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AN INTERMEDIATE RING BETWEEN A POLYNOMIAL RING AND A POWER SERIES RING

BY

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Abstract. Let $R[x]$ and $R[[x]]$ respectively denote the ring of polynomials and the ring of power series in one indeterminate x over a ring R. For an ideal I of R, denote by $[R;I][x]$ the following subring of $R[[x]]$:

$$[R;I][x] := \Big\{ \sum_{i\geq 0} r_i x^i \in R[[x]] : \exists 0 \leq n \in \mathbb{Z} \text{ such that } r_i \in I,\ \forall i \geq n \Big\}.$$

The polynomial and power series rings over R are extreme cases where $I = 0$ or R, but there are ideals I such that neither $R[x]$ nor $R[[x]]$ is isomorphic to $[R;I][x]$. The results characterizing polynomial rings or power series rings with a certain ring property suggest a similar study to be carried out for the ring $[R;I][x]$. In this paper, we characterize when the ring $[R;I][x]$ is semipotent, left Noetherian, left quasi-duo, principal left ideal, quasi-Baer, or left p.q.-Baer. New examples of these rings can be given by specializing to some particular ideals I, and some known results on polynomial rings and power series rings are corollaries of our formulations upon letting $I = 0$ or R.

1. Definitions and notations. Throughout, R is a ring with an identity unless specified otherwise, M is a left unitary R-module and $I \lhd R$ is an ideal. We write $J(R)$ for the Jacobson radical of the ring R. Let $R[x]$, $R[[x]]$, $R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ respectively denote the ring of polynomials, the ring of power series, the ring of Laurent polynomials and the ring of Laurent series in one indeterminate x over R. We denote by $[R;I][x]$ the subring $R[x] + I[[x]]$ of $R[[x]]$ where $I[[x]]$ is the set of power series all of whose coefficients belong to I, and by $[R;I][x, x^{-1}]$ the subring $R[x, x^{-1}] + I[[x, x^{-1}]]$ of $R[[x, x^{-1}]]$ where $I[[x, x^{-1}]]$ is the set of Laurent series all of whose coefficients belong to I (see [17]). That is,

$$[R;I][x] = \Big\{ \sum_{i\geq 0} r_i x^i \in R[[x]] : \exists 0 \leq n \in \mathbb{Z} \text{ such that } r_i \in I,\ \forall i \geq n \Big\}$$

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and

$$[R;I][x,x^{-1}] = \Big\{ \sum_{i\geq -s} r_i x^i \in R[[x,x^{-1}]] : \\ s \geq 0,\ \exists -s \leq n \in \mathbb{Z} \text{ such that } r_i \in I,\ \forall i \geq n \Big\}.$$

Let $M[x]$, $M[[x]]$, $M[x,x^{-1}]$ and $M[[x,x^{-1}]]$ respectively denote the module of formal polynomials, of formal power series, of formal Laurent polynomials and of formal Laurent series in x with coefficients from M. In a natural way, $M[x]$, $M[[x]]$, $M[x,x^{-1}]$ and $M[[x,x^{-1}]]$ are left modules over $R[x]$, $R[[x]]$, $R[x,x^{-1}]$ and $R[[x,x^{-1}]]$, respectively.

For a submodule N of M, define

$$[M;N][x] = \Big\{ \sum_{i\geq 0} v_i x^i \in M[[x]] : \exists 0 \leq n \in \mathbb{Z} \text{ such that } v_i \in N,\ \forall i \geq n \Big\}$$

and

$$[M;N][x,x^{-1}] = \Big\{ \sum_{i\geq -s} v_i x^i \in M[[x,x^{-1}]] : \\ s \geq 0,\ \exists -s \leq n \in \mathbb{Z} \text{ such that } v_i \in N,\ \forall i \geq n \Big\}.$$

It is easy to see that $IM \subseteq N$ iff $[M;N][x]$ is a left $[R;I][x]$-module under usual addition and multiplication of power series, and that $IM \subseteq N$ iff $[M;N][x,x^{-1}]$ is a left $[R;I][x,x^{-1}]$-module under usual addition and multiplication of Laurent series (see [17]). In particular, $[M;IM][x]$ is a left module over $[R;I][x]$, and $[M;IM][x,x^{-1}]$ is a left module over $[R;I][x,x^{-1}]$. Moreover, when $I = 0$ we have $[R;I][x] = R[x]$, $[M;IM][x] = M[x]$, $[R;I][x,x^{-1}] = R[x,x^{-1}]$ and $[M;IM][x,x^{-1}] = M[x,x^{-1}]$; when $I = R$ we have $[R;I][x] = R[[x]]$, $[M;IM][x] = M[[x]]$, $[R;I][x,x^{-1}] = R[[x,x^{-1}]]$ and $[M;IM][x,x^{-1}] = M[[x,x^{-1}]]$.

2. Semipotent rings. A ring is called *clean* if every element is the sum of a unit and an idempotent. It is known that a polynomial ring is never clean (see [23, Proposition 13]) and that $R[[x]]$ is clean iff R is clean (see [10, Proposition 5]). It is then natural to ask: When is the ring $[R;I][x]$ clean? We answer this by considering a basic but weaker concept. A ring R is called *semipotent* if every left (resp. right) ideal not contained in $J(R)$ contains a nonzero idempotent. Semipotent rings were named I_0-rings by Nicholson in [22]. It is easily seen that the quotient ring of a semipotent ring R modulo an ideal contained in $J(R)$ is again semipotent. The next lemma will be used several times.

LEMMA 1. *Let $S = [R;I][x]$. The following hold:*

(1) *$I[[x]] \lhd S$ and $S/I[[x]] \cong (R/I)[x]$.*
(2) *$J(S) \supseteq J(R) \cap I + I[[x]]x$.*

(3) *If* $I \subseteq J(R)$, *then* $J(S) = K[x] + I[[x]]$ *where* $K/I \lhd R/I$ *is a nil ideal. In particular,* $J([R; J(R)][x]) = J(R)[[x]]$.

Proof. (1) This is clear.

(2) We see that $\Delta := J(R) \cap I + I[[x]]x$ is an ideal of S. Let $\sum_{i\geq 0} a_i x^i \in \Delta$. Since $\Delta \subseteq J(R[[x]])$, there exists $\sum_{i\geq 0} b_i x^i \in R[[x]]$ such that

$$\Big(1 + \sum_{i\geq 0} a_i x^i\Big) \sum_{i\geq 0} b_i x^i = 1.$$

Thus, $b_0 = (1 + a_0)^{-1}$ and $b_n = -(1 + a_0)^{-1}(a_1 b_{n-1} + \cdots + a_n b_0) \in I$ for all $n \geq 1$. So $\sum_{i\geq 0} b_i x^i \in [R; I][x]$. This shows that $\Delta \subseteq J(S)$.

(3) By a result of Amitsur [1], $J((R/I)[x]) = (K/I)[x]$ where K/I is a nil ideal of R/I. As $(R/I)[x] \cong R[x]/I[x] \cong S/I[[x]]$, we have $J\big((R/I)[x]\big) \cong J\big(R[x]/I[x]\big) \cong J\big(S/I[[x]]\big)$. Hence $J(S/I[[x]]) = (K[x]+I[[x]])/I[[x]]$. Since $I \subseteq J(R)$, one sees that $I[[x]] \subseteq J(S)$ by (2); so $J(S/I[[x]]) = J(S)/I[[x]]$. Hence $J(S) = K[x] + I[[x]]$. ■

THEOREM 2. *The ring* $[R; I][x]$ *is semipotent if and only if* $I = R$ *and* R *is semipotent.*

Proof. ($\Rightarrow$) Let $S := [R; I][x]$. By Lemma 1, $I[[x]]x$ is an ideal of S contained in $J(S)$. So $S/I[[x]]x$ is a semipotent ring. Assume that $I \neq R$, i.e., $1 \notin I$. Write $\overline{\alpha} = \alpha + I[[x]]x \in S/I[[x]]x$ for any $\alpha \in S$. If $\overline{1} + \overline{x^2}$ is a unit of $S/I[[x]]x$, then there exists $f(x) = \sum_{i\geq 0} f_i x^i \in S$ such that $(1 + x^2) f(x) \in 1 + I[[x]]x$. It follows that $f_0 = 1$ and $f_n + f_{n+2} \in I$ for all $n \geq 0$. This shows that $f_{2n} \notin I$ for all $n \geq 0$, and this contradicts $f(x) \in S$. So $\overline{1} + \overline{x^2}$ is not a unit of $S/I[[x]]x$, and hence $\overline{x^2}$ is not in the Jacobson radical of $S/I[[x]]x$. Thus, $\overline{f(x)}\,\overline{x^2}$ is a nonzero idempotent of $S/I[[x]]x$ for some $f(x) \in S$, but this is clearly impossible. Hence $I = R$, and so $S = R[[x]]$. To see that R is semipotent, let $a \in R \setminus J(R)$. As $J(S) = J(R) + xR[[x]]$, $a \notin J(S)$. So $g(x)a$ is a nonzero idempotent for some $g(x) = \sum_{i\geq 0} b_i x^i \in S$. It follows that $b_0 a \in Ra$ is a nonzero idempotent. So R is semipotent.

($\Leftarrow$) Let $T = R[[x]]$, and let $f(x) := \sum_{i\geq 0} a_i x^i \in T \setminus J(T)$. We show that $Tf(x)$ contains a nonzero idempotent. Because $J(T) = J(R) + Tx$, $a_0 \in R \setminus J(R)$. So, by hypothesis, there exists $b \in R$ such that ba_0 is a nonzero idempotent. With $f(x)$ replaced by $bf(x)$, we can assume that a_0 is a nonzero idempotent of R. With $f(x)$ replaced by $a_0 f(x)$, we can further assume that $a_0 a_i = a_i$ for $i = 0, 1, \ldots$. We next define a sequence $\{b_i : i = 0, 1, \ldots\}$ inductively

$$b_0 = 1, \quad b_1 = -a_1, \quad b_n = -(a_n + b_1 a_{n-1} + \cdots + b_{n-1} a_1) \quad \text{for } n \geq 2.$$

Thus, for each $n \geq 1$, we see that $b_n \in a_0 R$ and

$$a_n + b_1 a_{n-1} + \cdots + b_{n-1} a_1 + b_n a_0 = -b_n(1 - a_0) = -a_0 b_n (1 - a_0).$$

So, for $g(x) := \sum_{i\geq 0} b_i x^i \in T$, we have

$$\begin{aligned} g(x)f(x) &= \sum_{i\geq 0} \big(a_i + b_1 a_{i-1} + \cdots + b_i a_0\big) x^i \\ &= a_0 + \sum_{i\geq 1} \big(a_i + b_1 a_{i-1} + \cdots + b_i a_0\big) x^i \\ &= a_0 - \sum_{i\geq 1} a_0 b_i (1-a_0) x^i = a_0 - a_0 \Big(\sum_{i\geq 1} b_i (1-a_0) x^i \Big), \end{aligned}$$

which is a nonzero idempotent of T. So T is semipotent. ∎

COROLLARY 3. *$R[x]$ is never semipotent, and $R[[x]]$ is semipotent iff R is semipotent.*

A semipotent ring is called *potent* if idempotents lift modulo its Jacobson radical. By [24], a semipotent ring need not be potent. One easily sees that $R/J(R) \cong R[[x]]/J(R[[x]])$ and that idempotents of $R/J(R)$ lift to idempotents of R iff idempotents of $R[[x]]/J(R[[x]])$ lift to idempotents of $R[[x]]$. Thus, it follows from [22, Proposition 1.4] that R is potent iff $R[[x]]$ is potent (this is observed in [19] and in [26]). The next corollary is clear.

COROLLARY 4. *The ring $[R;I][x]$ is a potent ring iff $I = R$ and R is a potent ring.*

COROLLARY 5. *The ring $[R;I][x]$ is a clean ring iff $I = R$ and R is a clean ring.*

Proof. This follows from Theorem 2 and [10, Proposition 5]. ∎

EXAMPLE 6. *Let R be a semipotent ring which is semiprimitive or countable, and I a nonzero proper ideal of R. Then $[R;I][x]$ is not isomorphic to either of $R[x]$ and $R[[x]]$.*

Proof. By Theorem 2, $[R;I][x]$ is not semipotent but $R[[x]]$ is semipotent, so $[R;I][x] \not\cong R[[x]]$. If R is semiprimitive, then $R[x]$ is semiprimitive by a well-known result of Amitsur [1]. So $[R;I][x] \not\cong R[x]$ as $[R;I][x]$ is not semiprimitive by Lemma 1. If R is countable, then $R[x]$ is countable but $[R;I][x]$ is uncountable. So $[R;I][x] \not\cong R[x]$. ∎

EXAMPLE 7. *Let R be a semipotent ring which is semiprimitive, and $R = I \oplus K$ a direct sum of nonzero ideals I and K. Then $[R;I][x]$ is never isomorphic to a polynomial ring or a power series ring.*

Proof. Since $R = I \oplus K$, it can be verified that $[R;I][x] \cong I[[x]] \oplus K[x]$.

If $[R;I][x] \cong T[x]$ for a ring T, then there exists a central idempotent e of $T[x]$ such that $e(T[x]) \cong I[[x]]$. But it is easily seen that $e \in T$ is central. So $e(T[x]) = (eT)[x]$, and hence $(eT)[x] \cong I[[x]]$. Since I is semipotent,

$I[[x]]$ is semipotent and $(eT)[x]$ is not semipotent by Corollary 3. This is a contradiction.

If $[R;I][x] \cong T[[x]]$ for a ring T, then there exists a central idempotent e of $T[[x]]$ such that $e(T[[x]]) \cong K[x]$. But it is easily seen that $e \in T$ is central. So $e(T[[x]]) = (eT)[[x]]$, and hence $(eT)[[x]] \cong K[x]$. Since K is semiprimitive, $K[x]$ is semiprimitive, but $(eT)[[x]]$ is clearly not semiprimitive. This is a contradiction. ■

3. Noetherian rings and modules. A ring R is left Noetherian iff $R[x]$ is left Noetherian (by Hilbert's Basis Theorem) iff $R[[x]]$ is left Noetherian (see Caruth [6]). It is natural to ask if R being left Noetherian also implies that $[R;I][x]$ is left Noetherian. We first mention here a relevant result due to Varadarajan [28]. Let W be a left module over a ring T not necessarily possessing an identity. Following [28], the module ${}_TW$ is said to have *property* (P) if $\{w \in W : Tw \subseteq U\} = U$ for any submodule U of W. One easily sees that ${}_TW$ has property (P) iff $w \in Tw$ for all $w \in W$, i.e., ${}_TW$ is an s-unital module in the sense of Tominaga [25]. It is proved in [28] that ${}_TW$ is a Noetherian module which is s-unital iff ${}_{T[x]}W[x]$ is a Noetherian module iff ${}_{T[x,x^{-1}]}W[x,x^{-1}]$ is a Noetherian module iff ${}_{T[[x]]}W[[x]]$ is a Noetherian module.

THEOREM 8. *Let M be a module over R and let $I \lhd R$ be such that ${}_I(IM)$ is an s-unital module. The following are equivalent:*

(1) *${}_RM$ is Noetherian.*
(2) *$[M;IM][x]$ is a Noetherian module over $[R;I][x]$.*
(3) *$[M;IM][x,x^{-1}]$ is a Noetherian module over $[R;I][x,x^{-1}]$.*

Proof. (1)⇔(2). Write $S = [R;I][x]$ and $V = [M;IM][x]$.

Suppose (2) holds. If $N_1 \subseteq N_2 \subseteq \cdots$ is a chain of submodules of ${}_RM$, then $[N_1;IN_1][x] \subseteq [N_2;IN_2][x] \subseteq \cdots$ is a chain of submodules of ${}_SV$ and so it is stable. This implies that the first chain is stable. So ${}_RM$ is Noetherian.

Suppose (1) holds. Then M/IM is a Noetherian module over R and hence over R/I. By [28, Theorem A], $\left(\frac{M}{IM}\right)[x]$ is a Noetherian module over $\left(\frac{R}{I}\right)[x]$. As the lattice of S-submodules of $\frac{V}{(IM)[[x]]}$ coincides with the lattice of $\frac{S}{I[[x]]}$-submodules of $\frac{V}{(IM)[[x]]}$, which is isomorphic to the lattice of $\left(\frac{R}{I}\right)[x]$-submodules of $\left(\frac{M}{IM}\right)[x]$, we see that $\frac{V}{(IM)[[x]]}$ is a Noetherian S-module. So to show that ${}_SV$ is a Noetherian module, it suffices to show that $(IM)[[x]]$ is a Noetherian S-module.

Let $W \subseteq (IM)[[x]]$ be an S-submodule. Next we show that ${}_SW$ is finitely generated. We introduce a notation: For $v = \sum_{i\geq 0} v_i x^i \in M[[x]]$, the coefficient v_i is denoted as $c_i(v)$. For each $i \geq 0$, let $W_i = \{z \in M : z = c_i(f)$ for some $f \in W \cap x^iV\}$. Then $W_0 \subseteq W_1 \subseteq \cdots$ is an ascending chain of S-submodules of M, so there exists $l \geq 0$ such that $W_l = W_{l+1} = \cdots$. Moreover, for each $0 \leq i \leq l$, W_i is generated as an R-module by $\{z_{ij} : j = 1, \ldots, n(i)\}$. Take $f_{ij} \in W \cap x^i V$ such that $c_i(f_{ij}) = z_{ij}$ for $i = 0, \ldots, l$ and $j = 1, \ldots, n(i)$. We claim that ${}_S W$ is generated by $\{f_{ij} : i = 1, \ldots, l;\ j = 1, \ldots, n(i)\}$.

Let $f \in W$. Then $c_0(f) \in W_0$, so $c_0(f) = \sum_{j=1}^{n(0)} a_{0j} z_{0j}$ with all a_{0j} in R. Since the module ${}_I(IM)$ is s-unital, $z_{0j} \in I z_{0j}$, so $z_{0j} = c_{0j} z_{0j}$ where $c_{0j} \in I$. Thus, $a_{0j} z_{0j} = (a_{0j} c_{0j}) z_{0j}$ with $a_{0j} c_{0j} \in I$. Hence we can assume that $c_0(f) = \sum_{j=1}^{n(0)} a_{0j} z_{0j}$ where all $a_{0j} \in I$. So $f_1 := f - \sum_{j=1}^{n(0)} a_{0j} f_{0j} \in W \cap xV$. As $c_1(f_1) \in W_1$, in the same manner, we have $c_1(f_1) = \sum_{j=1}^{n(1)} a_{1j} z_{1j}$ where all $a_{1j} \in I$. So $f_2 := f_1 - \sum_{j=1}^{n(1)} a_{1j} f_{1j} \in W \cap x^2 V$. By induction, we can find $\{a_{ij} \in I : 0 \leq i < l;\ 1 \leq j \leq n(i)\}$ and $\{b_{ij} \in I : i \geq l;\ 1 \leq j \leq n(l)\}$ such that

$$g := f - \sum_{j=1}^{n(0)} a_{0j} f_{0j} - \cdots - \sum_{j=1}^{n(l-1)} a_{l-1,j} f_{l-1,j} \in W \cap x^l V$$

and

$$g - \sum_{j=1}^{n(l)} b_{lj} f_{lj} - \sum_{j=1}^{n(l)} b_{l+1,j} x f_{lj} - \cdots - \sum_{j=1}^{n(l)} b_{l+k,j} x^k f_{lj} \in W \cap x^{l+k+1} V$$

for all $k \geq 0$. Let $g_j = b_{lj} + b_{l+1,j} x + \cdots + b_{l+k,j} x^k + \cdots \in I[[x]]$ for $j = 1, \ldots, n(l)$. Then $g = \sum_{j=1}^{n(l)} g_j f_{lj}$ and hence

$$\begin{aligned} f &= \sum_{j=1}^{n(0)} a_{0j} f_{0j} + \cdots + \sum_{j=1}^{n(l-1)} a_{l-1,j} f_{l-1,j} + g \\ &\in \sum_{j=1}^{n(0)} S f_{0j} + \cdots + \sum_{j=1}^{n(l-1)} S f_{l-1,j} + \sum_{j=1}^{n(l)} S f_{lj}. \end{aligned}$$

(1)⇔(3). Write $S = [R; I][x, x^{-1}]$ and $V = [M; IM][x, x^{-1}]$.

Suppose (3) holds. If $N_1 \subseteq N_2 \subseteq \cdots$ is a chain of submodules of ${}_R M$, then $[N_1; IN_1][x, x^{-1}] \subseteq [N_2; IN_2][x, x^{-1}] \subseteq \cdots$ is a chain of submodules of ${}_S V$ and so it is stable. This implies that the first chain is stable. So ${}_R M$ is Noetherian.

Suppose (1) holds. Then M/IM is a Noetherian module over R and hence over R/I. By [28, Theorem A], $\left(\frac{M}{IM}\right)[x, x^{-1}]$ is a Noetherian module over $\left(\frac{R}{I}\right)[x, x^{-1}]$. As the lattice of S-submodules of $\frac{V}{(IM)[[x,x^{-1}]]}$ coincides with the lattice of $\frac{S}{I[[x,x^{-1}]]}$-submodules of $\frac{V}{(IM)[[x,x^{-1}]]}$, which is isomorphic to the lattice of $\left(\frac{R}{I}\right)[x, x^{-1}]$-submodules of $\left(\frac{M}{IM}\right)[x, x^{-1}]$, we see that $\frac{V}{(IM)[[x,x^{-1}]]}$

is a Noetherian S-module. So to show that ${}_SV$ is a Noetherian module, it suffices to show that $(IM)[[x,x^{-1}]]$ is a Noetherian S-module.

Let $W \subseteq (IM)[[x,x^{-1}]]$ be an S-submodule. Next we show that ${}_SW$ is finitely generated. For each $k \geq 0$, let $W_k = \{z \in M : z = v_k \text{ for some } \sum_{i\geq k} v_i x^i \in W\}$. Then each W_k is a submodule of ${}_RM$, and $W_0 = W_1 = \cdots$ as x is invertible in S. By (1), we can assume that W_0 is generated as an R-module by $\{z_1,\ldots,z_s\}$. For each $1 \leq j \leq s$, take $h_j = \sum_{i\geq 0} v_{ji}x^i \in W$ such that $v_{j0} = z_j$. We claim that ${}_SW$ is generated by $\{h_j : j = 1,\ldots,s\}$.

Let $f \in W$. There exists $l \geq 0$ such that $f_0 := x^l f = \sum_{i\geq 0} v_i x^i$. So $v_0 \in W_0$, and $v_0 = \sum_{j=1}^s a_{0j}z_j$ where all $a_{0j} \in R$. Since the module ${}_I(IM)$ is s-unital, as above we can assume all a_{0j} are in I. So $f_1 := f_0 - \sum_{j=1}^s a_{0j}h_j = v_1'x + v_2'x^2 + \cdots \in W$. As $v_1' \in W_1$, in the same manner, we have $v_1' = \sum_{j=1}^s a_{1j}z_j$ where all $a_{1j} \in I$. So $f_2 := f_1 - x\sum_{j=1}^s a_{1j}h_j = v_2''x^2 + v_3''x^3 + \cdots$ is in W. By induction, we can find $\{a_{ij} \in I : 0 \leq i;\ 1 \leq j \leq s\}$ such that

$$f_{n+1} := f_n - x^n \sum_{j=1}^s a_{nj}h_j = v_{n+1}^{(n)}x^{n+1} + v_{n+2}^{(n)}x^{n+2} + \cdots \in W$$

for all $n \geq 0$. Let $g_j = a_{0j} + a_{1j}x + \cdots \in I[[x]]$ for $j = 1,\ldots,s$. Then

$$\begin{aligned} x^l f = f_0 &= (a_{01}h_1 + a_{02}h_2 + \cdots + a_{0s}h_s) \\ &\quad + x(a_{11}h_1 + a_{12}h_2 + \cdots + a_{1s}h_s) \\ &\quad + x^2(a_{21}h_1 + a_{22}h_2 + \cdots + a_{2s}h_s) + \cdots \\ &= g_1h_1 + g_2h_2 + \cdots + g_sh_s. \end{aligned}$$

So $f = (x^{-l}g_1)h_1 + (x^{-l}g_2)h_2 + \cdots + (x^{-l}g_s)h_s$. ∎

An ideal I of R is said to be *left s-unital* if $a \in Ia$ for all $a \in I$ (see [25]).

Corollary 9. *Let I be a left s-unital ideal of R. Then R is left Noetherian iff $[R;I][x]$ is left Noetherian iff $[R;I][x,x^{-1}]$ is left Noetherian.*

Corollary 10. *Let R be a countable ring and I an ideal of R. Then:*

(1) *$[R;I][x]$ is left Noetherian iff R is left Noetherian and I is left s-unital.*
(2) *$[R;I][x,x^{-1}]$ is left Noetherian iff R is left Noetherian and I is left s-unital.*

Proof. (1) The sufficiency is by Corollary 9.

Suppose that $S := [R;I][x]$ is left Noetherian. For $a \in I$, let $A = (Ra)[[x]]$ and $B = (Ia)[[x]]$. Then A, B are left ideals of S. Since S is left Noetherian, ${}_S(A/B)$ is Noetherian. Since $I[[x]] \cdot A \subseteq B$, we see that A/B is a left Noetherian module over $\frac{S}{I[[x]]}$. That is, $\left(\frac{Ra}{Ia}\right)[[x]]$ is a left Noetherian module over

$\left(\frac{R}{I}\right)[x]$. Hence there exist $f_1, \ldots, f_n \in \left(\frac{Ra}{Ia}\right)[[x]]$ such that

$$\left(\frac{Ra}{Ia}\right)[[x]] = f_1 \cdot \left(\frac{R}{I}\right)[x] + \cdots + f_n \cdot \left(\frac{R}{I}\right)[x].$$

If $a \notin Ia$, then Ra/Ia has a cardinality ≥ 2, so $\left(\frac{Ra}{Ia}\right)[[x]]$ is not countable. But since R is countable, R/I is countable and so is $\left(\frac{R}{I}\right)[x]$. Consequently, $f_1 \cdot \left(\frac{R}{I}\right)[x] + \cdots + f_n \cdot \left(\frac{R}{I}\right)[x]$ is countable, a contradiction. So $a \in Ia$.

(2) The proof is similar to the proof of (1). ■

QUESTION 11. *Is it true that* $[R;I][x]$ (*resp.* $[R;I][x,x^{-1}]$) *is left Noetherian iff* R *is left Noetherian and* I *is left* s*-unital?*

COROLLARY 12. *A module* ${}_RM$ *is Noetherian iff* ${}_{R[[x,x^{-1}]]}M[[x,x^{-1}]]$ *is Noetherian.*

EXAMPLE 13. *Let* $R = \mathbb{Z}_{p^n}$ *where* p *is a prime and* $n \geq 1$ *and* I *an ideal of* R. *Then* $[R;I][x]$ (*resp.* $[R;I][x,x^{-1}]$) *is left Noetherian iff* $I = 0$ *or* R.

EXAMPLE 14. *Let* I *be an ideal of* $\mathbb{Z}$. *Then* $[\mathbb{Z};I][x]$ (*resp.* $[\mathbb{Z};I][x,x^{-1}]$) *is left Noetherian iff* $I = 0$ *or* $\mathbb{Z}$.

EXAMPLE 15. *Let* V *be a left Noetherian ring with a left identity, and let* $R = \mathbb{I}(\mathbb{Z}, V)$ *be the ideal extension of* $\mathbb{Z}$ *by* V. *That is,* $(R,+) = \mathbb{Z} \oplus V$ *with multiplication defined by* $(m,v)(n,w) = (mn, mw + nv + vw)$. *Let* $I = 0 \oplus V$ (*an ideal of* R). *Then* $[R;I][x]$ *and* $[R;I][x,x^{-1}]$ *are left Noetherian rings.*

Proof. As $R/I \cong \mathbb{Z}$ is Noetherian, $(R/I)_R$ is Noetherian. As the lattice of submodules of I_R is isomorphic to the lattice of left ideals of V, ${}_RI$ is Noetherian by the assumption on V. Hence R is a left Noetherian ring. Since V has a left identity, I is a left s-unital ideal of R. So $[R;I][x]$ and $[R;I][x,x^{-1}]$ are left Noetherian by Corollary 9. ■

4. Quasi-duo rings. Following Yu [32], a ring is called *left quasi-duo* if every maximal left ideal is an ideal. Every factor ring of a left quasi-duo ring is again left quasi-duo (see [32]). In [15, Theorem 3.2], a characterization of a left quasi-duo ring is obtained: A ring R is left quasi-duo iff $Ra + R(ab-1) = R$ for all $a, b \in R$. It is easy to see that, for an ideal K of R with $K \subseteq J(R)$, R is left quasi-duo iff so is R/K. Hence R is left quasi-duo iff so is $R[[x]]$. In [18], the authors proved that $R[x]$ is left quasi-duo iff $J(R[x]) = N(R)[x]$ and $R/N(R)$ is commutative, where $N(R)$ denotes the nil radical of R. This result can be used to prove

THEOREM 16. *Let* $I \lhd R$ *and* $\overline{R} = R/I$. *The following are equivalent:*

(1) $[R;I][x]$ *is left quasi-duo.*
(2) R *and* $\overline{R}[x]$ *are left quasi-duo.*
(3) R *is left quasi-duo,* $J(\overline{R}[x]) = N(\overline{R})[x]$ *and* $\overline{R}/N(\overline{R})$ *is commutative.*

Proof. (1)⇒(2). Let $S = [R; I][x]$. Then $R \cong S/Sx$ and $\overline{R}[x] \cong S/I[[x]]$. So (1) clearly implies (2).

(2)⇒(1). By [18, Lemma 3.2], (2) implies that $R[x]/I[x]x$ is left quasi-duo. But $S/I[[x]]x = \big(R[x] + I[[x]]x\big)/I[[x]]x \cong R[x]/\big(R[x] \cap I[[x]]x\big) = R[x]/I[x]x$, so $S/I[[x]]x$ is left quasi-duo. Hence S is left quasi-duo, because $I[[x]]x \subseteq J(S)$ by Lemma 1.

(2)⇔(3). This is by [18, Corollary 4.3]. ∎

Corollary 17. *The ring* $[R; J(R)][x]$ *is left quasi-duo iff* $R/J(R)$ *is commutative.*

In [9], the authors proved that the transpose of every invertible matrix over R is invertible exactly when $R/J(R)$ is commutative.

Let δ_l denote the intersection of all essential maximal left ideals of R. Then δ_l is an ideal of R, and $\delta_l/S_l = J(R/S_l)$ where S_l denotes the left socle of R (see [33]). Hence $J(R/\delta_l) = 0$.

Corollary 18. $[R; \delta_l][x]$ *is left quasi-duo iff* R *is left quasi-duo and* R/δ_l *is commutative.*

5. Principal left ideal rings. Following Goldie [8], a ring R is called a *principal left ideal ring* (*pli-ring*) if every left ideal is principal. A *principal right ideal ring* (*pri-ring*) is defined similarly. In [13], Jategaonkar proved that a left skew polynomial ring $R[x; \varphi]$ is a prime pli-ring if R is a prime pli-ring and $\varphi : Q \to R$ is a monomorphism where Q is the simple Artinian left quotient ring of R. So a polynomial ring over a simple Artinian ring is a pli-ring. Jategaonkar also commented that this result and its proof can be adapted to left skew power series rings. In [27], Tuganbaev characterized the right skew polynomial rings $R[x, \varphi]$ which are pri-rings (where φ is an automorphism), and the right skew power series rings $R[[x, \varphi]]$ which are pli-rings (where φ is injective) or pri-rings (where φ is an automorphism). With $\varphi = 1_R$, these results state that $R[x]$ is a pli-ring iff $R[[x]]$ is a pli-ring iff R is semisimple Artinian.

Theorem 19. *Let* $I \lhd R$. *The following are equivalent:*

(1) $[R; I][x]$ *is a pri-ring.*
(2) $[R; I][x]$ *is a pli-ring.*
(3) R *is a semisimple Artinian ring.*

Proof. (1)⇒(3). Let $S = [R; I][x]$. Since a factor ring of a pri-ring is again a pri-ring, S/x^2S is a pri-ring by (1). So $R[x]/x^2R[x] \cong S/x^2S$ is a pri-ring. Thus R is semisimple Artinian by [27, Proposition 2.3].

(3)⇒(1). If $1 = e_1 + \cdots + e_n$ where $e_1, \ldots, e_n$ are orthogonal central idempotents of R, then $[R; I][x] \cong [e_1R; e_1I][x] \oplus \cdots \oplus [e_nR; e_nI][x]$. So we may assume that R is simple Artinian. If $I = 0$, then $[R; I][x] = R[x]$ is a pri-ring by [13, Theorem 3.1, p. 54]. If $I = R$, then $[R; I][x] = R[[x]]$ is a pri-ring by [31, Theorem 4.5]. ■

EXAMPLE 20. Let I be a nonzero proper ideal of a semisimple Artinian ring R. Then $[R; I][x]$ is a pli-ring and a pri-ring by Theorem 19, but it is not isomorphic to a polynomial ring or a power series ring by Example 7.

6. Hopfian modules. Following Hiremath [12], a module M over R is called *Hopfian* if every surjective endomorphism of M is an automorphism. One easily sees that the module ${}_RR$ is Hopfian iff R is a Dedekind finite ring, i.e., $ab = 1$ in R always implies $ba = 1$. Motivated by Theorem 2.1 in Varadarajan [29], we prove the following

THEOREM 21. *Let $I \lhd R$. Then a module ${}_RM$ is Hopfian iff $[M, IM][x]$ is a Hopfian module over $[R, I][x]$.*

Proof. Let $S = [R; I][x]$ and $V = [M; IM][x]$.

($\Rightarrow$) Let $p : V \to M$ be given by $p(\sum_{i\geq 0} v_i x^i) = v_0$. Then p is an R-homomorphism. Suppose that φ is a surjective endomorphism of ${}_SV$. For any $w_0 \in M$, there exists $v = \sum_{i\geq 0} v_i x^i \in V$ such that $\varphi(v) = w_0$. Thus,

$$w_0 = p(w_0) = p(\varphi(v)) = p\Big(\varphi(v_0) + x\varphi\Big(\sum_{i\geq 0} v_{i+1}x^i\Big)\Big) = p(\varphi(v_0)).$$

This shows that $p\varphi|_M : M \to M$ is surjective, so it is injective as ${}_RM$ is Hopfian.

Next we show that φ is injective. Assume that $\mathrm{Ker}(\varphi) \neq 0$. Then there exists $v = \sum_{i\geq k} v_i x^i \in V$ with $v_k \neq 0$ such that $\varphi(v) = 0$. Thus, $0 = \varphi(v) = \varphi(x^k \sum_{i\geq 0} v_{k+i}x^i) = x^k\varphi(\sum_{i\geq 0} v_{k+i}x^i)$; this shows that $\varphi(\sum_{i\geq 0} v_{k+i}x^i) = 0$. So $0 = p(0) = p(\varphi(\sum_{i\geq 0} v_{k+i}x^i)) = p(\varphi(v_k) + x\varphi(\sum_{i\geq 1} v_{k+i}x^i)) = p\varphi(v_k)$. Hence $v_k = 0$ as $p\varphi|_M$ is injective. This contradiction shows that φ is injective.

($\Leftarrow$) If f is a surjective endomorphism of ${}_RM$, then $f(IM) \subseteq IM$ and hence $\bar{f} : V \to V$, $\sum_{i\geq 0} v_i x^i \mapsto \sum_{i\geq 0} f(v_i)x^i$ is a surjective S-homomorphism, so it is injective by hypothesis. It follows that f is injective. ■

COROLLARY 22 ([29]). *A module ${}_RM$ is Hopfian iff ${}_{R[x]}M[x]$ is Hopfian iff ${}_{R[[x]]}M[[x]]$ is Hopfian.*

The question of Varadarajan [29] whether ${}_RM$ Hopfian implies that ${}_{R[x,x^{-1}]}M[x, x^{-1}]$ is Hopfian remains open. By Varadarajan [30], Corollary 22 holds true if R is a ring not necessarily possessing an identity and M is a left s-unital R-module.

7. Quasi-Baer rings and modules. Following Clark [7], a ring R is called *quasi-Baer* if for any ideal K of R, $\mathbf{l}_R(K) = Re$ where $e^2 = e \in R$. The definition of quasi-Baer rings is left-right symmetric by [7]. Following [16], a module M over R is called *quasi-Baer* if for any submodule N of M, $\mathbf{l}_R(N) = Re$ for some $e^2 = e \in R$. Thus R is a quasi-Baer ring iff ${}_RR$ is a quasi-Baer module. The following theorem is motivated by [5, Theorem 1.8] and [16, Corollary 2.14].

THEOREM 23. *Let $I \lhd R$. The following are equivalent:*

(1) *M is a quasi-Baer module over R.*
(2) *$[M; IM][x]$ is a quasi-Baer module over $[R; I][x]$.*
(3) *$[M; IM][x, x^{-1}]$ is a quasi-Baer module over $[R; I][x, x^{-1}]$.*

Proof. (1)⇒(2). Let $S = [R; I][x]$ and $V = [M; IM][x]$. Suppose that ${}_RM$ is a quasi-Baer module and let W be an S-submodule of V. We show that $\mathbf{l}_S(W)$ is generated by an idempotent as a left ideal of S. This is clearly true if $W = 0$. Assume that $W \neq 0$ and let

$$W_0 = \{0 \neq w \in M : w = \text{the coefficient of the lowest degree term of some } v(x) \in W\} \cup \{0\}.$$

Then W_0 is a submodule of M, so $\mathbf{l}_R(W_0) = Re$ where $e^2 = e \in R$. For any $v(x) = v_0 + v_1x + \cdots + v_kx^k + \cdots \in W$, we have $v_0 \in W_0$, so $ev_0 = 0$ holds. If $ev_i = 0$ for $0 \leq i \leq k$, then $ev(x) = ev_{k+1}x^{k+1} + ev_{k+2}x^{k+2} + \cdots \in W$, and so $ev_{k+1} \in W_0$. Hence $ev_{k+1} = e(ev_{k+1}) = 0$. By induction, we have $ev_i = 0$ for all $i \geq 0$. So $ev(x) = 0$ and hence $Se \subseteq \mathbf{l}_S(W)$. To show that $Se \supseteq \mathbf{l}_S(W)$, let $f(x) = a_0 + a_1x + \cdots \in \mathbf{l}_S(W)$. It suffices to show that $a_i = a_ie$ for all $i \geq 0$ (this gives $f(x) = f(x)e$). For any $w_0 \in W_0$, there exists $w(x) = w_0x^k + w_1x^{k+1} + \cdots \in W$ where $k \geq 0$. Then $f(x)w(x) = 0$, which implies that $a_0w_0 = 0$. Since w_0 is an arbitrary element of W_0, one finds that $a_0 \in \mathbf{l}_R(W_0) = Re$; so $a_0 = a_0e$. Let us assume that $a_i = a_ie$ for all $0 \leq i \leq k$. Thus $f(x) = (a_0 + a_1x + \cdots + a_kx^k)e + f_1(x)x^{k+1}$ where $f_1(x) = a_{k+1} + a_{k+2}x + \cdots$. So $f_1(x)x^{k+1}$, and hence $f_1(x)$ is in $\mathbf{l}_S(W)$. From $f_1(x)w(x) = 0$, it follows that $a_{k+1}w_0 = 0$. Hence $a_{k+1} \in \mathbf{l}_R(W_0) = Re$, so $a_{k+1} = a_{k+1}e$. An induction shows that $a_i = a_ie$ for all $i \geq 0$.

(2)⇒(1). Suppose that $V := [M; MI][x]$ is a quasi-Baer module over $S := [R; I][x]$. To show that ${}_RM$ is quasi-Baer, let N be a submodule of M. Then $U := [N; IN][x]$ is an S-submodule of V and therefore $\mathbf{l}_S(U) = Se(x)$ where $e(x)^2 = e(x) \in S$. Let e_0 be the constant term of $e(x)$. Then $e_0^2 = e_0$ and $e_0N = 0$ (as $e(x)U = 0$). So $Re_0 \subseteq \mathbf{l}_R(N)$. For any $a \in \mathbf{l}_R(N)$, $aU = 0$. Thus $a \in \mathbf{l}_S(U) = Se(x)$, so $a = ae(x)$. This gives $a = ae_0 \in Re_0$. So $\mathbf{l}_R(N) = Re_0$.

(1)⇒(3). Same as the proof of (1)⇒(2).

(3)⇒(1). Suppose that $V := [M; MI][x, x^{-1}]$ is a quasi-Baer module over $S := [R; I][x, x^{-1}]$. To show that ${}_RM$ is quasi-Baer, let N be a submodule of M. Then $U := [N; IN][x, x^{-1}]$ is an S-submodule of V and therefore

$\mathbf{l}_S(U) = Se(x)$ where $e(x)^2 = e(x) \in S$. Write $e(x) = \sum_{i\geq -l} e_i x^i$ where $e_i \in \mathbf{l}_R(N)$. For any $a \in \mathbf{l}_R(N)$, $a \in \mathbf{l}_S(U) = Se(x)$, so $a = ae(x)$. This shows that $a = ae_0$. Consequently, $e_0^2 = e_0$ and $\mathbf{l}_R(N) = Re_0$. ∎

COROLLARY 24 ([16]). *A module ${}_RM$ is quasi-Baer iff ${}_{R[x]}M[x]$ is quasi-Baer iff ${}_{R[[x]]}M[[x]]$ is quasi-Baer iff ${}_{R[x,x^{-1}]}M[x,x^{-1}]$ is quasi-Baer iff ${}_{R[[x,x^{-1}]]}M[[x,x^{-1}]]$ is quasi-Baer.*

COROLLARY 25. *Let $I \lhd R$. Then R is quasi-Baer iff $[R;I][x]$ is quasi-Baer iff $[R;I][x,x^{-1}]$ is quasi-Baer.*

COROLLARY 26 ([5]). *A ring R is quasi-Baer iff $R[x]$ is quasi-Baer iff $R[[x]]$ is quasi-Baer iff $R[x,x^{-1}]$ is quasi-Baer iff $R[[x,x^{-1}]]$ is quasi-Baer.*

EXAMPLE 27.

(1) *Let R be any countable quasi-Baer ring which is semipotent, and I a nonzero proper ideal of R. Then $[R;I][x]$ is a quasi-Baer ring by Corollary* 25*, but it is not isomorphic to either of $R[x]$ and $R[[x]]$ by Example* 6*.*

(2) *Let R be a primitive potent ring, and I a nonzero proper ideal of R. Then R is a quasi-Baer ring by* [3, Lemma 4.2]. *So $[R;I][x]$ is quasi-Baer by Corollary* 25*, but it is not isomorphic to either of $R[x]$ and $R[[x]]$ by Example* 6*.*

8. Principally quasi-Baer rings and modules. Following Birkenmeier, Kim and Park [4], a ring R is called *left principally quasi-Baer* (or simply *left p.q.-Baer*) if the left annihilator of a principal left ideal is generated as a left ideal by an idempotent. Following Başer and Harmanci [2], a module M over R is called *p.q.-Baer* if for any cyclic submodule N of M, $\mathbf{l}_R(N) = Re$ for some $e^2 = e \in R$. These rings and modules are extensions of quasi-Baer rings and modules.

LEMMA 28. *Let $f(x) = \sum_{i\geq -l} a_i x^i \in R[[x,x^{-1}]]$ and $v(x) = \sum_{i\geq -k} v_i x^i \in M[[x,x^{-1}]]$, where $l,k \geq 0$, be such that, for $j = -k, -(k-1), \ldots,$ the left annihilator of Rv_j in R is generated as a left ideal by an idempotent. If $f(x)Rv(x) = 0$ then $a_iRv_j = 0$ for all i and j.*

Proof. From $f(x)Rv(x) = 0$ it follows that $\big(x^l f(x)\big)R\big(x^k v(x)\big) = 0$. Thus we can assume that $l = k = 0$. Write $\mathbf{l}_R(Rv_0) = Re$ where $e^2 = e \in R$. From $f(x)Rv(x) = 0$, it follows that $a_0Rv_0 = 0$, so $a_0 \in \mathbf{l}_R(Rv_0)$ and hence $a_0 = a_0e$. Assume that $a_iRv_0 = 0$ for $i = 0, 1, \ldots, n$. Thus, $a_i = a_ie$ for $i = 0, 1, \ldots, n$. Since $f(x)Rv(x) = 0$, we have

$$a_0rv_{n+1} + a_1rv_n + \cdots + a_nrv_1 + a_{n+1}rv_0 = 0$$

for all $r \in R$. Replacing r by er in this formula yields $a_0rv_{n+1} + a_1rv_n + \cdots + a_nrv_1 = 0$ (as $eRv_0 = 0$), and hence $a_{n+1}rv_0 = 0$ for all $r \in R$. So

$a_{n+1}Rv_0 = 0$. By the induction principle, $a_iRv_0 = 0$ for all $i = 0, 1, \ldots$. Hence $f(x)Rv_0 = 0$. Assume that $f(x)Rv_j = 0$ for $j = 0, 1, \ldots, m-1$. It follows from $f(x)Rv(x) = 0$ that $f(x)R(\sum_{i\geq 0} v_{m+i}x^i) = 0$. As above we have $f(x)Rv_m = 0$. So $f(x)Rv_j = 0$ for all j by induction. ∎

The next lemma is implicitly contained in the proof of [5, Lemma 1.7].

LEMMA 29 ([5]). *Let* $e(x)^2 = e(x) = \sum_{i=-l}^{\infty} e_ix^i \in R[[x, x^{-1}]]$ *where* $l \geq 0$. *If* $e(x)ae(x) = e(x)a$ *for all* $a \in R$, *then* $e_0^2 = e_0$.

THEOREM 30. *Let* $I \lhd R$. *The following are equivalent:*

(1) $[M; IM][x]$ *is a p.q.-Baer module over* $[R; I][x]$.
(2) $[M; IM][x, x^{-1}]$ *is a p.q.-Baer module over* $[R; I][x, x^{-1}]$.
(3) *For any sequence* $\{v_0, v_1, \ldots\}$ *of elements of* M *with almost all* v_i *in* IM, $\mathbf{l}_R(\sum_{i\geq 0} Rv_i) = Re$ *for some* $e^2 = e \in R$.
(4) ${}_RM$ *is a p.q.-Baer module, and for any sequence* $\{v_0, v_1, \ldots\}$ *of elements of* IM, $\mathbf{l}_R(\sum_{i\geq 0} Rv_i) = Re$ *for some* $e^2 = e \in R$.

Proof. Let $S = [R; I][x, x^{-1}]$ and $V = [M; IM][x, x^{-1}]$.

(2)⇒(3). Let $w \in M$. By (2), $\mathbf{l}_S(Sw) = Se(x)$ where $e(x) = \sum_{i\geq -l} e_ix^i$ ($l \geq 0$) is an idempotent of S. As $Se(x)$ is an ideal of S, $e(x)S \subseteq Se(x)$, so $e(x)a = e(x)ae(x)$ for all $a \in R$. Then $e_0^2 = e_0$ by Lemma 29, and it follows that $e_0Rw = 0$, so $\mathbf{l}_R(Rw) \supseteq Re_0$. If $a \in \mathbf{l}_R(Rw)$, then $a \in \mathbf{l}_S(Sw)$, so $a = ae(x)$; hence $a = ae_0$. So $\mathbf{l}_R(Rw) = Re_0$. This shows that ${}_RM$ is a p.q.-Baer module.

Let $v_i \in M$ for $i = 0, 1, \ldots$ with $v_i \in IM$ for almost all i. Then $v(x) := \sum_{i\geq 0} v_ix^i \in V$, so $\mathbf{l}_S(Sv(x)) = Sg(x)$ where $g(x) = \sum_{i\geq -l} g_ix^i$ ($l \geq 0$) is an idempotent of S. By Lemma 29, $g_0^2 = g_0$. By Lemma 28, $g_iRv_j = 0$ for all i and j. Thus $\mathbf{l}_R(\sum_{i\geq 0} Rv_i) \supseteq Rg_0$. If $a \in \mathbf{l}_R(\sum_{i\geq 0} Rv_i)$, then $a \in \mathbf{l}_S(Sv(x))$. Thus $a = ag(x)$, so $a = ag_0 \in Rg_0$.

(3)⇒(4). This is clear.

(4)⇒(2). Let $v(x) = \sum_{i\geq -l} v_ix^i \in V$ where $l \geq 0$. Then there exists $n > -l$ such that $v_i \in IM$ for all $i \geq n$. By (4), there exist idempotents $e_{-l}, \ldots, e_{n-1}, e_n$ of R such that $\mathbf{l}_R(Rv_i) = Re_i$ for $i = -l, \ldots, n-1$ and $\mathbf{l}_R(\sum_{i\geq n} Rv_i) = Re_n$. Since Re_i is an ideal of R (for $i = -l, \ldots, n$), we have $e_iR \subseteq Re_i$, i.e., $e_ia = e_iae_i$ for all $a \in R$. It follows that $e := e_{-l} \cdots e_n$ is an idempotent and $\bigcap_{i=-l}^{n} Re_i \subseteq Re$. Moreover, for any $-l \leq i \leq n$, we have $e = ee_i \in Re_i$. Hence $\bigcap_{i=-l}^{n} Re_i = Re$. Thus,

$$\mathbf{l}_R\Big(\sum_{i\geq -l} Rv_i\Big) = \mathbf{l}_R(Rv_{-l}) \cap \cdots \cap \mathbf{l}_R(Rv_{n-1}) \cap \mathbf{l}_R\Big(\sum_{i\geq n} Rv_i\Big) = \bigcap_{i=-l}^{n} Re_i = Re.$$

Hence $\mathbf{l}_S(Sv(x)) \supseteq Se$. If $h(x) = \sum_{i\geq -s} h_ix^i \in \mathbf{l}_S(Sv(x))$ ($s \geq 0$), then

$h_i \in \mathbf{l}_R(\sum_{i\geq -l} Rv_i)$ for all $i \geq 0$ by Lemma 28. So $h_i = h_ie$ and hence $h(x) = h(x)e \in Se$. So $\mathbf{l}_S(Sv(x)) = Se$.

(1)⇔(3)⇔(4). The proof is similar to the proof of the equivalences (2)⇔(3)⇔(4), even without the use of Lemma 29. ■

Corollary 31 ([2]). *The module ${}_{R[x]}M[x]$ is p.q.-Baer iff ${}_RM$ is p.q.-Baer.*

Corollary 32 ([11]). *The module ${}_{R[[x]]}M[[x]]$ is p.q.-Baer iff the left annihilator in R of any countably generated submodule of M is generated as a left ideal by an idempotent.*

Corollary 33. *The module ${}_{R[x,x^{-1}]}M[x,x^{-1}]$ is p.q.-Baer iff ${}_RM$ is p.q.-Baer.*

Corollary 34. *The module ${}_{R[[x,x^{-1}]]}M[[x,x^{-1}]]$ is p.q.-Baer iff the left annihilator in R of any countably generated submodule of M is generated as a left ideal by an idempotent.*

Corollary 35. *Let $I \lhd R$. The following are equivalent:*

(1) $[R;I][x]$ *is left p.q.-Baer.*
(2) $[R;I][x,x^{-1}]$ *is left p.q.-Baer.*
(3) *For any sequence $\{a_0, a_1, \ldots\}$ of elements of R with almost all a_i in I, $\mathbf{l}_R(\sum_{i\geq 0} Ra_i) = Re$ for some $e^2 = e \in R$.*
(4) *R is left p.q.-Baer, and for any sequence $\{a_0, a_1, \ldots\}$ of elements of I, $\mathbf{l}_R(\sum_{i\geq 0} Ra_i) = Re$ for some $e^2 = e \in R$.*

Corollary 36 ([4]). *$R[x]$ is left p.q.-Baer if and only if R is left p.q.-Baer.*

Corollary 37 ([20]). *$R[x,x^{-1}]$ is left p.q.-Baer if and only if R is left p.q.-Baer.*

Corollary 38 ([21]). *$R[[x]]$ is left p.q.-Baer if and only if the left annihilator of any countably generated left ideal of R is generated as a left ideal by an idempotent.*

In [20], Liu discussed the question of when the ring $R[[x,x^{-1}]]$ is left p.q.-Baer. An idempotent e of R is called *right semi-central* if $er = ere$ for all $r \in R$. Following [20], a countable set $\{e_i : i \geq 0\}$ of idempotents of R is said to have a *generalized join* if there exists $e^2 = e \in R$ such that (1) $(1-e)Re_i = 0$ for all i and (2) $(1-f)Re = 0$ for any $f^2 = f \in R$ with $(1-f)Re_i = 0$ for all i. Liu [20, Theorem 4] proved: If $R[[x,x^{-1}]]$ is left p.q.-Baer, then any countable set of idempotents of R has a generalized join; the converse holds if every right semicentral idempotent of R is central. It was noticed in [20, Example 6] that for a ring R for which $R[[x,x^{-1}]]$ is left p.q.-Baer, right semicentral idempotents need not be central. Corollary 35 has an immediate consequence.

Corollary 39. *$R[[x, x^{-1}]]$ is left p.q.-Baer if and only if the left annihilator of any countably generated left ideal of R is generated as a left ideal by an idempotent.*

Example 40. *Let F be a field and $Q = \prod_{i=1}^{\infty} R_i$ a direct product of rings where $R_i = F$ for all i. Let $R = \langle \bigoplus_i R_i, 1_Q \rangle$ be the subring of Q generated by $\bigoplus_i R_i$ and 1_Q. Then* $\mathrm{soc}(R) := \bigoplus_i R_i$ *is the socle of R. Let I be an ideal of R. Then:*

(1) *$[R; I][x]$ is left p.q.-Baer iff I is a principal ideal of R contained in* $\mathrm{soc}(R)$.

(2) *For any nonzero principal ideal I of R contained in* $\mathrm{soc}(R)$, *$[R; I][x]$ is not isomorphic to any polynomial ring or any power series ring.*

Proof. (1)($\Rightarrow$) Assume that $I \not\subseteq \mathrm{soc}(R)$. Then there exist $k \in \mathbb{Z}$ and $y \in \mathrm{soc}(R)$ such that $k1_Q \neq 0$ and $k1_Q + y \in I$. Thus, $1_Q + z \in I$ for some $z \in \mathrm{soc}(R)$. We can assume that $z \in \bigoplus_{i=1}^{s} R_i$. Write $e_i = 1_{R_i}$. Then, for $i > s$, $e_i = e_i(1_Q + z) \in I$. But $\mathbf{l}_R(\sum_{i=1}^{\infty} Re_{s+2i}) = (\bigoplus_{i=1}^{s+1} Re_i) \oplus (\bigoplus_{i=1}^{\infty} Re_{s+2i+1})$, which is not generated by an idempotent. This is a contradiction by Corollary 35. So I is contained in $\mathrm{soc}(R)$. If I is not principal, then it is not finitely generated (as R is von Neumann regular), and so $e_i \in I$ for infinitely many i. But this gives a contradiction by arguing as above. Hence I is principal.

(1)($\Leftarrow$) Since R is a commutative regular ring, it is left p.q.-Baer. The hypothesis shows that $I = \bigoplus_{i \in L} R_i$ where L is a finite subset of $\mathbb{N}$. Let Z be any countable subset of I, and let $S = \{i \in L : \exists z \in Z$ such that the projection of z onto R_i is nonzero$\}$. Then $\mathbf{l}_R(\sum_{z \in Z} Rz) = Re$ where $e = 1_Q - \sum_{i \in S} e_i$ is an idempotent of R. So $[R; I][x]$ is left p.q.-Baer by Corollary 35.

(2) Since R is von Neumann regular, $[R; I][x]$ is not isomorphic to any polynomial ring or power series ring by Example 7. ∎

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