, an arbitrary projection, and y by λA with $\lambda \in \mathbb{R}$ and $A \in \mathcal{A}$ in (10), we have

$$P(H(\lambda A) - \lambda H(A)) - P(H(\lambda A) - \lambda H(A))P + (H(\lambda A) - \lambda H(A))P = 0.$$

Multiplying the above equation by P from the left and then from the right, we obtain $P(H(\lambda A) - \lambda H(A))P = 0$. Since P is an arbitrary finite-rank operator, we get $H(\lambda A) = \lambda H(A)$. Similarly, we have $G(\lambda A) = \lambda G(A)$ for all $A \in \mathcal{A}$. Thus we see that E and F are both real-linear on $\mathcal{A}$, which completes the proof. ∎

4. Further results. In this short section, we give two more applications of Proposition 3.1. The first one describes a characterization of a non-normal prime $*$-ring with identity 1 and elements $1/2, 1/3$. This result is motivated by the main result Theorem 3 in [1]. The second one is related to the representation of quasi-quadratic functionals. Details on quasi-quadratic functionals can be found in [4], [5], [8]. Recall that a $*$-ring R is called a *normal ring* provided that every element x in R is normal (that is, $xx^* = x^*x$), and that a mapping f of any ring R into itself is said to be *commuting* provided that $[f(x), x] = 0$ ($x \in R$). A Jordan $*$-derivation pair (E, F) on a $*$-ring R is *commuting* if both E and F are commuting.

PROPOSITION 4.1. *Let R be a non-commutative prime $*$-ring with identity* 1 *and elements* $1/2$ *and* $1/3$. *Then R is normal if and only if there exists a non-zero commuting Jordan $*$-derivation pair.*

Proof. If R is normal, let $E(x) = x^* - 2x$ and $F(x) = 2x^* - x$ for all $x \in R$. Then clearly (E, F) is a non-zero commuting Jordan $*$-derivation pair on R.

Let R be a prime $*$-ring with identity 1 and elements $1/2$ and $1/3$. To show the converse, it is equivalent to prove that if R is not normal, then every commuting Jordan $*$-derivation pair must be zero. For this, suppose that (E, F) is a commuting Jordan $*$-derivation pair on R. As shown in Proposition 3.1, the mapping $G := E + F$ is a Jordan $*$-derivation on R. Moreover, clearly it is commuting since both E and F are. By Theorem 3 in [1], we have

$$G(x) = 0 \quad (x \in R).$$

According to (12), we get

$$E(x) = ax^* + xa, \quad F(x) = -(ax^* + xa) \quad (x \in R) \tag{19}$$

where, in fact, $a = E(1)/2$. Since E is commuting, i.e., $[E(x), x] = 0$ ($x \in R$), linearizing x we get $[x, E(y)] = [E(x), y]$. Hence it follows from (19) that

$$[x, ay^* + ya] = [ax^* + xa, y] \quad (x, y \in R). \tag{20}$$

Let $y = 1$ in (20) to get $[x, 2a] = 0$. This implies $[x, a] = 0$ for all $x \in R$ as $1/2 \in R$. Thus a is in the centre $Z(R)$ of R. It follows that $a[x, y^*] = a[x^*, y]$ ($x, y \in R$). In particular, substituting $y = x$ in the identity just got yields

$a[x, x^*] = a[x^*, x]$, i.e., $2a[x, x^*] = 0$, which clearly implies that

$$a[x, x^*] = 0 \quad (x \in R).$$

Hence $Ra[x, x^*] = \{0\}$. Thus $aR[x, x^*] = \{0\}$ as $a \in Z(R)$. Since R is prime and not normal, we have $a = 0$. The proof is completed directly from (19). ∎

From the proof of the representation theorem in Section 4 of [8], one can easily get

COROLLARY 4.2. *Let R be a $*$-ring with* 1 *and elements* 1/2, 1/3. *Suppose that $ax^* = xa$ $(x \in R)$ implies $a = 0$. If every Jordan $*$-derivation pair on R is inner, then every quasi-quadratic functional on a unitary R-bimodule $\mathcal{M}$ can be represented by some sesquilinear form.*

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