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WEAKLY AMENABLE GROUPS AND THE RNP FOR SOME BANACH ALGEBRAS RELATED TO THE FOURIER ALGEBRA

BY

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Abstract. It is shown that if G is a weakly amenable unimodular group then the Banach algebra $A_p^r(G) = A_p \cap L^r(G)$, where $A_p(G)$ is the Figà-Talamanca–Herz Banach algebra of G, is a dual Banach space with the Radon–Nikodym property if $1 \le r \le \max(p, p')$. This does not hold if $p = 2$ and $r > 2$.

Let G be a locally compact group and let $A_p(G)$ denote the Figà-Talamanca–Herz Banach algebra of G as defined in [Hz1], thus generated by $L^{p'} * \check{L}^p(G)$. Hence $A_2(G)$ is the Fourier algebra of G as defined and studied in Eymard [Ey1]. If G is abelian then $A_2(G) = L^1(\hat{G})\hat{}$.

Denote $A_p^r(G) = A_p \cap L^r(G)$ for $1 \le r \le \infty$, $1 < p < \infty$, equipped with the norm $\|u\|_{A_p^r} = \|u\|_{A_p} + \|u\|_{L^r}$. If G is abelian then $A_2^r(\hat{G}) = \{f \in L^1(G) : \hat{f} \in L^r(\hat{G})\}$, with the norm $\|u\| = \|f\|_{L^1(G)} + \|\hat{f}\|_{L^r(\hat{G})}$ if $u = \hat{f}$.

A Banach space has the **RNP** if its unit ball *wants to be weakly compact but just cannot make it*, as beautifully put by Jerry Uhl.

A Banach space X has the *Krein–Milman Property* (**KMP**) [*Radon Nikodym Property* (**RNP**)] if each closed convex bounded subset is the norm closed convex hull of its extreme points [strongly exposed points] (see [DU, p. 138]). If X is a dual Banach space, then the **RNP** *and* **KMP** *are equivalent* (see [DU, p. 190 and p. 218]).

Strongly exposed points are extreme points which are very "smooth" (they are certainly weak-to-norm continuity points), and the fact that we can take the above as the *definition* of the **RNP**, is owed to the valiant efforts of many mathematicians (see [DU]).

The Fourier algebra of the torus, $A_2(\mathbb{T})$, which is in fact $\ell_1(\mathbb{Z})$, has the **RNP**, a property possessed by any Banach space which is isomorphic to an ℓ_1 space (see [DU]), while $A_2(\mathbb{R})$ does not possess the **RNP**.

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And yet, for any compact subset K of $\mathbb{R}$, $A^2_K(\mathbb{R}) = \{u \in A_2(\mathbb{R}) : \operatorname{spt} u \subset K\}$ does have the **RNP** (and $\mathbb{R}$ can be replaced by any abelian G; here spt denotes support).

We have proved in [Gr1] that *for any G and any compact $K \subset G$ and any $1 < p < \infty$, $A^p_K(G) = \{u \in A_p(G) : \operatorname{spt} u \subset K\}$ has the* **RNP**. Tools in abelian harmonic analysis are not available to prove this latter result.

It has been proved by W. Braun in an unpublished preprint [Br] that if G is amenable then $A^1_p(G)$ is a dual Banach space with the **RNP**. The result in [Br] uses the method in [Gr1] and the involved machinery of [BrF], which is avoided below and in [Gr3]. We have proved in [Gr3] the following

THEOREM 0.1.

(A) *Let G be unimodular and $1 < p < \infty$. If G is amenable then $A^r_p(G)$ is a dual Banach space with the **RNP** for all $1 \le r \le \max(p, p')$.*

(B) *Let G be unimodular and $A_2(G)$ have a multiplier bounded approximate identity. Then $A^r_2(G)$ is a dual Banach space with the **RNP** for all $1 \le r \le 2$.*

(C) *If G is* $\mathrm{SL}(2,\mathbb{R})$ *or* $\mathrm{SL}(2,\mathbb{C})$ *then, for any $2 < r \le \infty$, $A^r_2(G)$ does not have the **RNP*** (*see* [Gr3, p. 4382])*, even though these groups are unimodular, weakly amenable* (*and nonamenable; see* [DCH, Thm. 3.7 and Remark 3.8(b)])*. Hence the above interval for r is the best possible.*

A group G is *weakly amenable* if $A_2(G)$ has an approximate identity bounded in the (Herz–Schur multiplier) $B_2(G)$ norm (see below).

It is the main purpose of this paper to show that Theorem 0.1(A) is true if G is merely weakly amenable.

It has been proved by De Cannière and Haagerup [DCH, pp. 481–486] that any closed subgroup G of any finite extension of the general Lorenz group $\mathrm{SO}_0(n,1)$ for all $n \ge 2$ (hence in particular $G = F_N$, the free group on $N > 1$ generators) is weakly amenable. Thus there exists a multitude of *nonamenable groups which are weakly amenable*. And yet, Haagerup [Ha] has proved that $G = \mathrm{SL}(2,\mathbb{R}) \rtimes \mathbb{R}^2$ *is not weakly amenable* (see also [Do]).

One will note that, in proving the main result, some difficulties need to be overcome to prove that $W^r_p(G)$ is a dual Banach space for all $r \ge 1$. This is done in Section 1.

The main result is proved in Section 2.

In Section 3 we prove that the Banach algebras $A^r_p(G)$ do not factorise for any noncompact G and $1 \le r < \infty$, a result announced earlier.

1. Definitions and notations. Denote by $PM_p(G) = A_p(G)^*$ the Banach space dual of $A_p(G)$. We will omit G at times and write A_p, L^r, PM_p, etc., instead of $A_p(G)$, $L^r(G)$, $PM_p(G)$, etc.

Let PF_p denote the norm closure of L^1 in PM_p and set $W_p(G) = PF_p(G)^*$. Then W_p is a Banach algebra of bounded continuous functions on G, studied by M. Cowling [Co1].

Define $W_p^r(G) = W_p \cap L^r(G)$, with the norm $\|w\|_{W_p^r} = \|w\|_{W_p} + \|w\|_{L^r}$.

Denote by $M(A_p)$ the set of multipliers of A_p with the norm

$$\|u\|_{M(A_p)} = \sup\{\|uv\|_{A_p};\ v \in A_p,\ \|v\|_{A_p} \le 1\}, \qquad u \in M(A_p).$$

If $v \in A_p$, let $\|v\|_{W_p}$ be the norm of v as an element of $PF_p(G)^* = W_p(G)$.

If $u \in M(A_2)$ let $m_u : A_2 \to A_2$ be given by $m_u v = uv$. Let $M_u = m_u^* : PM_2 \to PM_2$ denote the adjoint of m_u. The multiplier $u \in M(A_2)$ is *completely bounded* (and $M_0(A_2)$ is the algebra of all such multipliers) if the operator $M_u : PM_2 \to PM_2$ is completely bounded on the W^* algebra PM_2. The set $M_0(A_2)$ is equipped with the norm $\|u\|_{M_0(A_2)} = \|M_u\|_{cb}$, the completely bounded norm of the operator M_u (see [DCH], [CH], [Jo], where all the above notions are defined).

It has been proved by Bożejko and Fendler [BF] that $M_0(A_2)$ coincides with the space $B_2 = B_2(G)$ of Herz–Schur multipliers and $\|u\|_{M_0(A_2)} = \|u\|_{B_2}$.

A group G is *weakly amenable* if $A_2(G)$ has an approximate identity (A.I.) bounded in the $\|\ \|_{B_2(G)}$ norm.

In an important ground-breaking paper [DCH], De Cannière and Haagerup have studied weakly amenable groups G.

2. $W_p(G) \cap L^r(G)$ **is a dual Banach space for all** $r \ge 1$**.** We have proved this result, for any group G, in [Gr3, Prop. 2.1] *only for* $r > 1$. The proof there fails in case $r = 1$. This case requires an entirely different proof, which is given below.

The result in the title of this section is needed to prove that for all, $1 \le r \le \max(p, p')$, A_p^r is a dual Banach space if G is unimodular and weakly amenable.

Remark (for $r = 1$). Let $Z = X \times Y$, $X = PF_p$, $Y = L^\infty$, with norm $\|(x, y)\| = \max(\|x\|, \|y\|)$. Hence $Z^* = X^* \times Y^* = W_p \times L^{\infty *}$, with norm $\|(x^*, y^*)\| = \|x^*\| + \|y^*\|$. Let $D = \{(w, w);\ w \in W_p \cap L^1\} \subset W_p \cap L^{\infty *}$. Let $U = w^*\text{-}\operatorname{cl} D \subset Z^*$. If U_0 $[(U_0)^0]$ is the annihilator of U $[U_0]$ in Z $[Z^*]$, respectively, then, since U is w^*-closed, $U = (U_0)^0 = (Z/U_0)^*$ (see [Da, p. 822]). Thus U is a dual Banach space.

Let now $P : X \to X \times Y$ be given by $Px = (x, 0)$. Then $P^* : X^* \times Y^* \to X^* = W_p$ is onto, in fact $P^*(x^*, y^*) = x^*$.

Lemma 2.1.

(a) $W_p \cap L^{\infty *} = W_p \cap L^1$.
(b) $P^* U = W_p \cap L^1$.

Proof. Let $w \in W_p \cap L^{\infty *}$. Clearly $W_p = PF_p^*$ and $L^1 \subset PF_p$. If $f \in L^1$ then $w(f) = \int wf\,dx$. If $f \in L^1 \cap L^\infty$ then $|w(f)| = |\int wf\,dx| \leq \|w\|_{L^{\infty *}} \|f\|_{L^\infty}$. Hence if $\|f\|_{L^\infty} \leq 1$ then $|\int wf\,dx| \leq \|w\|_{L^{\infty *}}$. Let now $K \subset G$ be compact and $f = (\overline{w}/|w|)1_K$; then $\int_K |w|\,dx \leq \|w\|_{L^{\infty *}}$. Hence $w \in L^1$, which proves (a).

(b) Let $(w_\alpha, w_\alpha) \in D \subset W_p \times L^{\infty *}$ satisfy $w^*\text{-}\lim(w_\alpha, w_\alpha) = (w, z) \in W_p \times L^{\infty *}$. Then for $f \in L^1 \subset PF_p$ one has $\int w_\alpha f\,dx = w_\alpha(f) \to \int wf\,dx$. And for $f \in L^\infty$, $\int w_\alpha f\,dx = w_\alpha(f) \to z(f)$. Thus $w(f) = z(f)$ for all $f \in L^1 \cap L^\infty$. Hence by (a), $w \in L^1$. Thus $U = w^*\text{-}\operatorname{cl} D \subset \big(W_p \cap L^1\big) \times L^{\infty *}$. It follows that $P^*(U) = W_p \cap L^1$, since $D = \{(w, w);\ w \in W_p \cap L^1\}$. ∎

Remark. Let $N = \{u \in U;\ P^*(u) = 0\} = U \cap (0, Y^*)$. Then U/N is isomorphic to $W_p \cap L^1$, where U is a dual space and N is a w^*-closed subspace.

Theorem 2.2. *$W_p(G) \cap L^r(G)$ with the norm $\|w\|_{W_p} + \|w\|_{L^r}$ is a dual Banach space for all $1 \leq r \leq \infty$, and for all locally compact groups G.*

Proof. If $r > 1$ this is just our Proposition 2.1 in [Gr3]. If $r = 1$, then $U = X^*$ for some Banach space X and $N = (N_0)^0$. Hence by [Da, p. 822, Theorem A.3.47(i)], $U/N \approx X^*/(N_0)^0 = N_0^*$. Thus $W_p \cap L^1(G)$ is norm isomorphic to a dual Banach space, thus is a dual space, by the use of the main theorem of Kaijser [Ka]. ∎

Remark. Note that a dual Banach space may have *two non-norm isomorphic preduals* (see [BL]).

3. Weakly amenable groups and the RNP. It is the purpose of this section to prove the main result of this paper, namely:

Theorem 3.1. *Let G be unimodular and weakly amenable, and let $1 < p < \infty$. Then for all $1 \leq r \leq \max(p, p')$, $A_p^r(G)$ is a dual Banach space which has the **RNP**.*

*If G is $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$ then $A_2^r(G)$, for any $2 < r \leq \infty$, does not have the **RNP**, a fortiori is not a dual Banach space, by* [Gr3].

If G is amenable this is part of [Gr3, Theorem 2.2, p. 4380].

The proofs in [Gr3] will work for proving our main result once we show that *if G is weakly amenable, the W_p norm restricted to A_p is equivalent to the A_p norm.*

It has been proved by M. Cowling [Co1] that the group $\mathrm{SL}(2,\mathbb{R})$ satisfies the assumption of the next result.

Proposition 3.2. *Assume that $A_p(G)$ has an approximate identity $\{u_\alpha\}$ such that $\|u_\alpha\|_{M(A_p)} \leq K$. Then*

$$\forall u \in A_p, \qquad \|u\|_{A_p} \leq (1 + K)\|u\|_{W_p} \leq (1 + K)\|u\|_{A_p}.$$

REMARK. It has been proved by Haagerup [Ha] (see [Do] for a different proof) that if $G = \mathrm{SL}(2,\mathbb{R}) \rtimes \mathbb{R}^2$, then $A_2(G)$ has no multiplier bounded approximate identity. It is not clear to us if Theorem 3.1 holds for this group.

Proof of Proposition 3.2. Clearly $\|u\|_{W_p} \le \|u\|_{A_p}$ for all $u \in A_p$ by the definition of these norms, hence only the left hand inequality needs proof.

Let $e_\alpha \in A_p \cap C_c$ satisfy $\|e_\alpha - u_\alpha\|_{A_p} \to 0$. As is readily seen, $\{e_\alpha\}$ is an A.I. for A_p and for some α_0, $\|e_\alpha\|_{M(A_p)} \le 1 + K$ if $\alpha > \alpha_0$. Hence for any $T \in PM_p$, $\|e_\alpha T\|_{PM_p} \le 1 + K$ if $\|T\|_{PM_p} \le 1$. And if $v \in A_p$ then $|(e_\alpha T, v) - (T, v)| = |(T, e_\alpha v - v)| \to 0$, thus $e_\alpha T \to T$ in $\sigma(PM_p, A_p)$, i.e. in w^*. Also $\operatorname{spt} e_\alpha T \subset \operatorname{spt} e_\alpha$, which is compact.

Hence if $u_0 \in A_p$ then

$$\|u_0\|_{A_p} \le \sup\{|(u_0, T)|;\ \|T\| \le 1 + K,\ \operatorname{spt} T \text{ is compact}\}.$$

However if $\operatorname{spt} T$ is compact there exists a net $f_\alpha \in C_c(G)$ such that $\|\lambda_p f_\alpha\| \le \|T\|$ and $\lambda_p f_\alpha \to T$ ultrastrongly by [Hz1, Prop. 9, p. 117]. Hence

$$\|u_0\|_{A_p} \le \sup\{|(u_0, \lambda_p f)|;\ f \in C_c,\ \|\lambda_p f\| \le 1 + K\} = (1 + K)\|u_0\|_{W_p}. \ \blacksquare$$

The algebra $B_p(G)$ of Herz–Schur multipliers of G, for $1 < p < \infty$, has been investigated by Eymard [Ey1] and Herz [Hz1]–[Hz3]. As shown in these papers (see also [Fu, p. 581])

$$A_p(G) \subset W_p(G) \subset B_p(G) \subset MA_p(G)$$

and each imbedding is contractive.

DEFINITION 3.3. G is *p-weakly amenable* if $A_p(G)$ has an A.I. bounded in the $\|\ \|_{B_p(G)}$ norm. Thus 2-weak amenability and weak amenability are identical.

We need the fact that 2-*weak amenability implies p-weak amenability for all p*. This is hinted in [Fu, p. 586], in different terminology, without proof. We give a proof based on Furuta's useful theorem [Fu, Theorem 2.4]:

THEOREM 3.4. *For any $1 < p < \infty$, $B_2(G) \subset B_p(G)$ and $\|u\|_{B_p} \le \|u\|_{B_2}$ for all $u \in B_2(G)$.*

Furuta's proof of this theorem is based on an unpublished theorem of J. E. Gilbert:

THEOREM 3.5. *Let w be a function on G. Then $w \in B_2(G)$ iff there exists a Hilbert space K and bounded continuous functions u, v from G to K such that $w(x^{-1}y) = \langle u(y), v(x)\rangle$ for all $x, y \in G$.*

A proof of this theorem has been given in [Ha], and for a different proof see P. Jolissaint [Jo].

The proof of the next result is very different from the suggestion, with no proof, given in [Fu, p. 586].

Proposition 3.6. *If G is 2-weak amenable then it is p-weak amenable for all $1 < p < \infty$.*

Proof. Let $\{v_\alpha\}$ be an A.I. in $A_2(G)$ such that $\|v_\alpha\|_{B_2} \le C$. Let $\{u_\alpha\} \subset A_2 \cap C_c \subset A_p \cap C_c$ satisfy $\|u_\alpha - v_\alpha\|_{A_2} \to 0$. If $v \in A_2$ then

$$\|u_\alpha v - v\|_{A_2} \le \|u_\alpha - v_\alpha\|_{A_2}\|v\|_{A_2} + \|v_\alpha v - v\|_{A_2} \to 0.$$

Moreover

$$\|u_\alpha\|_{B_p} \le \|u_\alpha\|_{B_2} \le \|u_\alpha - v_\alpha\|_{A_2} + \|v_\alpha\|_{B_2} \le 2C$$

if $\alpha > \alpha_0$, for some α_0. It follows by Furuta's theorem that $\{u_\alpha\}$ is an A.I. for $A_p(G)$ equipped with the $\| \ \|_{B_p}$ norm, *while we need that it be in the* $\| \ \|_{A_p}$ *norm.*

In contrast to the hint in [Fu, p. 586] we proceed as follows:

Let K_β be compact subsets of G whose interiors satisfy $\operatorname{int} K_\beta \uparrow G$. Let $V = V^{-1}$ be a neighborhood of the unit e of G with compact closure. Let

$$e_\beta(x) = \lambda(V)^{-1}(1_{K_\beta V} \ {}^*1_V(x)) = \lambda(V)^{-1}\lambda(xV \cap K_\beta V).$$

Then $e_\beta(x) = 1$ [0] if $x \in K_\beta$ [$x \notin K_\beta V^2$], respectively. Choose a subnet $\{u_\beta\}$ of $\{u_\alpha\}$ such that $\|u_\beta e_\beta - e_\beta\|_{A_2} \le 1$ and let $s_\beta(x) = (u_\beta + e_\beta - u_\beta e_\beta)(x)$. Then $s_\beta(x) = 1$ [0] if $x \in K_\beta$ [$x \notin K_\beta V^2 \cup \operatorname{spt} u_\beta$]. Also

$$\|s_\beta\|_{B_p} \le \|s_\beta\|_{B_2} \le \|u_\beta\|_{B_2} + 1 \le 2C + 1,$$

by Furuta's theorem. If $v \in A_p \cap C_c$ and $K = \operatorname{spt} v$, then $K \subset K_\beta$ if $\beta > \beta_0$, for some β_0. Let now $v \in A_p$ and $\epsilon > 0$. Let $u \in A_p \cap C_c$ satisfy $\|v - u\|_{A_p} < \epsilon$. If $\beta > \beta_0$ then

$$\begin{aligned}\|s_\beta v - v\|_{A_p} &\le \|s_\beta(v-u)\|_{A_p} + \|s_\beta u - u\|_{A_p} + \|v-u\|_{A_p}\\ &\le \|s_\beta\|_{M(A_p)}\epsilon + 0 + \epsilon < (2C+2)\epsilon.\end{aligned}$$

Thus $\{s_\beta\} \subset A_p$ is an A.I. for A_p, bounded in B_p norm, i.e. G is p-weakly amenable. ∎

Corollary 3.7. *If G is 2-weakly amenable then the W_p norm restricted to A_p is equivalent to the A_p norm.*

Proof. As noted, $\|s_\beta\|_{M(A_p)} \le \|s_\beta\|_{B_p}$, hence one can apply Propositions 3.2 and 3.6. ∎

Proof of Theorem 3.1. We only need to use Corollary 3.7 to prove that the $W_p = PF_p^*$ norm restricted to A_p is equivalent to the A_p norm, a fact well known if G is amenable. Then the proof of Theorem 2.1 in [Gr3] carries over verbatim to prove that $A_p^r = W_p^r$ if $1 \le r \le \max(p, p')$, and is hence a dual Banach space, by Theorem 3.1 (note that G need not be weak amenable if $p = 2$).

The **RNP** part is based on the fact that, if G is separable metric, then $A_p^r(G)$ is norm separable, a fact implied by the existence in $A_p^r(G)$ of a multiplier bounded approximate identity (since Theorem 2.2 in [Gr3] is only based on Theorem 2.1, see [Gr3, p. 4380]). ∎

4. Nonfactorisation. Improving a result of Burnham [Bu], Lai and Chen [LCh, Thm. 3.3] have proved that for any noncompact locally compact group G the algebra $A_p^1(G)$ does not factorise. We extend this result to the algebras $A_p^r(G)$ for all $1 \le r < \infty$.

THEOREM 4.1. *For any noncompact locally compact group G and any $1 \le r < \infty$, the algebra $A_p^r(G)$ does not factorise.*

Proof. Assume at first that $1 < r < \infty$. If $A_p^r \cdot A_p^r = A_p^r$, let $u \in A_p^r$. Then for any n, there exist $u_1, \dots, u_n$ in A_p^r such that $u = u_1 \dots u_n$, where $u_i \in A_p \cap L^r$. By [Bu, Lemma A], $u \in L_1$. It follows that $A_p^r(G) = A_p^1(G)$. Since $r > 1$, it follows from our Strong Containment Theorem 3.3 in [Gr2] that this cannot be. The Lai–Chen result completes the proof. ∎

REMARK. M. Leinert [Le] has given an example of a commutative semisimple Banach algebra which factorises but does not even have unbounded approximate units. Hence the fact that $A_p^r(G)$ has no bounded approximate identity [Gr2] does not imply that it does not factorise.

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