WEAKLY AMENABLE GROUPS AND THE RNP FOR SOME BANACH ALGEBRAS RELATED TO THE FOURIER ALGEBRA

by

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Abstract. It is shown that if \( G \) is a weakly amenable unimodular group then the Banach algebra \( A^r_p(G) = A_p \cap L^r(G) \), where \( A_p(G) \) is the Figà-Talamanca–Herz Banach algebra of \( G \), is a dual Banach space with the Radon–Nikodym property if \( 1 \leq r \leq \max(p, p') \). This does not hold if \( p = 2 \) and \( r > 2 \).

Let \( G \) be a locally compact group and let \( A_p(G) \) denote the Figà-Talamanca–Herz Banach algebra of \( G \) as defined in [Hz1], thus generated by \( L^p(G) \). Hence \( A_2(G) \) is the Fourier algebra of \( G \) as defined and studied in Eymard [Ey1]. If \( G \) is abelian then \( A_2(G) = L^1(\hat{G})^\sim \).

Denote \( A^r_p(G) = A_p \cap L^r(G) \) for \( 1 \leq r \leq \infty, 1 < p < \infty \), equipped with the norm \( \|u\|_{A^r_p} = \|u\|_{A_p} + \|u\|_{L^r} \). If \( G \) is abelian then \( A^r_2(\hat{G}) = \{\hat{f} \in L^1(\hat{G}) : \hat{f} \in L^r(\hat{G})\} \), with the norm \( \|u\| = \|f\|_{L^1(G)} + \|\hat{f}\|_{L^r(\hat{G})} \) if \( u = \hat{f} \).

A Banach space has the RNP if its unit ball wants to be weakly compact but just cannot make it, as beautifully put by Jerry Uhl.

A Banach space \( X \) has the Krein–Milman Property (KMP) [Radon Nikodym Property (RNP)] if each closed convex bounded subset is the norm closed convex hull of its extreme points [strongly exposed points] (see [DU, p. 138]). If \( X \) is a dual Banach space, then the RNP and KMP are equivalent (see [DU, p. 190 and p. 218]).

Strongly exposed points are extreme points which are very “smooth” (they are certainly weak-to-norm continuity points), and the fact that we can take the above as the definition of the RNP, is owed to the valiant efforts of many mathematicians (see [DU]).

The Fourier algebra of the torus, \( A_2(\mathbb{T}) \), which is in fact \( \ell_1(\mathbb{Z}) \), has the RNP, a property possessed by any Banach space which is isomorphic to an \( \ell_1 \) space (see [DU]), while \( A_2(\mathbb{R}) \) does not possess the RNP.

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And yet, for any compact subset $K$ of $\mathbb{R}$, $A^2_K(\mathbb{R}) = \{u \in A_2(\mathbb{R}) : \text{spt } u \subset K\}$ does have the RNP (and $\mathbb{R}$ can be replaced by any abelian $G$; here spt denotes support).

We have proved in [Gr1] that for any $G$ and any compact $K \subset G$ and any $1 < p < \infty$, $A^p_K(G) = \{u \in A_p(G) : \text{spt } u \subset K\}$ has the RNP. Tools in abelian harmonic analysis are not available to prove this latter result.

It has been proved by W. Braun in an unpublished preprint [Br] that if $G$ is amenable then $A^1_p(G)$ is a dual Banach space with the RNP. The result in [Br] uses the method in [Gr1] and the involved machinery of [BrF], which is avoided below and in [Gr3]. We have proved in [Gr3] the following

**Theorem 0.1.**

(A) Let $G$ be unimodular and $1 < p < \infty$. If $G$ is amenable then $A^r_p(G)$ is a dual Banach space with the RNP for all $1 \leq r \leq \max(p, p')$.

(B) Let $G$ be unimodular and $A_2(G)$ have a multiplier bounded approximate identity. Then $A^r_2(G)$ is a dual Banach space with the RNP for all $1 \leq r \leq 2$.

(C) If $G$ is $\text{SL}(2, \mathbb{R})$ or $\text{SL}(2, \mathbb{C})$ then, for any $2 < r \leq \infty$, $A^r_2(G)$ does not have the RNP (see [Gr3, p. 4382]), even though these groups are unimodular, weakly amenable (and nonamenable; see [DCH] Thm. 3.7 and Remark 3.8(b))). Hence the above interval for $r$ is the best possible.

A group $G$ is weakly amenable if $A_2(G)$ has an approximate identity bounded in the (Herz–Schur multiplier) $B_2(G)$ norm (see below).

It is the main purpose of this paper to show that Theorem 0.1(A) is true if $G$ is merely weakly amenable.

It has been proved by De Cannière and Haagerup [DCH] pp. 481–486 that any closed subgroup $G$ of any finite extension of the general Lorenz group $\text{SO}_0(n, 1)$ for all $n \geq 2$ (hence in particular $G = F_N$, the free group on $N > 1$ generators) is weakly amenable. Thus there exists a multitude of nonamenable groups which are weakly amenable. And yet, Haagerup [Ha] has proved that $G = \text{SL}(2, \mathbb{R}) \times \mathbb{R}^2$ is not weakly amenable (see also [Do]).

One will note that, in proving the main result, some difficulties need to be overcome to prove that $W^r_p(G)$ is a dual Banach space for all $r \geq 1$. This is done in Section 1.

The main result is proved in Section 2.

In Section 3 we prove that the Banach algebras $A^r_p(G)$ do not factorise for any noncompact $G$ and $1 \leq r < \infty$, a result announced earlier.

1. **Definitions and notations.** Denote by $PM_p(G) = A_p(G)^*$ the Banach space dual of $A_p(G)$. We will omit $G$ at times and write $A_p$, $L^r$, $PM_p$, etc., instead of $A_p(G)$, $L^r(G)$, $PM_p(G)$, etc.
Let $PF_p$ denote the norm closure of $L^1$ in $PM_p$ and set $W_p(G) = PF_p(G)^*$. Then $W_p$ is a Banach algebra of bounded continuous functions on $G$, studied by M. Cowling [Co1].

Define $W_p^r(G) = W_p \cap L^r(G)$, with the norm $\|w\|_{W_p^r} = \|w\|_{W_p} + \|w\|_{L^r}$.

Denote by $M(A_p)$ the set of multipliers of $A_p$ with the norm

$$\|u\|_{M(A_p)} = \sup\{|uv|; v \in A_p, \|v\|_{A_p} \leq 1\}, \quad u \in M(A_p).$$

If $v \in A_p$, let $\|v\|_{W_p}$ be the norm of $v$ as an element of $PF_p(G)^* = W_p(G)$.

If $u \in M(A_2)$ let $m_u : A_2 \to A_2$ be given by $m_u v = uv$. Let $M_u = m_u^* : PM_2 \to PM_2$ denote the adjoint of $m_u$. The multiplier $u \in M(A_2)$ is completely bounded (and $M_0(A_2)$ is the algebra of all such multipliers) if the operator $M_u : PM_2 \to PM_2$ is completely bounded on the $W^*$ algebra $PM_2$. The set $M_0(A_2)$ is equipped with the norm $\|u\|_{M_0(A_2)} = \|M_u\|_{cb}$, the completely bounded norm of the operator $M_u$ (see [DCH], [CH], [Jo], where all the above notions are defined).

It has been proved by Bożejko and Fendler [BF] that $M_0(A_2)$ coincides with the space $B_2 = B_2(G)$ of Herz–Schur multipliers and $\|u\|_{M_0(A_2)} = \|u\|_{B_2}$.

A group $G$ is weakly amenable if $A_2(G)$ has an approximate identity (A.I.) bounded in the $\|\|_{B_2(G)}$ norm.

In an important ground-breaking paper [DCH], De Cannière and Haagerup have studied weakly amenable groups $G$.

2. $W_p(G) \cap L^r(G)$ is a dual Banach space for all $r \geq 1$. We have proved this result, for any group $G$, in [Gr3, Prop. 2.1] only for $r > 1$. The proof there fails in case $r = 1$. This case requires an entirely different proof, which is given below.

The result in the title of this section is needed to prove that for all $1 \leq r \leq \max(p,p')$, $A_p^r$ is a dual Banach space if $G$ is unimodular and weakly amenable.

**Remark** (for $r = 1$). Let $Z = X \times Y$, $X = PF_p$, $Y = L^\infty$, with norm $\|(x,y)|| = \max(||x||, ||y||)$. Hence $Z^* = X^* \times Y^* = W_p \times L^\infty^*$, with norm $\|(x^*,y^*)|| = ||x^*|| + ||y^*||$. Let $D = \{(w,w); w \in W_p \cap L^1\} \subset W_p \cap L^\infty^*$. Let $U = w^*- \text{cl} D \subset Z^*$. If $U_0 = \text{cl} U_0$ is the annihilator of $U$ in $Z^*$, respectively, then, since $U$ is $w^*$-closed, $U = (U_0)^0 = (Z/U_0)^*$ (see [Da] p. 822]). Thus $U$ is a dual Banach space.

Let now $P : X \to X \times Y$ be given by $Px = (x,0)$. Then $P^* : X^* \times Y^* \to X^* = W_p$ is onto, in fact $P^*(x^*,y^*) = x^*$.

**Lemma 2.1.**

(a) $W_p \cap L^\infty^* = W_p \cap L^1$.

(b) $P^* U = W_p \cap L^1$. 
Proof. Let \( w \in W_p \cap L^\infty \). Clearly \( W_p = PF_p^* \) and \( L^1 \subset PF_p \). If \( f \in L^1 \) then \( w(f) = \int w f \, dx \). If \( f \in L^1 \cap L^\infty \) then \( |w(f)| = |\int w f \, dx| \leq \|w\|_{L^\infty} \|f\|_{L^\infty} \). Hence if \( \|f\|_{L^\infty} \leq 1 \) then \( \int w f \, dx \leq \|w\|_{L^\infty} \). Let now \( K \subset G \) be compact and \( f = (\bar{w}/|w|)1_K \); then \( \int_K |w| \, dx \leq \|w\|_{L^\infty} \). Hence \( w \in L^1 \), which proves (a).

(b) Let \((w_\alpha, w_\alpha) \in D \subset W_p \times L^\infty \) satisfy \( w^*\)-\( \lim (w_\alpha, w_\alpha) = (w, z) \in W_p \times L^\infty \). Then for \( f \in L^1 \subset PF_p \) one has \( \int w_\alpha f \, dx = w_\alpha(f) \to \int w \, dx \). And for \( f \in L^\infty \), \( \int w_\alpha f \, dx = w_\alpha(f) \to z(f) \). Thus \( w(f) = z(f) \) for all \( f \in L^1 \cap L^\infty \). Hence by (a), \( w \in L^1 \). Thus \( U = w^*\)-\( \cl D \subset (W_p \cap L^1) \times L^\infty \). It follows that \( P^*(U) = W_p \cap L^1 \), since \( D = \{(w, w); w \in W_p \cap L^1\} \).

REMARK. Let \( N = \{u \in U; P^*(u) = 0\} = U \cap (0,Y^*) \). Then \( U/N \) is isomorphic to \( W_p \cap L^1 \), where \( U \) is a dual space and \( N \) is a \( w^* \)-closed subspace.

**Theorem 2.2.** \( W_p(G) \cap L^r(G) \) with the norm \( \|w\|_{W_p} + \|w\|_{L^r} \) is a dual Banach space for all \( 1 \leq r \leq \infty \), and for all locally compact groups \( G \).

**Proof.** If \( r > 1 \) this is just our Proposition 2.1 in [Gr3]. If \( r = 1 \), then \( U = X^* \) for some Banach space \( X \) and \( N = (N_0)^0 \). Hence by [Da, p. 822, Theorem A.3.47(i)], \( U/N \approx X^*/(N_0)^0 = N_0^* \). Thus \( W_p \cap L^1(G) \) is norm isomorphic to a dual Banach space, thus is a dual space, by the use of the main theorem of Kajijser [Ka].

REMARK. Note that a dual Banach space may have two non-norm isomorphic preduals (see [BL]).

3. Weakly amenable groups and the RNP. It is the purpose of this section to prove the main result of this paper, namely:

**Theorem 3.1.** Let \( G \) be unimodular and weakly amenable, and let \( 1 < p < \infty \). Then for all \( 1 \leq r \leq \max(p,p') \), \( A_p^r(G) \) is a dual Banach space which has the **RNP**.

If \( G \) is \( SL(2, \mathbb{R}) \) or \( SL(2, \mathbb{C}) \) then \( A_2^r(G) \), for any \( 2 < r < \infty \), does not have the **RNP**, a fortiori is not a dual Banach space, by [Gr3].

If \( G \) is amenable this is part of [Gr3, Theorem 2.2, p. 4380].

The proofs in [Gr3] will work for proving our main result once we show that if \( G \) is weakly amenable, the \( W_p \) norm restricted to \( A_p \) is equivalent to the \( A_p \) norm.

It has been proved by M. Cowling [Co1] that the group \( SL(2, \mathbb{R}) \) satisfies the assumption of the next result.

**Proposition 3.2.** Assume that \( A_p(G) \) has an approximate identity \( \{u_\alpha\} \) such that \( \|u_\alpha\|_{M(A_p)} \leq K \). Then
\[
\forall u \in A_p, \quad \|u\|_{A_p} \leq (1 + K)\|u\|_{W_p} \leq (1 + K)\|u\|_{A_p}.
\]
Remark. It has been proved by Haagerup [Ha] (see [Do] for a different proof) that if \( G = \text{SL}(2, \mathbb{R}) \times \mathbb{R}^2 \), then \( A_2(G) \) has no multiplier bounded approximate identity. It is not clear to us if Theorem 3.1 holds for this group.

Proof of Proposition 3.2. Clearly \( \|u\|_{W_p} \leq \|u\|_{A_p} \) for all \( u \in A_p \) by the definition of these norms, hence only the left hand inequality needs proof.

Let \( e_\alpha \in A_p \cap C_c \) satisfy \( \|e_\alpha - u_\alpha\|_{A_p} \to 0 \). As is readily seen, \( \{e_\alpha\} \) is an A.I. for \( A_p \) and for some \( \alpha_0 \), \( \|e_\alpha\|_{C(A_p)} \leq 1 + K \) if \( \alpha > \alpha_0 \). Hence for any \( T \in PM_p \), \( \|e_\alpha T\|_{PM_p} \leq 1 + K \) if \( \|T\|_{PM_p} \leq 1 \). And if \( v \in A_p \) then \( |(e_\alpha T, v) - (T, v)| = |(T, e_\alpha v - v)| \to 0 \), thus \( e_\alpha T \to T \) in \( \sigma(PM_p, A_p) \), i.e. in \( w^* \). Also \( \text{spt } e_\alpha T \subset \text{spt } e_\alpha \), which is compact.

Hence if \( u_0 \in A_p \) then
\[
\|u_0\|_{A_p} \leq \sup\{|(u_0, T)|; \|T\| \leq 1 + K, \text{spt } T \text{ is compact}\}.
\]
However if \( spt T \) is compact there exists a net \( f_\alpha \in C_c(G) \) such that \( \|\lambda_p f_\alpha\| \leq \|T\| \) and \( \lambda_p f_\alpha \to T \) ultrastrongly by [Hz1, Prop. 9, p. 117]. Hence
\[
\|u_0\|_{A_p} \leq \sup\{|(u_0, \lambda_p f)|; f \in C_c, \|\lambda_p f\| \leq 1 + K\} = (1 + K)\|u_0\|_{W_p}.
\]

The algebra \( B_p(G) \) of Herz–Schur multipliers of \( G \), for \( 1 < p < \infty \), has been investigated by Eymard [Ey1] and Herz [Hz1–Hz3]. As shown in these papers (see also [Fu, p. 581])

\[
A_p(G) \subset W_p(G) \subset B_p(G) \subset MA_p(G)
\]

and each imbedding is contractive.

Definition 3.3. \( G \) is \( p \)-weakly amenable if \( A_p(G) \) has an A.I. bounded in the \( \|\|_{B_p(G)} \) norm. Thus \( 2 \)-weak amenability and weak amenability are identical.

We need the fact that \( 2 \)-weak amenability implies \( p \)-weak amenability for all \( p \). This is hinted in [Fu, p. 586], in different terminology, without proof. We give a proof based on Furuta’s useful theorem [Fu, Theorem 2.4]:

Theorem 3.4. For any \( 1 < p < \infty \), \( B_2(G) \subset B_p(G) \) and \( \|u\|_{B_p} \leq \|u\|_{B_2} \) for all \( u \in B_2(G) \).

Furuta’s proof of this theorem is based on an unpublished theorem of J. E. Gilbert:

Theorem 3.5. Let \( w \) be a function on \( G \). Then \( w \in B_2(G) \) iff there exists a Hilbert space \( K \) and bounded continuous functions \( u, v \) from \( G \) to \( K \) such that \( w(x^{-1}y) = \langle u(y), v(x) \rangle \) for all \( x, y \in G \).

A proof of this theorem has been given in [Ha], and for a different proof see P. Jolissaint [Jo].
The proof of the next result is very different from the suggestion, with no proof, given in \cite{Fu} p. 586.

**Proposition 3.6.** If $G$ is 2-weak amenable then it is $p$-weak amenable for all $1 < p < \infty$.

**Proof.** Let $\{v_\alpha\}$ be an A.I. in $A_2(G)$ such that $\|v_\alpha\|B_2 \leq C$. Let $\{u_\alpha\} \subset A_2 \cap C_c \subset A_p \cap C_c$ satisfy $\|u_\alpha - v_\alpha\|A_2 \to 0$. If $v \in A_2$ then

$$\|u_\alpha v - v\|A_2 \leq \|u_\alpha - v_\alpha\|A_2 \|v\|A_2 + \|v_\alpha v - v\|A_2 \to 0.$$ 

Moreover

$$\|u_\alpha\|B_p \leq \|u_\alpha\|B_2 \leq \|u_\alpha - v_\alpha\|A_2 + \|v_\alpha\|B_2 \leq 2C$$

if $\alpha > \alpha_0$, for some $\alpha_0$. It follows by Furuta’s theorem that $\{u_\alpha\}$ is an A.I. for $A_p(G)$ equipped with the $\|\|B_p$ norm, while we need that it be in the $\|\|A_p$ norm.

In contrast to the hint in \cite{Fu} p. 586 we proceed as follows:

Let $K_\beta$ be compact subsets of $G$ whose interiors satisfy $\text{int } K_\beta \uparrow G$. Let $V = V^{-1}$ be a neighborhood of the unit $e$ of $G$ with compact closure. Let

$$e_\beta(x) = \lambda(V)^{-1}(1_{K_\beta} V^* 1_V(x)) = \lambda(V)^{-1} \lambda(xV \cap K_\beta V).$$

Then $e_\beta(x) = 1 [0]$ if $x \in K_\beta [x \notin K_\beta V^2]$, respectively. Choose a subnet $\{u_\beta\}$ of $\{u_\alpha\}$ such that $\|u_\beta e_\beta - e_\beta\|A_2 \leq 1$ and let $s_\beta(x) = (u_\beta + e_\beta - u_\beta e_\beta)(x)$. Then $s_\beta(x) = 1 [0]$ if $x \in K_\beta [x \notin K_\beta V^2 \cup \text{spt } u_\beta]$. Also

$$\|s_\beta\|B_p \leq \|s_\beta\|B_2 \leq \|u_\beta\|B_2 + 1 \leq 2C + 1,$$

by Furuta’s theorem. If $v \in A_p \cap C_c$ and $K = \text{spt } v$, then $K \subset K_\beta$ if $\beta > \beta_0$, for some $\beta_0$. Let now $v \in A_p$ and $\epsilon > 0$. Let $u \in A_p \cap C_c$ satisfy $\|v - u\|A_p < \epsilon$. If $\beta > \beta_0$ then

$$\|s_\beta v - u\|A_p \leq \|s_\beta (v - u)\|A_p + \|s_\beta u - u\|A_p + \|v - u\|A_p \leq \|s_\beta\|M(A_p)\epsilon + 0 + \epsilon < (2C + 2)\epsilon.$$

Thus $\{s_\beta\} \subset A_p$ is an A.I. for $A_p$, bounded in $B_p$ norm, i.e. $G$ is $p$-weakly amenable. ■

**Corollary 3.7.** If $G$ is 2-weakly amenable then the $W_p$ norm restricted to $A_p$ is equivalent to the $A_p$ norm.

**Proof.** As noted, $\|s_\beta\|M(A_p) \leq \|s_\beta\|B_p$, hence one can apply Propositions 3.2 and 3.6. ■

**Proof of Theorem 3.1.** We only need to use Corollary 3.7 to prove that the $W_p = PF_p^*$ norm restricted to $A_p$ is equivalent to the $A_p$ norm, a fact well known if $G$ is amenable. Then the proof of Theorem 2.1 in \cite{Gr3} carries over verbatim to prove that $A'_p = W_r$ if $1 \leq r \leq \max(p, p')$, and is hence a
dual Banach space, by Theorem 3.1 (note that \( G \) need not be weak amenable if \( p = 2 \)).

The RNP part is based on the fact that, if \( G \) is separable metric, then \( A_p^r(G) \) is norm separable, a fact implied by the existence in \( A_p^r(G) \) of a multiplier bounded approximate identity (since Theorem 2.2 in [Gr3] is only based on Theorem 2.1, see [Gr3, p. 4380]).

4. Nonfactorisation. Improving a result of Burnham [Bu], Lai and Chen [LCh, Thm. 3.3] have proved that for any noncompact locally compact group \( G \) the algebra \( A_1^1(G) \) does not factorise. We extend this result to the algebras \( A_r^p(G) \) for all \( 1 \leq r < \infty \).

**Theorem 4.1.** For any noncompact locally compact group \( G \) and any \( 1 \leq r < \infty \), the algebra \( A_r^p(G) \) does not factorise.

**Proof.** Assume at first that \( 1 < r < \infty \). If \( A_p^r \cdot A_p^r = A_p^r \), let \( u \in A_p^r \). Then for any \( n \), there exist \( u_1, \ldots, u_n \) in \( A_p^r \) such that \( u = u_1 \ldots u_n \), where \( u_i \in A_p \cap L^r \). By [Bu, Lemma A], \( u \in L_1 \). It follows that \( A_p^r(G) = A_1^1(G) \). Since \( r > 1 \), it follows from our Strong Containment Theorem 3.3 in [Gr2] that this cannot be. The Lai–Chen result completes the proof.

**Remark.** M. Leinert [Le] has given an example of a commutative semi-simple Banach algebra which factorises but does not even have unbounded approximate units. Hence the fact that \( A_r^p(G) \) has no bounded approximate identity [Gr2] does not imply that it does not factorise.

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