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ON THE LUCAS SEQUENCE EQUATIONS $V_n = kV_m$ AND $U_n = kU_m$

BY

REFİK KESKİN (Sakarya) and ZAFER ŞİAR (Bilecik)

Abstract. Let P and Q be nonzero integers. The sequences of generalized Fibonacci and Lucas numbers are defined by $U_0 = 0$, $U_1 = 1$ and $U_{n+1} = PU_n - QU_{n-1}$ for $n \geq 1$, and $V_0 = 2$, $V_1 = P$ and $V_{n+1} = PV_n - QV_{n-1}$ for $n \geq 1$, respectively. In this paper, we assume that $P \geq 1$, Q is odd, $(P, Q) = 1$, $V_m \neq 1$, and $V_r \neq 1$. We show that there is no integer x such that $V_n = V_r V_m x^2$ when $m \geq 1$ and r is an even integer. Also we completely solve the equation $V_n = V_m V_r x^2$ for $m \geq 1$ and $r \geq 1$ when $Q \equiv 7 \pmod 8$ and x is an even integer. Then we show that when $P \equiv 3 \pmod 4$ and $Q \equiv 1 \pmod 4$, the equation $V_n = V_m V_r x^2$ has no solutions for $m \geq 1$ and $r \geq 1$. Moreover, we show that when $P > 1$ and $Q = \pm 1$, there is no generalized Lucas number V_n such that $V_n = V_m V_r$ for $m > 1$ and $r > 1$. Lastly, we show that there is no generalized Fibonacci number U_n such that $U_n = U_m U_r$ for $Q = \pm 1$ and $1 < r < m$.

1. Introduction. Let P and Q be nonzero integers with $P^2 - 4Q \neq 0$. The sequences of generalized Fibonacci and Lucas numbers, $(U_n(P, Q))$ and $(V_n(P, Q))$, are defined as follows:

$$U_0(P,Q) = 0, \quad U_1(P,Q) = 1,$$
$$U_{n+1}(P,Q) = PU_n(P,Q) - QU_{n-1}(P,Q) \quad \text{for } n \geq 1$$

and

$$V_0(P,Q) = 2, \quad V_1(P,Q) = P,$$
$$V_{n+1}(P,Q) = PV_n(P,Q) - QV_{n-1}(P,Q) \quad \text{for } n \geq 1,$$

respectively. $U_n(P, Q)$ and $V_n(P, Q)$ are called nth *generalized Fibonacci number* and nth *generalized Lucas number*, respectively. Generalized Fibonacci and Lucas numbers for negative subscripts are defined to be

$$(1.1) \qquad U_{-n}(P,Q) = \frac{-U_n(P,Q)}{Q^n} \quad \text{and} \quad V_{-n}(P,Q) = \frac{V_n(P,Q)}{Q^n},$$

respectively.

Then it is well known that

$$U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n(P,Q) = \alpha^n + \beta^n,$$

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where

$$\alpha = \frac{P + \sqrt{P^2 - 4Q}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{P^2 - 4Q}}{2}.$$

The above formulas are known as *Binet's formulas.* Since

$$U_n(-P, Q) = (-1)^{n-1} U_n(P, Q) \quad \text{and} \quad V_n(-P, Q) = (-1)^n V_n(P, Q),$$

it will be assumed that $P \geq 1$. Moreover, we will assume that $P^2 - 4Q > 0$. Instead of $U_n(P, Q)$ and $V_n(P, Q)$, we will use U_n and V_n, respectively.

For $(P, Q) = (1, -1)$, we have the classical Fibonacci and Lucas number sequences, (F_n) and (L_n). For $(P, Q) = (2, -1)$, we have the Pell and Pell–Lucas number sequences (P_n) and (Q_n), respectively. For more information about the sequences of generalized Fibonacci and Lucas numbers one can consult [5, 8, 9, 11].

In this paper, we assume that $P \geq 1$, Q is odd, $(P, Q) = 1$, $V_m \neq 1$, and $V_r \neq 1$. We will show that when m and r are natural numbers with r even, there is no integer x such that $V_n = V_r V_m x^2$. Also when $Q \equiv 7 \pmod 8$ and x is an even integer, we completely solve the equation $V_n = V_m V_r x^2$ for $m \geq 1$ and $r \geq 1$. In addition, when $P \equiv 3 \pmod 4$ and $Q \equiv 1 \pmod 4$ we solve the equation $V_n = V_m V_r x^2$ for $m \geq 1$ and $r \geq 1$. Moreover, we show that when $P > 1$ and $Q = \pm 1$, there is no generalized Lucas number V_n such that $V_n = V_m V_r$ for $m > 1$ and $r > 1$. Lastly, we show that there is no generalized Fibonacci number U_n such that $U_n = U_m U_r$ for $Q = \pm 1$ and $1 < r < m$.

In Section 2, we give the necessary identities and theorems. Then in Section 3, we present our main theorems.

2. Preliminaries.

In this section, we give some theorems and some identities concerning generalized Fibonacci and Lucas numbers, which will be used later. Then we present our theorems in the third section. Throughout the paper, $\left(\frac{a}{m}\right)$ represents the Jacobi symbol.

The proofs of the following two theorems are given in [13].

THEOREM 1. *Let $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, and $r \in \mathbb{Z}$ with $mn + r \geq 0$. Then*

$$U_{2mn+r} \equiv Q^{mn} U_r \pmod{U_m}, \tag{2.1}$$

$$V_{2mn+r} \equiv Q^{mn} V_r \pmod{U_m}. \tag{2.2}$$

THEOREM 2. *Let $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, and $r \in \mathbb{Z}$ with $mn + r \geq 0$. Then*

$$U_{2mn+r} \equiv (-Q^m)^n U_r \pmod{V_m}, \tag{2.3}$$

$$V_{2mn+r} \equiv (-Q^m)^n V_r \pmod{V_m}. \tag{2.4}$$

When P is odd, we see that $8 \mid U_6$. Thus, using (2.1) and (2.2), we get

$$U_{12q+r} \equiv U_r \pmod 8, \tag{2.5}$$

$$V_{12q+r} \equiv V_r \pmod 8, \tag{2.6}$$

for any nonnegative integers q and r. If $Q \equiv 7 \pmod 8$, then

$$8 \nmid V_n \tag{2.7}$$

for every natural number n, and if $Q \equiv 1, 5 \pmod 8$, then

$$4 \nmid V_n \tag{2.8}$$

for every natural number n. Moreover, when P is odd and $Q \equiv 3, 7 \pmod 8$, we get

$$4 \mid V_3. \tag{2.9}$$

The following lemma can be found in [12].

Lemma 3. *Let P and m be odd positive integers, and $r \geq 1$.*

(a) *If $3 \nmid m$, then*

$$V_{2^r m} \equiv \begin{cases} 3 \pmod 8 & \text{if } r = 1 \text{ and } Q \equiv 3 \pmod 4, \\ 7 \pmod 8 & \text{otherwise.} \end{cases}$$

(b) *If $3 \mid m$, then $V_{2^r m} \equiv 2 \pmod 8$.*

The following lemma can be proved by induction.

Lemma 4. *If n is a positive odd integer, then*

$$V_n \equiv nP(-Q)^{(n-1)/2} \pmod{P^2},$$

and if n is a positive even integer, then

$$V_n \equiv 2(-Q)^{n/2} \pmod{P^2}.$$

The following three theorems are given in [7, 10, 12].

Theorem 5. *If $U_m \neq 1$, then $U_m \mid U_n$ if and only if $m \mid n$.*

Theorem 6. *If $V_m \neq 1$, then $V_m \mid V_n$ if and only if $m \mid n$ and n/m is odd.*

Theorem 7. *If $V_m \neq 1$, then $V_m \mid U_n$ if and only if $m \mid n$ and n/m is even.*

Now we give some properties of generalized Fibonacci and Lucas numbers:

$$U_{2n} = U_n V_n, \tag{2.10}$$

$$V_{2n} = V_n^2 - 2Q^n, \tag{2.11}$$

(2.12) if P is odd and $n \geq 1$, then $2 \mid V_n \Leftrightarrow 2 \mid U_n \Leftrightarrow 3 \mid n$,

(2.13) $$(V_n, V_{2n}) = 1 \text{ or } 2,$$

(2.14) $$(U_n, U_m) = U_{(n,m)} \quad \text{for all natural numbers } m \text{ and } n,$$

(2.15) $$V_{3n} = V_n(V_n^2 - 3Q^n),$$

(2.16) $$\left(\frac{U_3}{V_{2^k}}\right) = 1 \;\Leftrightarrow\; k > 1, \text{ or } k = 1 \text{ and } Q \equiv 1 \pmod 4,$$

(2.17) $$\text{if } n \geq 1, \text{ then } (U_n, Q) = (V_n, Q) = 1,$$

(2.18) $$\text{if } 3 \mid P \text{ and } (3, Q) = 1, \text{ then } 3 \mid V_n \Leftrightarrow n \text{ is odd},$$

(2.19) $$\text{if } 3 \nmid P, \text{ then } 3 \mid V_n \Leftrightarrow n \equiv 2 \pmod 4 \text{ and } Q \equiv -1 \pmod 3.$$

Moreover, when P is even, it can be easily seen that

$$U_n \text{ is odd } \Leftrightarrow n \text{ is an odd natural number},$$
$$U_n \text{ is even } \Leftrightarrow n \text{ is an even natural number},$$
$$V_n \text{ is even for all natural numbers } n.$$

The properties (2.10)–(2.17) can be found in [10, 12, 13]. The proofs of the others are easy and will be omitted.

3. Main theorems. Throughout, we will assume that n is a natural number.

In [6], the authors showed that there is no integer x such that $L_n = L_m L_r x^2$ when m and r are natural numbers with r even. Now we solve the same problem for generalized Lucas numbers.

THEOREM 8. *Let $m \geq 1$, $k \geq 1$ and r be an odd positive integer. Then there is no integer x such that $V_n = V_{2^k r} V_m x^2$.*

Proof. Assume that $V_n = V_{2^k r} V_m x^2$.

Firstly, we assume that P is odd. Now we divide the proof into two cases.

CASE 1: $3 \nmid r$, $k = 1$, $Q \equiv 3 \pmod 4$. We have $V_n = V_{2r} V_m x^2$. Since $V_m \mid V_n$ and $V_{2r} \mid V_n$, there exist odd positive integers t and s such that $n = mt$ and $n = 2rs$ by Theorem 6. Thus $m = 2c$ for some odd positive integer c. By Lemma 3, we get $V_{2r} \equiv 3 \pmod 8$ and

$$V_m = V_{2c} \equiv \begin{cases} 3 \pmod 8 & \text{if } 3 \nmid c, \\ 2 \pmod 8 & \text{if } 3 \mid c. \end{cases}$$

Thus

$$V_n = V_{2r} V_m x^2 \equiv \begin{cases} 9x^2 \pmod 8 & \text{if } 3 \nmid c, \\ 6x^2 \pmod 8 & \text{if } 3 \mid c. \end{cases}$$

But

$$V_n = V_{2rs} \equiv \begin{cases} 3 \pmod 8 & \text{if } 3 \nmid rs, \\ 2 \pmod 8 & \text{if } 3 \mid rs, \end{cases}$$

a contradiction, since x^2 and $6x^2$ are not congruent to 3 or 2 respectively modulo 8.

Case 2. Not all of $3 \nmid r$, $k = 1$, $Q \equiv 3 \pmod 4$ hold. Since $V_m \mid V_n$ and $V_{2^k r} \mid V_n$, there exist odd integers t and s such that $n = mt$ and $n = 2^k rs$ by Theorem 6. Thus we have $m = 2^k c$ for some odd positive integer c. Then $V_n \equiv 2, 7 \pmod 8$, $V_m \equiv 2, 7 \pmod 8$ and $V_{2^k r} \equiv 2, 7 \pmod 8$ by Lemma 3.

Assume that $V_{2^k r} \equiv 2 \pmod 8$. Then it follows that $V_n = V_{2^k r} V_m x^2 \equiv 2V_m x^2 \pmod 8$. Moreover, since $2x^2 \equiv 0, 2 \pmod 8$ and $V_m \equiv 2, 7 \pmod 8$, it is seen that $2V_m x^2 \equiv 0, 4, 6 \pmod 8$. This contradicts the fact that $V_n \equiv 2, 7 \pmod 8$.

Now assume that $V_{2^k r} \equiv 7 \pmod 8$. Then $V_n = V_{2^k r} V_m x^2 \equiv 7V_m x^2 \pmod 8$. Moreover, $7x^2 \equiv 0, 4, 7 \pmod 8$ and $V_m \equiv 2, 7 \pmod 8$. This shows that $7V_m x^2 \equiv 0, 1, 4, 6 \pmod 8$, which contradicts the fact that $V_n \equiv 2, 7 \pmod 8$.

Secondly, we assume that P is even. Then since n is even and Q is odd, it is seen that $V_n \equiv 2 \pmod 4$ by Lemma 4. Similarly, we see that $V_m \equiv 2 \pmod 4$ and $V_{2r} \equiv 2 \pmod 4$. This shows that $V_n \equiv 0 \pmod 4$, which contradicts the fact that $V_n \equiv 2 \pmod 4$. This completes the proof. ∎

Now the following corollary follows easily.

Corollary 9. *Let m and r be two natural numbers. If r is an even integer, then there is no integer x such that $V_n = V_m V_r x^2$.*

In the next theorem we will use the following known lemma from number theory.

Lemma 10. *Let $a, b, c, x \in \mathbb{Z}$, $\gcd(a, b) = 1$ and $ab = cx^2$. Then $a = ru^2$ and $b = sv^2$ with $rs = c$ for some positive integers u and v.*

In [6], the authors showed that for $m > 1$ and $r > 1$, there is no even integer x such that $L_n = L_m L_r x^2$. If $Q \equiv 1, 5 \pmod 8$ and x is even, then the equation $V_n = V_m V_r x^2$ has no solutions by (2.8). Now, we consider the same problem for $P \geq 1$ and $Q \equiv 7 \pmod 8$.

Theorem 11. *Let x be an even integer and $Q \equiv 7 \pmod 8$. If m and r are positive integers and $V_n = V_m V_r x^2$, then $m = r = 1$, $n = 3$, and $P = 3$.*

Proof. If one of m and r is even, the assertion follows from Corollary 9. Assume that m and r are odd.

Firstly, we assume that P is odd. Since x is even, it follows that $4 \mid V_n$ and therefore $3 \mid n$ by (2.12). If $3 \mid m$ or $3 \mid r$, we see that V_m or V_r is even by (2.12). Thus we get $8 \mid V_n$. This is impossible by (2.7). Therefore we have $3 \nmid m$ and $3 \nmid r$. Since $V_m \mid V_n$ and $V_r \mid V_n$, there exist odd positive integers t and s such that $n = mt$ and $n = rs$ by Theorem 6. Then n is odd. As a result, $n = mt$, $n = rs$, $3 \mid n$, $3 \nmid m$ and $3 \nmid r$. Therefore $t = 3a$ and $s = 3b$

for some odd positive integers a and b. This shows that $n = 3ma = 3rb$, i.e., $ma = rb$. Thus since n is odd, we get

$$V_m V_r x^2 = V_n = V_{3ma} = V_{ma}(V_{ma}^2 - 3Q^{ma})$$

by (2.15). This shows that

$$\frac{V_{ma}}{V_m}(V_{ma}^2 - 3Q^{ma}) = V_r x^2. \tag{3.1}$$

It can be seen that $(V_{ma}/V_m, V_{ma}^2 - 3Q^{ma}) = 1$ or 3 by (2.17). In both cases, by Lemma 10, we have either

$$\frac{V_{ma}}{V_m} = wu_1^2 \quad \text{and} \quad V_{ma}^2 - 3Q^{ma} = yu_2^2 \tag{3.2}$$

or

$$\frac{V_{ma}}{3V_m} = wu_1^2 \quad \text{and} \quad \frac{V_{ma}^2 - 3Q^{ma}}{3} = yu_2^2 \tag{3.3}$$

with $wy = V_r$ for some positive integers w, y, u_1, and u_2. Using the fact that $ma = rb$ in (3.2) and (3.3), we get $V_{rb}^2 - 3Q^{rb} = yu_2^2$ and $V_{rb}^2 - 3Q^{rb} = 3yu_2^2$, respectively. Thus $y \mid V_{rb}^2 - 3Q^{rb}$. Since $y \mid V_r$ and $V_r \mid V_{rb}$, it is seen that $y \mid 3Q^{rb}$. Since $y \mid V_r$, we get $y \mid 3$ by (2.17). This shows that $y = 1$ or $y = 3$. As a result, $V_{rb}^2 - 3Q^{rb} = v^2$ or $V_{rb}^2 - 3Q^{rb} = 3v^2$ for some integer v.

Assume that $V_{rb}^2 - 3Q^{rb} = v^2$. Since $V_{2rb} = V_{rb}^2 - 2Q^{rb}$ by (2.11), we obtain $V_{2rb} = v^2 + Q^{rb}$. Assume that $rb > 1$. Then we can write $2rb = 2(4q \pm 1) = 2(2^k z) \pm 2$ for some positive odd integer z with $k \geq 2$. Hence we get either

$$V_{2rb} \equiv -Q^{rb-1} V_2 \pmod{V_{2^k}}$$

or

$$V_{2rb} \equiv -Q^{rb+1} V_{-2} \pmod{V_{2^k}}$$

by (2.4). It is seen that

$$v^2 + Q^{rb} \equiv -Q^{rb-1} V_2 \pmod{V_{2^k}}$$

by (1.1). That is,

$$v^2 \equiv -Q^{rb-1}(V_2 + Q) \equiv -Q^{rb-1} U_3 \pmod{V_{2^k}}.$$

This shows that

$$J = \left(\frac{-Q^{rb-1} U_3}{V_{2^k}}\right) = 1.$$

On the other hand, $\left(\frac{U_3}{V_{2^k}}\right) = 1$ by (2.16). Moreover, $V_{2^k} \equiv 7 \pmod 8$ by Lemma 3 and therefore $\left(\frac{-1}{V_{2^k}}\right) = -1$. Also since $rb - 1$ is even, it is seen that $\left(\frac{Q^{rb-1}}{V_{2^k}}\right) = 1$. Thus we get

$$J = \left(\frac{-Q^{rb-1}U_3}{V_{2^k}}\right) = \left(\frac{-1}{V_{2^k}}\right)\left(\frac{Q^{rb-1}}{V_{2^k}}\right)\left(\frac{U_3}{V_{2^k}}\right) = -1,$$

which contradicts the fact that $J = 1$.

Assume now that $V_{rb}^2 - 3Q^{rb} = 3v^2$. Then $3 \mid V_{rb}$. This shows that $3 \mid P$ by (2.18) since rb is odd. Moreover, $V_{2rb} = V_{rb}^2 - 2Q^{rb}$ by (2.11). Thus we obtain $V_{2rb} = 3v^2 + Q^{rb}$.

Assume that $rb > 1$. Then we can write $2rb = 2(4q \pm 1) = 2(2^k z) \pm 2$ for some positive odd integer z with $k \geq 2$. Hence we get either

$$V_{2rb} \equiv -Q^{rb-1}V_2 \pmod{V_{2^k}}$$

or

$$V_{2rb} \equiv -Q^{rb+1}V_{-2} \pmod{V_{2^k}}$$

by (2.4). It is seen that

$$3v^2 + Q^{rb} \equiv -Q^{rb-1}V_2 \pmod{V_{2^k}}$$

by (1.1). That is,

$$3v^2 \equiv -Q^{rb-1}(V_2 + Q) \equiv -Q^{rb-1}U_3 \pmod{V_{2^k}}.$$

This shows that

$$J = \left(\frac{-3Q^{rb-1}U_3}{V_{2^k}}\right) = 1.$$

Since $V_{2^k} \equiv 7 \pmod 8$ by Lemma 3 and $\left(\frac{U_3}{V_{2^k}}\right) = 1$ by (2.16), it is seen that $\left(\frac{-1}{V_{2^k}}\right) = -1$. Since rb is odd, we get $\left(\frac{Q^{rb-1}}{V_{2^k}}\right) = 1$. Also since $3 \mid P$ and $k \geq 2$, it can be easily seen that $V_{2^k} \equiv 2(-Q)^{2^k/2} \equiv 2Q^{2^k/2} \pmod 3$ by Lemma 4. Therefore

$$\left(\frac{3}{V_{2^k}}\right) = \left(\frac{V_{2^k}}{3}\right)(-1)^{\left(\frac{3-1}{2}\right)\left(\frac{V_{2^k}-1}{2}\right)} = -\left(\frac{2Q^{2^k/2}}{3}\right) = -\left(\frac{2}{3}\right)\left(\frac{Q^{2^k/2}}{3}\right) = 1.$$

Thus we get

$$J = \left(\frac{-3Q^{rb-1}U_3}{V_{2^k}}\right) = \left(\frac{-1}{V_{2^k}}\right)\left(\frac{3}{V_{2^k}}\right)\left(\frac{Q^{rb-1}}{V_{2^k}}\right)\left(\frac{U_3}{V_{2^k}}\right) = -1.$$

But this contradicts the fact that $J = 1$.

Therefore $rb = 1$, i.e., $r = b = 1$. This shows that $m = r = 1$ and $n = 3$. Thus we obtain $V_3 = V_1V_1x^2 = (Px)^2$, i.e., $P(P^2 - 3Q) = P^2x^2$. Hence $P \mid P^2 - 3Q$ and therefore $P \mid 3Q$. Since $(P, Q) = 1$, it follows that $P \mid 3$. This shows that $P = 3$ since $P = V_1 = V_m > 1$ by assumption.

Secondly, we assume that P is even. Since x is even, it is seen that $4 \mid V_n$ and therefore n is odd by Lemma 4. This shows that m and r are also odd. Hence we have $V_n \equiv nP(-Q)^{(n-1)/2} \pmod{P^2}$, $V_m \equiv mP(-Q)^{(m-1)/2} \pmod{P^2}$ and $V_r \equiv rP(-Q)^{(r-1)/2} \pmod{P^2}$ by Lemma 4. This implies that

$$nP(-Q)^{(n-1)/2} \equiv mrP^2(-Q)^{(\frac{m+r-2}{2})}x^2 \pmod{P^2}.$$

Thus it follows that

$$n(-Q)^{(n-1)/2} \equiv mrP(-Q)^{(\frac{m+r-2}{2})}x^2 \pmod{P},$$

which is impossible since n and Q are odd integers. This completes the proof. ∎

Theorem 12. *Let $P \equiv 3 \pmod 4$ and $Q \equiv 1 \pmod 4$. If m and r are positive integers, then there is no integer x such that $V_n = V_mV_rx^2$.*

Proof. Assume that $V_n = V_mV_rx^2$. When m or r is even, the assertion follows from Corollary 9. Now assume that m and r are odd. Then n is also odd.

Firstly, assume that P is odd. If $3 \mid m$ and $3 \mid r$, then V_m and V_r are even by (2.12). Thus it follows that $4 \mid V_n$. This is impossible by (2.8). Therefore $3 \nmid m$ or $3 \nmid r$. Since $4 \nmid V_n$, x is odd integer.

Assume that $3 \nmid m$ and $3 \nmid r$. Thus $3 \nmid n$. Since n, m and r are odd, it is seen that $V_n \equiv P, 5P \pmod 8$, $V_m \equiv P, 5P \pmod 8$ and $V_r \equiv P, 5P \pmod 8$ by (2.6). Thus we get $V_n = V_mV_rx^2 \equiv 1, 5 \pmod 8$. Then either $P \equiv 1, 5 \pmod 8$ or $5P \equiv 1, 5 \pmod 8$, which is impossible since $P \equiv 3 \pmod 4$.

Assume that $3 \mid m$ and $3 \nmid r$. Then $3 \mid n$. If $Q \equiv 1 \pmod 8$, then it follows that $V_n \equiv 6P \pmod 8$, $V_m \equiv 6P \pmod 8$, $V_r \equiv P \pmod 8$ by (2.6) and if $Q \equiv 5 \pmod 8$, then $V_n \equiv 2P \pmod 8$, $V_m \equiv 2P \pmod 8$, $V_r \equiv P, 5P \pmod 8$ by (2.6). In both cases, from the equation $V_n = V_mV_rx^2$ we get $P \equiv 1 \pmod 4$, which is impossible since $P \equiv 3 \pmod 4$.

Secondly, assume that P is even. In this case, since the proof is identical to that for P even in Theorem 11, we omit the details. ∎

The following theorem was proved by Keskin and Demirtürk in [6] when $(P, Q) = (1, -1)$.

Theorem 13. *Let m and r be positive integers, $P > 1$ and $Q = -1$. Then there is no generalized Lucas number V_n such that $V_n = V_mV_r$.*

Proof. Assume that $V_n = V_mV_r$. If one of m and r is even, then the statement follows from Corollary 9. Now assume that m and r are odd.

Firstly, we assume that P is odd. Since $V_m \mid V_n$ and $V_r \mid V_n$, there exist odd integers t and s such that $n = mt$ and $n = rs$ by Theorem 6. It is obvious that $t > 1$ and $s > 1$. Hence $t = 4q \pm 1$ for some $q \geq 1$. Therefore

$n = mt = 4mq \pm m = 2(2mq) \pm m$. Then it follows that

$$V_m V_r = V_n \equiv \pm V_m \pmod{V_{2m}} \tag{3.4}$$

by (2.4). Similarly, it can be seen that

$$V_m V_r \equiv \pm V_r \pmod{V_{2r}}. \tag{3.5}$$

If $3 \mid m$ and $3 \mid r$, then, since m and r are odd, we get $V_3 \mid V_m$ and $V_3 \mid V_r$ by Theorem 6. It follows that $8 \mid V_n$ by (2.9), which is impossible by (2.7). Therefore $3 \nmid m$ or $3 \nmid r$. Assume that $3 \nmid m$ and $3 \nmid r$. Then $(V_m, V_{2m}) = (V_r, V_{2r}) = 1$ by (2.12) and (2.13). Hence we get

$$V_r \equiv \pm 1 \pmod{V_{2m}} \tag{3.6}$$

by (3.4) and

$$V_m \equiv \pm 1 \pmod{V_{2r}} \tag{3.7}$$

by (3.5). Thus we obtain

$$V_{2m} \leq V_r \pm 1 \leq V_r + 1 \quad \text{and} \quad V_{2r} \leq V_m \pm 1 \leq V_m + 1$$

by (3.6) and (3.7), respectively. It follows that

$$V_{2m} + V_{2r} \leq V_m + V_r + 2. \tag{3.8}$$

Using (2.11) and (3.8), we get $V_m^2 + V_r^2 + 2 \leq V_m + V_r$, which is impossible.

Assume that $3 \mid m$ and $3 \nmid r$. Then $(V_m, V_{2m}) = 2$ and $(V_r, V_{2r}) = 1$ by (2.12) and (2.13). Hence

$$V_r \equiv \pm 1 \pmod{V_{2m}/2}, \tag{3.9}$$

$$V_m \equiv \pm 1 \pmod{V_{2r}}, \tag{3.10}$$

by (3.4) and (3.5), respectively. By (3.9) and (3.10), it can be seen that

$$V_{2m} \leq 2V_r + 2, \tag{3.11}$$

$$V_{2r} \leq V_m + 1. \tag{3.12}$$

Thus we get

$$V_{2m} + V_{2r} \leq V_m + 2V_r + 3 \tag{3.13}$$

by (3.11) and (3.12). Using (2.11) in (3.13), we obtain $V_m^2 - V_m + V_r^2 - 2V_r \leq -1$. This shows that $V_m(V_m - 1) + V_r(V_r - 2) \leq -1$, which is impossible since $V_m \geq 2$ and $V_r \geq 2$.

Secondly, we assume that P is even. Since n, m, and r are odd, $V_n \equiv nP \pmod{P^2}$, $V_m \equiv mP \pmod{P^2}$, and $V_r \equiv rP \pmod{P^2}$ by Lemma 4. This shows that $nP \equiv mrP^2 \pmod{P^2}$. Hence $n \equiv mrP \pmod{P}$, which is impossible since n is odd. This completes the proof. ∎

Now we prove the same statement for $Q = 1$.

THEOREM 14. *Let m and r be positive integers, $P > 1$ and $Q = 1$. Then there is no generalized Lucas number V_n such that $V_n = V_m V_r$.*

Proof. Assume that $V_n = V_m V_r$. If one of m and r is even, then the conclusion follows from Corollary 9. Now assume that m and r are odd.

Firstly, we assume that P is odd. Then since $V_m \mid V_n$ and $V_r \mid V_n$, there exist odd integers t and s such that $n = mt$ and $n = rs$ by Theorem 6. It is obvious that $t > 1$ and $s > 1$. Hence $t = 4q \pm 1$ for some $q \geq 1$. Therefore $n = mt = 4mq \pm m = 2(2mq) \pm m$. Then it follows that

$$V_m V_r = V_n \equiv \pm V_m \pmod{V_{2m}} \tag{3.14}$$

by (2.4) and (1.1). Similarly, it can be seen that

$$V_m V_r \equiv \pm V_r \pmod{V_{2r}}. \tag{3.15}$$

If $3 \mid m$ and $3 \mid r$, then V_m and V_r are even by (2.12). This shows that $4 \mid V_n$, which is impossible by (2.8). Therefore $3 \nmid m$ or $3 \nmid r$. Assume that $3 \nmid m$ and $3 \nmid r$. Then $(V_m, V_{2m}) = (V_r, V_{2r}) = 1$ by (2.12) and (2.13). Hence

$$V_r \equiv \pm 1 \pmod{V_{2m}} \tag{3.16}$$

by (3.14) and

$$V_m \equiv \pm 1 \pmod{V_{2r}} \tag{3.17}$$

by (3.15). Thus we obtain

$$V_{2m} \leq V_r \pm 1 \leq V_r + 1$$

and

$$V_{2r} \leq V_m \pm 1 \leq V_m + 1$$

by (3.16) and (3.17), respectively. Then it follows that

$$V_{2m} + V_{2r} \leq V_m + V_r + 2. \tag{3.18}$$

Using (2.11) and (3.18), we get $V_m(V_m - 1) + V_r(V_r - 1) \leq 6$, which is impossible since $V_m \geq P \geq 3$ and $V_r \geq P \geq 3$.

Assume now that $3 \mid m$ and $3 \nmid r$. Then $(V_m, V_{2m}) = 2$ and $(V_r, V_{2r}) = 1$ by (2.12) and (2.13). Hence

$$V_r \equiv \pm 1 \pmod{V_{2m}/2}, \tag{3.19}$$

$$V_m \equiv \pm 1 \pmod{V_{2r}}, \tag{3.20}$$

by (3.14) and (3.15), respectively. It can be seen that

$$V_{2m} \leq 2V_r + 2, \tag{3.21}$$

$$V_{2r} \leq V_m + 1, \tag{3.22}$$

by (3.19) and (3.20). Thus we get

$$V_{2m} + V_{2r} \leq V_m + 2V_r + 3 \tag{3.23}$$

by (3.21) and (3.22). Using (2.11) in (3.23), we obtain $V_m^2 - V_m + V_r^2 - 2V_r \leq 7$. This shows that $V_m(V_m - 1) + V_r(V_r - 2) \leq 7$, which is impossible since $V_m \geq P \geq 3$ and $V_r \geq P \geq 3$.

Secondly, we assume that P is even. Since $n, m,$ and r are odd, $V_n \equiv \pm nP \pmod{P^2}$, $V_m \equiv \pm mP \pmod{P^2}$ and $V_r \equiv \pm rP \pmod{P^2}$ by Lemma 4. This shows that $nP \equiv \pm mrP^2 \pmod{P^2}$, implying $n \equiv \pm mrP \pmod{P}$, which is impossible since n is odd. This completes the proof. ∎

Since the proof of the following lemma is easy, we omit it.

LEMMA 15. *If $Q = \pm 1$ and $0 < r < n$, then $V_n > 2U_r$.*

In [4], Farrokhi showed that the equation $F_n = F_m F_r$ has no solutions for $m > 2$ and $r > 2$. Now we give a similar result for generalized Fibonacci numbers when $P > 1$ and $Q = \pm 1$.

THEOREM 16. *Let $P > 1$, $Q = \pm 1$ and $m > r > 1$. Then there is no generalized Fibonacci number U_n such that $U_n = U_m U_r$.*

Proof. Assume that $U_n = U_m U_r$, $Q = \pm 1$ and $m > r > 1$. Then since $U_m \mid U_n$ and $U_r \mid U_n$, it follows that $n = mt$ and $n = rs$ for some positive integers t and s by Theorem 5.

Firstly, we assume that t is even, i.e., $t = 2a$ for some positive integer a. Then $n = mt = 2ma$. Hence $U_m U_r = U_n = U_{2ma} = U_{ma} V_{ma}$ by (2.10). This shows that $(U_{ma}/U_m) V_{ma} = U_r$ by Theorem 5. Therefore $V_{ma} \mid U_r$. By Theorem 7, we obtain $r = 2mac = nc$ for some natural number c. This shows that $n \mid r$. Since $r \mid n$, we get $n = r$. Therefore $U_m = 1$, which is impossible since $m > 1$ and $P > 1$.

Secondly, we assume that t is odd. It is obvious that $t > 1$. Then we can write $t = 4q \pm 1$ with $q \geq 1$. Therefore we get $n = mt = 2(2mq) \pm m$. Then

$$U_n = U_{2(2mq) \pm m} \equiv U_{\pm m} \pmod{U_{2m}},$$

by (2.1). Thus we get

$$U_m U_r \equiv \pm U_m \pmod{U_{2m}} \tag{3.24}$$

by (1.1). Since $U_{2m} = U_m V_m$ by (2.10), we obtain

$$U_r \equiv \pm 1 \pmod{V_m}.$$

Hence $V_m \leq U_r \pm 1 \leq U_r + 1$. Moreover, since $m > r > 1$, we have $V_m > 2U_r$ by Lemma 15. Thus it is seen that $U_r + 1 \geq V_m > 2U_r$, which is impossible. This completes the proof. ∎

It is well known that the greatest common divisor of U_m and U_n is again a generalized Fibonacci number by (2.14). But the least common multiple of U_m and U_n may not be a generalized Fibonacci number. This follows from the following theorem. Since the proof is similar to that of Theorem 16, we omit it.

THEOREM 17. *Let $Q = \pm 1$, $1 < m < n$, and $P > 1$. Then $[U_m, U_n]$, the least common multiple of U_m and U_n, is a generalized Fibonacci number if and only if $U_m \mid U_n$.*

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Refik Keskin
Department of Mathematics
Sakarya University
TR54187 Sakarya, Turkey
E-mail: rkeskin@sakarya.edu.tr

Zafer Şiar
Department of Mathematics
Bilecik Şeyh Edebali University
TR11030 Bilecik, Turkey
E-mail: zafer.siar@bilecik.edu.tr

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