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A q-ANALOGUE OF COMPLETE MONOTONICITY

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Abstract. The aim of this paper is to give a q-analogue for complete monotonicity. We apply a classical characterization of Hausdorff moment sequences in terms of positive definiteness and complete monotonicity, adapted to the q-situation. The method due to Maserick and Szafraniec that does not need moments turns out to be useful. A definition of a q-moment sequence appears as a by-product.

The aim of this paper is to find a q-analogue of complete monotonicity and relate it to an appropriate notion of q-positive definiteness as in the classical case. It turns out that the q-positive definiteness so defined is related to the one that has already appeared in [5] in the context of the q-oscillator. Let us mention that, in the vast literature concerning q-commuting variables, the paper [3] seems to be close to ours in flavour.

In the classical case positive definiteness and complete monotonicity are related via the Hausdorff moment problem. However, in the q-situation there is no standard understanding of moment problems. To omit this obstacle we use the connection established by Maserick and Szafraniec. More precisely, the paper [4] contains a proof (of the already known result) that a sequence $\{a_n\}_n$ is completely monotonic if and only if both $\{a_n\}_n$ and $\{a_n - a_{n+1}\}_n$ are positive definite. The novelty of the proof is that it avoids any integral representation of $\{a_n\}_n$. Following that method we get a characterization of q-completely monotonic sequences in terms of q-positive definiteness.

In Section 1 we collect the basic definitions and results from [4] that we will use. The definition of q-positive definite sequences is given and discussed in Sections 2 and 3. In Section 4 we define q-completely monotonic sequences and characterize them in terms of q-positive definiteness. Section 5 deals with relations between the classical properties and their q-analogues.

We set $\mathbb{N} = \{0, 1, 2, \ldots\}$. Whenever a sequence appears it is understood that its indices range from 0 to $+\infty$. Unless otherwise stated, we consider $q > 0$.

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1. Preliminaries. Let $\mathcal{R}$ be a commutative algebra with identity e and involution *. Call a subset $\tau \subset \mathcal{R}$ *admissible* if the following conditions are satisfied:

(1) $x^* = x$ for all $x \in \tau$;
(2) $1 - x \in \mathrm{Alg}^+(\tau)$ for all $x \in \tau$, where $\mathrm{Alg}^+(\tau)$ is the set of all nonnegative combinations of (finite) products of members of τ;
(3) $\mathcal{R} = \mathrm{Alg}(\tau)$, i.e. every $x \in \mathcal{R}$ is a combination of (finite) products of members of τ.

Let $\tau \subset \mathcal{R}$ be admissible. A linear functional f on $\mathcal{R}$ is called *τ-positive* if $f(x) \geq 0$ for all $x \in \mathrm{Alg}^+(\tau)$. Following standard conventions, f is called *positive* if $f(x^*x) \geq 0$ for all $x \in \mathcal{R}$. If f is positive then we set

$$|x|_f^2 = \sup_{y \in \mathcal{R}} \frac{f(x^*xy^*y)}{f(y^*y)}$$

$(0/0 = 0)$ and we call f *bounded* whenever $|x|_f < \infty$ for all $x \in \mathcal{R}$.

For all $x \in \mathcal{R}$ define the *shift operator* E_x on the set of all linear functionals on $\mathcal{R}$ by

$$E_x f(y) = f(xy), \qquad y \in \mathcal{R}.$$

THEOREM 1.1 (Maserick and Szafraniec [4]).

(1) *Let f be a bounded positive linear functional on $\mathcal{R}$. If $\tau \subset \mathcal{R}$ is admissible and $E_x f$ is positive for all $x \in \tau$, then f is τ-positive.*
(2) *If f is τ-positive for an admissible τ, then f is positive and bounded and $E_x f$ is positive for all $x \in \tau$.*

Take $\mathcal{R} = \mathrm{Lin}\{E_m;\, m \in \mathbb{N}\}$ and $\tau = \{E_1, I - E_1\}$, where $(E_m\mu)(n) := \mu(n+m)$ for a sequence $\{\mu(n)\}_n$. Then the above theorem implies a classical result: a sequence $\{\mu(n)\}_n$ is completely monotonic if and only if $\{\mu(n)\}_n$ and $\{\mu(n) - \mu(n+1)\}_n$ are positive definite (see [4] for details).

One may also apply the theorem to any other admissible set τ provided it generates $\mathcal{R}$. In particular, for $\tau = \{E_m, I - E_m;\, m \in \mathbb{N}\}$ we get the following implication.

COROLLARY 1.2. *If $\{\mu(n)\}_n$ is completely monotonic then $\{\mu(n+m)\}_n$ and $\{\mu(n) - \mu(n+m)\}_n$ (for all $m \in \mathbb{N}$) are positive definite.*

2. q-positive definite sequences. Recall that a (Hamburger) *moment sequence* is a sequence $\{\mu(n)\}_n$ that has an integral representation of the form

$$\mu(n) = \int_{\mathbb{R}} t^n \, d\mu(t), \qquad n \in \mathbb{N},$$

where μ is a Borel measure on $\mathbb{R}$. According to the Hamburger theorem (cf. [9] or [6]), a sequence $\{\mu(n)\}_n$ is a (Hamburger) moment sequence if and only if it is positive definite (PD), i.e. for every $n \in \mathbb{N}$ and any scalars $\alpha_1, \ldots, \alpha_n$,

$$\sum_{i,j=0}^{n} \alpha_i \alpha_j \mu(i+j) \geq 0.$$

The q-analogue of positive definite sequences is the following.

DEFINITION 1. A sequence $\{\varphi(n)\}_n$ is called *q-positive definite* (qPD) if for all $n \in \mathbb{N}$ and all scalars $\alpha_1, \ldots, \alpha_n$,

$$\sum_{i,j=0}^{n} q^{-ij} \alpha_i \alpha_j \varphi(i+j) \geq 0.$$

REMARK. A sequence is q-positive definite in the sense of Definition 1 if and only if it is q^{-1}-positive definite in the sense of the definition given by Ôta and Szafraniec [5].

3. q-shifts. The aim of this section is to express q-positive definiteness in terms of some properties of the corresponding linear functional. For this, let $\mathcal{F}$ be the linear space of all real sequences with the identity involution $\{\varphi(n)\}_n^* = \{\varphi(n)\}_n$. For each sequence $\{\varphi(n)\}_n \in \mathcal{F}$ define

$$F_m \varphi(k) := q^{-mk} \varphi(k+m).$$

The operator $F_m : \mathcal{F} \to \mathcal{F}$ will be called the *q-shift*.

PROPOSITION 3.1. *Let $\mathcal{R} = \mathrm{Lin}\{F_m;\, m \in \mathbb{N}\}$. Then $\mathcal{R}$ is a commutative algebra with identity $I = F_0$ and involution $F_i^* = F_i$.*

Proof. By an easy calculation we get

$$F_m F_n = q^{-nm} F_{m+n} = F_n F_m. \quad \blacksquare \tag{1}$$

Since any linear functional f on $\mathcal{R}$ is uniquely determined by its values on the basis $\{F_m;\, m \in \mathbb{N}\}$ via the formula

$$f\Big(\sum \alpha_n F_n\Big) = \sum \alpha_n f(F_n),$$

f can be identified with the sequence $\{\varphi(n)\}_n$ where

$$\varphi(n) = f(F_n).$$

PROPOSITION 3.2. *A linear functional f on $\mathcal{R}$ is positive if and only if the sequence $\{\varphi(n)\}_n$ is q-positive definite.*

Proof. It is sufficient to note that for $p = \sum \alpha_i F_i$ we have

$$f(p^* p) = \sum_{i,j=0}^{n} \alpha_i \alpha_j f(F_i^* F_j) = \sum_{i,j=0}^{n} \alpha_i \alpha_j F_i F_j \varphi(0) = \sum_{i,j=0}^{n} \alpha_i \alpha_j \, q^{-ij} \varphi(i+j). \quad \blacksquare$$

4. q-complete monotonicity. Recall that a sequence $\{\varphi(n)\}_n$ is called *completely monotonic* (CM) if

$$\sum_{m=0}^{k}(-1)^{m+k}\binom{k}{m}\varphi(n+k-m)\geq 0.$$

Another way to say this is that the (classical) mth differences, i.e.

$$\begin{aligned}&\Delta_0^{(1)}\varphi(n_0)=\varphi(n_0),\\&\Delta_{m+1}^{(1)}\varphi(n_0;n_1,\ldots,n_{m+1})\\&\qquad\qquad=\Delta_m^{(1)}\varphi(n_0;n_1,\ldots,n_m)-\Delta_m^{(1)}\varphi(n_0+n_{m+1};n_1,\ldots,n_m),\end{aligned}$$

are nonnegative for all $m\in\mathbb{N}$ and $n_0,\ldots,n_m\in\mathbb{N}$ (cf. [9], [1]).

For a sequence $\{\varphi(n)\}_n$ we define a q-generalization of mth differences by the formula

$$\begin{aligned}&\Delta_0\varphi(n_0)=\Delta_0^{(q)}\varphi(n_0)=\varphi(n_0),\\&\Delta_{m+1}\varphi(n_0;n_1,\ldots,n_{m+1})=\Delta_{m+1}^{(q)}\varphi(n_0;n_1,\ldots,n_{m+1})\\&\quad=\Delta_m\varphi(n_0;n_1,\ldots,n_m)-q^{-n_0n_{m+1}}\Delta_m\varphi(n_0+n_{m+1};n_1,\ldots,n_m).\end{aligned}$$

Definition 2. The sequence $\{\varphi(n)\}_n$ is called *q-completely monotonic* (qCM) if $\Delta_m\varphi(n_0;n_1,\ldots,n_m)\geq 0$ for all $m\in\mathbb{N}$ and $n_0,\ldots,n_m\in\mathbb{N}$.

The q-complete monotonicity can be expressed by means of q-shifts. Note that for $q\to 1$ the definition above leads to the classical one.

Proposition 4.1.

$$\Delta_m\varphi(n_0;n_1,\ldots,n_m)=F_{n_0}\prod_{k=1}^{m}(I-F_{n_k})\varphi(0)\quad \textit{for all } m,n_0,\ldots,n_m\in\mathbb{N}.$$

Proof. By induction on m, for any $n_0,\ldots,n_m\in\mathbb{N}$ we see that

$$\begin{aligned}&\Delta_{m+1}\varphi(n_0;n_1,\ldots,n_{m+1})\\&\quad=\Delta_m\varphi(n_0;n_1,\ldots,n_m)-q^{-n_0n_{m+1}}\Delta_m\varphi(n_0+n_{m+1};n_1,\ldots,n_m)\\&\quad=F_{n_0}\prod_{k=1}^{m}(I-F_{n_k})\varphi(0)-q^{-n_0n_{m+1}}F_{n_0+n_{m+1}}\prod_{k=1}^{m}(I-F_{n_k})\varphi(0)\\&\quad=(F_{n_0}-F_{n_0}F_{n_{m+1}})\prod_{k=1}^{m}(I-F_{n_k})\varphi(0)=F_{n_0}\prod_{k=1}^{m+1}(I-F_{n_k})\varphi(0).\ \blacksquare\end{aligned}$$

The formula above, which is the q-analogue of the formula in the classical case (see [4]), gives a description of the linear functionals corresponding to the qCM sequences.

PROPOSITION 4.2. *A sequence $\{\varphi(n)\}_n$ is q-completely monotonic if and only if the corresponding functional f is τ-positive with respect to the set $\tau = \{F_1, I - F_m;\ m \in \mathbb{N}\}$.*

Proof. 1. First, we show that τ is admissible. Condition (1) in the definition of an admissible set is obvious, while the other two conditions follow from the fact

$$F_m = q^{1(m-1)} F_1 F_{m-1} = q^{\sum_{j=1}^{m-1} j} F_1 \dots F_1 = q^{m(m-1)/2} F_1 \dots F_1 \in \mathrm{Alg}^+(\tau).$$

2. Suppose $\{\varphi(n)\}_n$ is qCM. Let f be the linear functional corresponding to the sequence $\{\varphi(n)\}_n$ via the formula $f(F_n) = \varphi(n) = F_n\varphi(0)$. By Proposition 4.1, for all $m, n_0, \dots, n_m \in \mathbb{N}$ we have

$$f\Big(F_{n_0} \prod_{k=1}^{m} (I - F_{n_k})\Big) = F_{n_0} \prod_{k=1}^{m} (I - F_{n_k})\varphi(0) \geq 0,$$

hence f is positive on every finite product of members of τ. So for x in $\mathrm{Alg}^+(\tau)$, i.e.

$$x = \sum_{i=1}^{n} \alpha_i x_i, \quad \text{where} \quad \alpha_i \geq 0,\ x_i = F_1^{n_{0,i}} \prod_{k=1}^{m_i} (I - F_{n_{k,i}}),$$

we get

$$f(x) = f\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) = \sum_{i=1}^{n} \alpha_i f(x_i) \geq 0.$$

Therefore f is τ-positive.

3. Suppose now that f is τ-positive with respect to $\tau = \{F_1, I - F_m;\ m \in \mathbb{N}\}$. Then

$$\begin{aligned} \Delta_m\varphi(n_0; n_1, \dots, n_m) &= F_{n_0} \prod_{k=1}^{m} (I - F_{n_k})\varphi(0) = f\Big(F_{n_0} \prod_{k=1}^{m} (I - F_{n_k})\Big) \\ &= q^{n_0(n_0-1)/2} f\Big(F_1^{n_0} \prod_{k=1}^{m} (I - F_{n_k})\Big) \geq 0. \ \blacksquare \end{aligned}$$

Now we state the main theorem which gives a characterization of q-completely monotonic sequences in terms of q-positive definiteness.

THEOREM 4.3. *A sequence $\{\varphi(n)\}_n$ is qCM if and only if the sequences $\{\varphi(n)\}_n$, $\{q^{-n}\varphi(n+1)\}_n$ and $\{\varphi(n) - q^{-nm}\varphi(n+m)\}_n$, for all $m \in \mathbb{N}$, are qPD.*

Proof. Suppose $\{\varphi(n)\}_n$ is qCM. It follows from Proposition 4.2 that the functional f on $\mathcal{R}$ given by

$$f(F_n) = F_n\varphi(0) = \varphi(n)$$

is τ-positive with respect to the admissible set $\tau = \{F_1, I - F_m;\, m \in \mathbb{N}\}$. Then Theorem 1.1 states that f is positive and bounded and $E_x f$ is positive for every $x \in \tau$. The positivity of f means (see Proposition 3.2) that $\{\varphi(n)\}_n$ is qPD.

If $x = F_1$ and $y = \sum_{i=1}^{n} \alpha_i F_i \in \mathcal{R}$, then

$$0 \le E_x f(y^* y) = \sum_{i,j=0}^{n} \alpha_i \alpha_j f(F_1 F_i F_j) = \sum_{i,j=0}^{n} q^{-(ij+i+j)} \alpha_i \alpha_j f(F_{i+j+1})$$
$$= \sum_{i,j=0}^{n} q^{-ij} \alpha_i \alpha_j [q^{-(i+j)} \varphi(i+j+1)].$$

Thus $\{q^{-n}\varphi(n+1)\}_n$ is qPD.

Let now $x = I - F_m \in \tau$ and set $y = \sum_{i=1}^{n} \alpha_i F_i \in \mathcal{R}$. Then

$$0 \le E_x f(y^* y) = \sum_{i,j=0}^{n} \alpha_i \alpha_j f((I - F_m) F_i F_j) = \sum_{i,j=0}^{n} \alpha_i \alpha_j f(F_i F_j - F_m F_i F_j)$$
$$= \sum_{i,j=0}^{n} q^{-ij} \alpha_i \alpha_j f(F_{i+j}) - \sum_{i,j=0}^{n} q^{-ij-m(i+j)} \alpha_i \alpha_j f(F_{i+j+m})$$
$$= \sum_{i,j=0}^{n} q^{-ij} \alpha_i \alpha_j [\varphi(i+j) - q^{-m(i+j)} \varphi(i+j+m)],$$

hence $\{\varphi(n) - q^{-nm}\varphi(n+m)\}_n$ is qPD.

Suppose the converse, i.e. $\{\varphi(n)\}_n$ is such that

(qCM1) $\{\varphi(n)\}_n$ is qPD,

(qCM2) $\{q^{-n}\varphi(n+1)\}_n$ is qPD,

(qCM3) $\forall_{m\in\mathbb{N}}\ \{\varphi(n) - q^{-nm}\varphi(n+m)\}_n$ is qPD.

Let f be the linear functional corresponding to $\{\varphi(n)\}_n$ as before. Condition (qCM1) implies that f is positive, while the other two conditions and the calculations above imply positivity of $E_x f$ for every $x \in \tau$. Now, it is enough to show that f is bounded. If this is the case, Theorem 1.1 shows that f is τ-positive, which is equivalent to $\{\varphi(n)\}_n$ being qCM.

For $m \in \mathbb{N}$ put $\alpha_m = 1$ and $\alpha_i = 0$ for $i \ne m$. Then (qCM1) states that

$$q^{-m^2}\varphi(2m) = \sum_{i,j=0}^{n} \alpha_i \alpha_j\, q^{-ij} \varphi(i+j) \ge 0 \quad \text{for } n \ge m,$$

while (qCM2) means that

$$q^{-(2m+m^2)}\varphi(2m+1) = \sum_{i,j=0}^{n} \alpha_i \alpha_j\, q^{-ij-i-j} \varphi(i+j+1) \ge 0 \quad \text{for } n \ge m.$$

Finally, for every $m \in \mathbb{N}$, $n \geq m$ and $\alpha_0 = 1$, $\alpha_i = 0$ for $i \in \{1, \ldots, n\}$ condition (qCM3) gives

$$\varphi(0) - q^{-0\cdot m}\varphi(m) = \sum_{i,j=0}^{n} \alpha_i\alpha_j\, q^{-ij}[\varphi(i+j) - q^{-(i+j)m}\varphi(i+j+m)] \geq 0.$$

Thus $|f(F_m)| = |\varphi(m)| \leq \varphi(0)$, i.e. f is bounded. ∎

5. Relation between complete monotonicity and q-complete monotonicity. In this section we investigate the relations between the classical and q-properties. It turns out that a description of the class of q-positive definite sequences in terms of some integral representation can be easily obtained due to the Hamburger theorem. A description of q-completely monotonic sequences is not so apparent, though possible as well. We start with the easier observation.

PROPOSITION 5.1. *A sequence $\{\varphi_n\}_n$ is qPD if and only if the sequence $\{\mu_n\}_n$, where $\mu_n = q^{-n(n-1)/2}\varphi_n$, is PD.*

Proof. This follows from

$$\begin{aligned}\sum_{n,m=0}^{N} a_n a_m \mu_{m+n} &= \sum_{n,m=0}^{N} a_n a_m q^{-(m+n)(m+n-1)/2}\varphi_{m+n} \\ &= \sum_{n,m=0}^{N} a_n a_m q^{-m(m-1)/2} q^{-n(n-1)/2} q^{-mn}\varphi_{m+n} \\ &= \sum_{n,m=0}^{N} (q^{-n(n-1)/2}a_n)(q^{-m(m-1)/2}a_m) q^{-mn}\varphi_{m+n} \\ &= \sum_{n,m=0}^{N} b_n b_m q^{-mn}\varphi_{m+n},\end{aligned}$$

where $N \in \mathbb{N}$ and $b_n = q^{-n(n-1)/2}a_n$. ∎

This proposition together with the Hamburger theorem gives us a description of the class of q-positive definite sequences.

COROLLARY 5.2. *Any q-positive definite sequence may be represented in the form*

$$\varphi_n = \int_{\mathbb{R}} q^{n(n-1)/2} t^n \, d\mu(t), \qquad n \in \mathbb{N},$$

where μ is a representing measure for the sequence $\{q^{-n(n-1)/2}\varphi_n\}_n$.

We now deal with the question whether a similar description (with measure concentrated on some compact interval) is true for q-complete monotonic sequences. One implication may be shown by direct calculation in case $q \in (0,1)$.

PROPOSITION 5.3. *Let $q \in (0,1)$. If a sequence $\{\mu_n\}_n$ is CM, then $\{q^{n(n-1)/2}\mu_n\}_n$ is qCM.*

Proof. Define $\varphi_n = q^{n(n-1)/2}\mu_n$. According to the classical theory of moment sequences we know that $\{\mu_n\}_n$ and $\{\mu_n - \mu_{n+1}\}_n$ are PD. Moreover, Corollary 1.2 states that for every $k \in \mathbb{N}$ the sequences $\{\mu_{n+k}\}_n$ and $\{\mu_n - \mu_{n+k}\}_n$ are PD as well. By Proposition 5.1, the first condition is equivalent to q-positive definiteness of $\{\varphi_n\}_n$.

Now we show that positive definiteness of $\{\mu_{n+1}\}_n$ is equivalent to q-positive definiteness of $\{q^{-n}\varphi_{n+1}\}_n$.

Observe that for all $n, m, k \in \mathbb{N}$ we have

$$\frac{1}{2}(n+m+k)(n+m+k-1) = \frac{n(n-1)}{2} + \frac{m(m-1)}{2} + \frac{k(k-1)}{2} + nm + k(n+m).$$

Thus for every $k \in \mathbb{N}$,

$$\begin{aligned}
\sum_{n,m=0}^{N} a_n a_m \mu_{m+n+k}
&= \sum_{n,m=0}^{N} a_n a_m q^{-(m+n+k)(m+n+k-1)/2}\varphi_{m+n+k} \\
&= \sum_{n,m=0}^{N} a_n a_m q^{-m(m-1)/2} q^{-n(n-1)/2} q^{-k(k-1)/2} q^{-mn-(n+m)k}\varphi_{m+n+k} \\
&= q^{-k(k-1)/2} \sum_{n,m=0}^{N} (q^{-n(n-1)/2}a_n)(q^{-m(m-1)/2}a_m) q^{-mn-(m+n)k}\varphi_{m+n+k} \\
&= q^{-k(k-1)/2} \sum_{n,m=0}^{N} b_n b_m q^{-mn} q^{-(m+n)k}\varphi_{m+n+k},
\end{aligned}$$

where $N \in \mathbb{N}$ and $b_n = q^{-n(n-1)/2}a_n$. In particular, for $k = 1$ the aforesaid equivalence is true. Moreover, if $\{\mu_n\}_n$ is completely monotonic, then

$$\sum_{n,m=0}^{N} b_n b_m q^{-mn} q^{-(m+n)k}\varphi_{m+n+k} = q^{k(k-1)/2} \sum_{n,m=0}^{N} a_n a_m \mu_{m+n+k} \geq 0. \tag{2}$$

Finally, observe that

$$\begin{aligned}\sum_{n,m=0}^{N} b_n b_m q^{-mn}\varphi_{m+n} &= \sum_{n,m}^{N} a_n a_m \mu_{m+n} \ge \sum_{n,m}^{N} a_n a_m \mu_{m+n+k}\\ &= \sum_{n,m}^{N} a_n a_m q^{-(m+n+k)(m+n+k-1)/2}\varphi_{m+n+k}\\ &= \sum_{n,m}^{N} (q^{-m(m-1)/2}a_n)(q^{-n(n-1)/2}a_m) q^{-mn-k(m+n)} q^{-k(k-1)/2}\varphi_{m+n+k}\\ &= q^{-k(k-1)/2}\sum_{n,m}^{N} b_n b_m q^{-mn} q^{-k(m+n)}\varphi_{m+n+k}\\ &\ge \sum_{n,m}^{N} b_n b_m q^{-mn} q^{-k(m+n)}\varphi_{m+n+k},\end{aligned}$$

The last inequality follows from (2) and the fact that $q^{-k(k-1)/2} \ge 1$ for $q \in (0,1)$. This means that

$$\sum_{n,m}^{N} b_n b_m q^{-mn}[\varphi_{m+n} - q^{-k(m+n)}\varphi_{m+n+k}] \ge 0.$$

Summarizing, we have shown that $\{\varphi_n\}_n$ is qPD, $\{q^{-n}\varphi_{n+1}\}_n$ is qPD and for all $m \in \mathbb{N}$, $\{\varphi_n - q^{-nm}\varphi_{n+m}\}_n$ is qPD. According to Theorem 4.3 this is equivalent to the fact that $\{\varphi_n\}_n$ is qCM. ∎

To get the opposite implication, we need more advanced arguments: the RKHS technique used as in [5] and [7] (for more on this subject see [8]). This yields the result for all $q > 0$.

THEOREM 5.4. *If a sequence $\{\varphi_n\}_n$ is qCM, then there exists a measure μ on $[0,1]$ such that*

$$\varphi_n = \int_{[0,1]} q^{n(n-1)/2} t^n \, d\mu(t), \quad n \in \mathbb{N}.$$

Proof. By Theorem 4.3 the sequence $\{\varphi_n\}_n$ satisfies conditions (qCM1)–(qCM3). Define the kernel on $\mathbb{N}$ by the formula

$$K(n,m) := q^{-mn}\varphi_{n+m}, \quad n,m \in \mathbb{N}.$$

The assumption (qCM1) means that this kernel is positive definite, i.e.

$$\sum_{n,m=0}^{N} K(n,m)\lambda_n\bar{\lambda}_m \ge 0, \quad \lambda_0,\dots,\lambda_N \in \mathbb{C},\ N \in \mathbb{N}.$$

The factorization theorem of Aronszajn (cf. [8], for example) implies that there exists a Hilbert space $\mathcal{H}$ and a mapping $\mathbb{N} \ni n \mapsto \gamma_n \in \mathcal{H}$ such that

$$\mathcal{H} = \overline{\mathrm{Lin}}\{\gamma_n;\, n \in \mathbb{N}\}, \qquad K(n,m) = \langle \gamma_n, \gamma_m \rangle.$$

Next, we set

$$\mathcal{D} := \mathrm{Lin}\{\gamma_n;\, n \in \mathbb{N}\}, \qquad T : \mathcal{D} \ni \sum_n \alpha_n \gamma_n \mapsto \sum_n \alpha_n q^{-n} \gamma_{n+1} \in \mathcal{D}.$$

Observe that for $u = \sum_{n=1}^N \alpha_n \gamma_n$ and $v = \sum_{n=1}^N \beta_n \gamma_n$ we have

$$\begin{aligned}\langle Tu, v\rangle &= \Big\langle \sum_n \alpha_n q^{-n} \gamma_{n+1}, \sum_m \beta_m \gamma_m \Big\rangle = \sum_{n,m} \alpha_n \beta_m q^{-n} \langle \gamma_{n+1}, \gamma_m \rangle \\ &= \sum_{n,m} \alpha_n \beta_m q^{-n} q^{-(n+1)m} \varphi_{n+m+1} = \sum_{n,m} \alpha_n \beta_m q^{-m-n(m+1)} \varphi_{n+m+1} \\ &= \sum_{n,m} \alpha_n \beta_m q^{-m} \langle \gamma_n, \gamma_{m+1} \rangle = \langle u, Tv \rangle.\end{aligned}$$

Now, suppose $v = \sum_{n=1}^N \beta_n \gamma_n = 0$. Then for every γ_k, we have $\langle Tv, \gamma_k \rangle = \langle v, T\gamma_k \rangle = 0$, so Tv is orthogonal to the total set $\{\gamma_n;\, n \in \mathbb{N}\}$ and must be zero. This means that T is well-defined and symmetric.

The operator T is obviously densely defined ($\mathcal{D}$ dense in $\mathcal{H}$) and closable, being a symmetric operator. It is easy to see that T has a cyclic vector γ_0. Indeed,

$$T^n \gamma_0 = q^{-n(n-1)/2} \gamma_n, \qquad n \in \mathbb{N}.$$

Since the operator $\overline{T}$ is closed, symmetric and has a cyclic vector, it admits a self-adjoint extension S in the same space $\mathcal{H}$ (cf. [2]). Thus by the spectral theorem for self-adjoint operators (cf. [2]) there exists a spectral measure E such that

$$S = \int_{\mathbb{R}} t\, dE(t).$$

Moreover,

$$S^n = \int_{\mathbb{R}} t^n\, dE(t).$$

Now we define $\mu(\sigma) := \langle E(\sigma)\gamma_0, \gamma_0 \rangle$ for all Borel sets $\sigma \subset \mathbb{R}$. Then

$$\begin{aligned}\varphi_n = \langle \gamma_n, \gamma_0 \rangle &= \langle q^{n(n-1)/2} T^n \gamma_0, \gamma_0 \rangle = q^{n(n-1)/2} \langle S^n \gamma_0, \gamma_0 \rangle \\ &= \int_{\mathbb{R}} q^{n(n-1)/2} t^n \langle dE(t)\gamma_0, \gamma_0 \rangle = \int_{\mathbb{R}} q^{n(n-1)/2} t^n\, d\mu(t).\end{aligned}$$

Now we show that $S \geq 0$, or equivalently that the measure μ is concentrated on $[0, \infty)$. For this, let $u = \sum_{n=1}^N \alpha_n \gamma_n$. By (qCM2) we have

$$\langle Su, u\rangle = \Big\langle \sum_{n=1}^{N} \alpha_n q^{-n}\gamma_{n+1}, \sum_{m=1}^{N} \alpha_m\gamma_m \Big\rangle = \sum_{m,n=1}^{N} \alpha_n\alpha_m q^{-n}\langle\gamma_{n+1},\gamma_m\rangle$$
$$= \sum_{m,n=1}^{N} \alpha_n\alpha_m q^{-n-m(n+1)}\varphi_{n+m+1} = \sum_{m,n=1}^{N} \alpha_n\alpha_m q^{-mn}q^{-(n+m)}\varphi_{n+m+1} \geq 0.$$

To prove that the measure is concentrated on $[0,1]$ we only need to show that $\|S\| \leq 1$. Since $\{\varphi_n - q^{-n}\varphi_{n+1}\}$ is qPD (see (qCM3) for $m = 1$), we have

$$\sum_{m,n=1}^{N} \alpha_n\alpha_m q^{-mn}q^{-(n+m)}\varphi_{n+m+1} \leq \sum_{m,n=1}^{N} \alpha_n\alpha_m q^{-mn}\varphi_{n+m}.$$

Thus for $u = \sum_{n=1}^{N}\alpha_n\gamma_n$ we get

$$\begin{aligned}\langle Su, u\rangle &= \Big\langle \sum_{n=1}^{N} \alpha_n q^{-n}\gamma_{n+1}, \sum_{m=1}^{N} \alpha_m\gamma_m \Big\rangle = \sum_{m,n=1}^{N} \alpha_n\alpha_m q^{-n}\langle\gamma_{n+1},\gamma_m\rangle \\ &= \sum_{m,n=1}^{N} \alpha_n\alpha_m q^{-n-m(n+1)}\varphi_{n+m+1} \\ &= \sum_{m,n=1}^{N} \alpha_n\alpha_m q^{-mn}q^{-(n+m)}\varphi_{n+m+1} \\ &\leq \sum_{m,n=1}^{N} \alpha_n\alpha_m q^{-mn}\varphi_{n+m} = \langle u, u\rangle.\end{aligned}$$

This gives the operator inequality $0 \leq S \leq I$ and therefore $\|S\| \leq 1$. ∎

Corollary 5.5. *Let $q \in (0,1)$. For a sequence $\{\varphi_n\}_n$ the following conditions are equivalent*:

(1) $\{\varphi_n\}_n$ *is qCM*,
(2) $\{q^{-n(n-1)/2}\varphi_n\}_n$ *is CM*,
(3) *there exists a measure μ on $[0,1]$ such that*

$$\varphi_n = \int_{[0,1]} q^{n(n-1)/2}t^n\,d\mu(t), \qquad n \in \mathbb{N}.$$

Proof. The implications (2)⇒(1)⇒(3) follow from Proposition 5.3 and Theorem 5.4, while (3)⇒(2) is a consequence of the Hausdorff theorem which states that a sequence admits an integral representation with a measure concentrated on $[0,1]$ if and only if it is completely monotonic ([9]). ∎

Remark. Observe that the first part of the proof of Theorem 5.4 gives the implication (already proved in Corollary 5.2) that if a sequence $\{\varphi_n\}$ is

q-positive definite then it may be represented in the form

$$\varphi_n = \int_{\mathbb{R}} q^{n(n-1)/2} t^n \, d\mu(t).$$

The result above suggests the following definition of q-moment sequences.

DEFINITION 3. Call $\{\varphi_n\}_n$ a *q-moment sequence* if there exists a Borel measure μ on some set $X \subset \mathbb{R}$ such that

$$\varphi_n = \int_X q^{n(n-1)/2} t^n \, d\mu(t), \qquad n \in \mathbb{N}.$$

REMARK. In the general case (for $q > 0$) the relations between conditions (1)–(3) in Corollary 5.5 are as follows:

$$(1) \Rightarrow (2) \Leftrightarrow (3)$$

and cannot be improved for $q > 1$. Indeed, a weighted sequence need not be qCM even if a classical sequence is CM. For example, take the sequence

$$\varphi_n = \int_0^1 q^{n(n-1)/2} t^n \, dt, \qquad n \in \mathbb{N},$$

corresponding to Lebesgue measure. Then $\{\varphi_n\}_n$ is not qCM.

Suppose to the contrary that for all $n, m = 0, \ldots, N$ and $k \in \mathbb{N}$ we have

$$\begin{aligned} 0 &\le \sum_{n,m} a_n a_m q^{-nm} [\varphi_{m+n} - q^{-(n+m)k} \varphi_{n+m+k}] \\ &= \int_0^1 (1 - q^{k(k-1)/2} t^k) \Big(\sum_n q^{n(n-1)/2} a_n t^n \Big)^2 \, dt. \end{aligned}$$

Now choose $i = k$ and set $a_i = q^{-k(k-1)/2}$, $a_n = 0$ for $n \neq i$. Then

$$\int_0^1 [1 - q^{k(k-1)/2} t^k] t^{2k} \, dt = \frac{1}{2k+1} - q^{k(k-1)/2} \frac{1}{3k+1} \ge 0,$$

and hence $q^{k(k-1)/2} \le 1 + k/(2k+1)$. But if $k \to \infty$ then the right hand side tends to $3/2$ while the left hand side tends to $+\infty$.

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