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EXACT KRONECKER CONSTANTS OF HADAMARD SETS

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Abstract. A set S of integers is called ε-Kronecker if every function on S of modulus one can be approximated uniformly to within ε by a character. The least such ε is called the ε-Kronecker constant, $\kappa(S)$. The angular Kronecker constant is the unique real number $\alpha(S) \in [0, 1/2]$ such that $\kappa(S) = |\exp(2\pi i\alpha(S)) - 1|$. We show that for integers $m > 1$ and $d \geq 1$,

$$\alpha\{1, m, \ldots, m^{d-1}\} = \frac{m^{d-1} - 1}{2(m^d - 1)} \quad \text{and} \quad \alpha\{1, m, m^2, \ldots\} = 1/(2m).$$

1. Introduction. A subset S of the dual of a compact, abelian group G is called an *ε-Kronecker set* if for every continuous function f mapping S into $\mathbb{T}$, the set of complex numbers of modulo 1, there exists $x \in G$ such that

$$|\gamma(x) - f(\gamma)| < \varepsilon \quad \text{for all } \gamma \in S.$$

The infimum of such ε is called the *Kronecker constant*, $\kappa(S)$.

Sets whose Kronecker constants are zero are called *Kronecker sets* and have been much studied (see [GM] and the references cited therein). The concepts were discussed in the Séminaire Bourbaki (1964–1966) without formal naming ([Kah]), were introduced by Varopoulos [Var] and were called ε-free in [GK].

Infinite ε-Kronecker sets (for small ε) are known to exist in many groups (cf. [GaHe], [GL], [GH4]). For instance, Hadamard sets $\{n_j\} \subseteq \mathbb{N}$ with ratio $m > 2$ (meaning $\inf_j\{n_{j+1}/n_j\} = m$) have Kronecker constant at most $|1 - e^{i\pi/(m-1)}|$ ([GH1], [KR]). Various properties of ε-Kronecker sets were established in [GH1], [GHK], [GH2] and [GH3]. For example, if $k(S) < \sqrt{2}$, then S is a Sidon set, meaning that every bounded function defined on S is the restriction to S of the Fourier transform of a measure on G. In fact, the interpolating measure can be chosen to be discrete, positive and supported on a small set.

However, many open problems remain. It is not known if every Hadamard set is ε-Kronecker for some $\varepsilon < 2$, for example, or if S is necessarily Sidon

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if $\sqrt{2} \leq \kappa(S) < 2$. There is a simple formula for $\kappa\{a,b\}$ when $a,b \in \mathbb{Z}$, but for larger subsets calculating Kronecker constants is generally very difficult. Other than for small examples calculated by computer, Kronecker constants have been determined only for certain classes of three-element subsets, such as arithmetic progressions [HR1]. Kronecker constants of finite sets are of interest because the Kronecker constant of an infinite set is the supremum of the Kronecker constants of its finite subsets.

In this paper we calculate the Kronecker constants for finite or infinite Hadamard sets $\{m^j\}$ with integer $m \geq 2$. Kronecker constants (or upper bounds on the Kronecker constants) are also obtained for certain closely related multiplicative sets.

To state our precise results, it is more convenient to identify $\mathbb{T}$ with the quotient space $[-1/2, 1/2)$ where $\pm 1/2$ are identified, and to calculate the angular Kronecker constant.

Definition 1. The *angular Kronecker constant* of a set S, denoted $\alpha(S)$, is the infimum of α such that for all $f : S \to \mathbb{R}$ there is some $x \in [-1/2, 1/2)$ with the property that for all $n \in S$, $\langle f(n) - nx\rangle < \alpha$, where $\langle u\rangle$ is the distance from u to the nearest integer.

It is easy to check that $\alpha(S)$ is the unique real number in $[0, 1/2]$ such that $\kappa(S) = |1 - e^{2\pi i\alpha(S)}|$. It is known that $\alpha(S)$ is rational when S is a finite subset of $\mathbb{Z}$ and that $\alpha\{a,b\} = \gcd(a,b)/2(|a|+|b|)$ for non-zero integers a, b ([HR1]).

In this note we show that

$$\alpha\{1, m, \dots, m^{d-1}\} = \frac{m^{d-1} - 1}{2(m^d - 1)}.$$

Consequently, $\alpha\{m^j\}_{j=0}^{\infty} = 1/(2m)$. Moreover, we find examples of functions f for which the bounds are sharp. We also consider sets of the form $S = \{1, a_1, a_1a_2, \dots, a_1 \dots a_d\}$ where $a_j \in \mathbb{N}$ and show, for example, that $\alpha(S) = 1/(2(1 + a_d))$ if $a_j > a_d$ for $j < d$.

2. Upper bounds for Kronecker constants

Proposition 2. *For $S = \{1, m, m^2, \dots, m^{d-1}\}$, with $m > 1$ an integer,*

$$\alpha(S) \leq \frac{m^{d-1} - 1}{2(m^d - 1)} =: s_d.$$

Proof. This is trivial for $d = 1$ and is known, as remarked above, for $d = 2$. Note that $s_d < 1/(2m)$ for all d.

Let $f : S \to \mathbb{R}$ and set $f(m^{j-1}) = \theta_j$ for $j = 1, \dots, d$. Put $D_1 = [\theta_1 - E, \theta_1 + E]$ and for $j = 2, \dots, d$ inductively define

$$D_j = \{z \in [\theta_j - E, \theta_j + E] : (\exists k \in \mathbb{Z})(z - k \in mD_{j-1})\}.$$

The definition of D_j ensures that given any $z_j \in D_j$, there is an integer k_j and $z_{j-1} \in D_{j-1}$ such that $z_j = k_j + mz_{j-1}$. Thus

$$z_j = \sum_{t=2}^{j} m^{j-t} k_t + m^{j-1} z_1 \quad \text{for } 1 \leq j \leq d. \tag{2.1}$$

Because each z_j is in $[\theta_j - E, \theta_j + E]$ and k_t, m are integers,

$$\langle \theta_j - m^{j-1} z_1 \rangle = \langle \theta_j - z_j \rangle \leq E.$$

Thus, provided $D_d \neq \emptyset$, there will be some x (here labeled as z_1) such that $\|\langle f - z_1 \mathbf{m} \rangle\|_\infty \leq E$ for $\mathbf{m} = (1, m, \ldots, m^{d-1})$. Since f was arbitrary, that will prove $\alpha(S) \leq E$. Of course, it will be enough to prove this for $s_d \leq E < 1/(2m)$.

To show this, we will prove that the Lebesgue measure of D_j, denoted $|D_j|$, satisfies

$$|D_j| \geq \frac{2(m^j - 1)}{m - 1} E - \frac{m^{j-1} - 1}{m - 1} \quad \text{for } 1 \leq j \leq d. \tag{2.2}$$

As $s_j \leq s_d$, the right hand side of (2.2) is strictly positive for all j.

Note that when $j = 1$ we have equality since $|D_1| = 2E$. Next, consider the case of $j = 2$. Let $D_2' = [\theta_2 - 1/2, \theta_2 + 1/2)$. Because D_2' has length 1 and is half-open, for each $y \in mD_1$ there is a unique $\omega(y) \in D_2'$ such that $y \equiv \omega(y) \bmod 1$. Since mD_1 is an interval of length $2mE < 1$, ω is 1-1 and piecewise a translation. Hence $|\omega(mD_1)| = 2mE$. We also note that as $D_2 \subset D_2'$, we have

$$D_2 = [\theta_2 - E, \theta_2 + E] \cap \omega(mD_1).$$

But $\omega(mD_1)$ misses (in measure) $1 - 2mE$ of D_2', thus

$$|D_2| \geq 2E - (1 - 2mE) = \frac{2(m^2 - 1)}{m - 1} E - \frac{m - 1}{m - 1},$$

showing (2.2) is satisfied for $j = 2$.

We proceed inductively. Suppose that (2.2) holds for some $2 \leq j < d$ and consider the case D_{j+1}. As $D_j \subset [\theta_j - E, \theta_j + E]$, with width $2E < 1/m$, mD_j is a subset of $[m\theta_j - mE, m\theta_j + mE]$ whose length is $2mE < 1$. Let $D_{j+1}' = [\theta_{j+1} - 1/2, \theta_{j+1} + 1/2)$. For each $y \in mD_j$ there is a unique $\omega(y) \in D_{j+1}'$ such that $y \equiv \omega(y) \bmod 1$. Since mD_j is a subset of an interval whose length is less than 1, ω is 1-1 and piecewise a translation on mD_j. Thus $|\omega(mD_j)| = m|D_j|$. As $D_{j+1} \subset D_{j+1}'$,

$$D_{j+1} = [\theta_{j+1} - E, \theta_{j+1} + E] \cap \omega(mD_j),$$

hence

$$|D_{j+1}| \geq 2E - \left\{1 - m\left[\frac{2(m^j-1)}{m-1}E - \frac{m^{j-1}-1}{m-1}\right]\right\}$$
$$= \frac{2(m^{j+1}-1)}{m-1}E - \frac{m^j-1}{m-1},$$

proving (2.2). That completes the proof. ∎

A similar argument gives the following related result.

PROPOSITION 3. *Let* $T = \{1, m, m^2, \ldots, m^d, m^d(m+1)\}$ *for integers* $d \geq 1$ *and* $m > 1$. *Then*

$$\alpha(T) \leq \frac{(m+1)(1+\cdots+m^{d-1})}{2(1+\cdots+m^d+m^2+\cdots+m^{d+1})} =: t_d.$$

Proof. Note that $2mt_d < 1$. Pick E with $t_d < E < 1/(2m)$ and assume $f : T \to \mathbb{R}$. We identify f with $\{\theta_j\}_{j=1}^{d+2}$, let $D_1 = [\theta_1 - E, \theta_1 + E]$ and for $1 \leq j \leq d$ inductively define

$$D_{j+1} = [\theta_{j+1} - E, \theta_{j+1} + E] \cap \{z : (\exists k \in \mathbb{Z})(z - k \in mD_j)\}.$$

Similar arguments to those used in the proof of Prop. 2 show that since $2mE < 1$, we have

$$|D_j| \geq 2E(1+\cdots+m^{j-1}) - (1+\cdots+m^{j-2}) \tag{2.3}$$

and this is easily seen to be strictly positive given the assumptions on E. Now let

$$D_{d+2} = [\theta_{d+2} - E, \theta_{d+2} + E] \cap \{z : (\exists k \in \mathbb{Z})(z - k \in (m+1)D_{d+1})\}.$$

For any $z_{d+2} \in D_{d+2}$, there is an integer, k_{d+2}, such that

$$z_{d+2} = k_{d+2} + (m+1)z_{d+1} \quad \text{for some } z_{d+1} \in D_{d+1}.$$

As in the previous proof there are integers k_t and $z_t \in D_t$ such that

$$z_{d+2} = k_{d+2} + (m+1)\Big(\sum_{t=2}^{d+1} m^{d+1-t}k_t + m^d z_1\Big).$$

It follows that for $j = 1, \ldots, d+1$,

$$\langle \theta_j - m^{j-1}z_1 \rangle = \langle \theta_j - z_j \rangle \leq E$$

and

$$\langle \theta_{d+2} - m^d(m+1)z_1 \rangle = \langle \theta_{d+2} - z_{d+2} \rangle \leq E.$$

It remains to check that D_{d+2} is non-empty. Of course, $(m+1)D_{d+1} \subseteq (m+1)[\theta_{d+1} - E, \theta_{d+1} + E] =: I_d$ and the length of I_d is $2(m+1)E > 1$. Thus every point of $[\theta_{d+2} - E, \theta_{d+2} + E]$ is congruent mod 1 to an element of I_d. Indeed, there is an integer N and $\beta \in [\theta_{d+2} - E, \theta_{d+2} + E]$ such that the intervals $(\beta, \theta_{d+2} + E] + N - 1$ and $[\theta_{d+2} - E, \beta] + N$ are disjoint and contained in I_d. It will suffice to prove that

$$I' := (m+1)D_{d+1} \cap \big([\beta, \theta_{d+2} + E] + N - 1 \cup [\theta_{d+2} - E, \beta] + N\big)$$

is non-empty. Since both intervals, $[\beta, \theta_{d+2}+E]+N-1$ and $[\theta_{d+2}-E, \beta]+N$, are contained in I_d, and $(m+1)D_{d+1}$ misses a subset of I_d of measure $2(m+1)E - (m+1)|D_{d+1}|$, it follows that the measure of I' is at least

$$\begin{aligned}&2E - (2(m+1)E - (m+1)|D_{d+1}|)\\ &\quad \geq 2E\big(1 - (m+1) + (m+1)(1 + \cdots + m^d)\big) - (m+1)(1 + \cdots + m^{d-1}).\end{aligned}$$

This is positive provided

$$E > \frac{(m+1)(1 + \cdots + m^{d-1})}{2((m+1)(1 + \cdots + m^d) - m)},$$

and that is true by the choice of E. This completes the proof. ∎

REMARK 4. One can similarly show that for any integer $p > 1$,

$$\alpha\{1, m, \ldots, m^d, m^d(m+p)\} \leq \frac{(m+p)(1 + \cdots + m^{d-1})}{2(p(m + \cdots + m^d) + 1 + m^2 + \cdots + m^{d+1})}$$

3. Lower bounds for Kronecker constants. The paper [HR1] provides an alternative calculation of $\alpha(S)$ for $S = \{n_1, \ldots, n_d\} \subset \mathbb{Z} \setminus \{0\}$:

$$\alpha(S) = \max\{\alpha_S(f) : f \in \mathbb{R}^{d-1}\}$$

where $\alpha_S(f)$ is the distance of f to a particular discrete subgroup $\mathcal{K} \subset \mathbb{R}^{d-1}$ (determined by S), with respect to the norm

$$\|\mathbf{z}\| = \max\left\{\frac{|n_j z_i - n_i z_j|}{|n_i| + |n_j|} : 1 \leq i < j \leq d\right\}$$

where $\mathbf{z} = (z_1, \ldots, z_d)$ and $z_d = 0$. We call $\frac{|n_j z_i - n_i z_j|}{|n_i| + |n_j|}$ the (i,j)-*form* of the norm. This formulation will be used to show that the upper bound of Prop. 2 is sharp.

THEOREM 5. *For $S = \{1, m, m^2, \ldots, m^{d-1}\}$, with $m > 1$ an integer,*

$$\alpha(S) = \frac{m^{d-1} - 1}{2(m^d - 1)}.$$

Proof. From the first proposition, we have already seen that $\alpha(S) \leq (m^{d-1}-1)/(2(m^d-1))$. Thus it will suffice to show there exists some $f \in \mathbb{R}^{d-1}$ such that $\alpha_S(f) = (m^{d-1} - 1)/(2(m^d - 1))$. We will show this is true for

$$f = \left\{\frac{(1 + m^{d-1})(m^{d-i} - 1)}{2m^{d-i}(m^d - 1)}\right\}_{i=1}^{d-1}.$$

By Proposition 3 of [HR1], a basis for $\mathcal{K}$ consists of P_i, $1 \le i \le d-1$, given by

$$(P_i)_j = \begin{cases} m^{-(i-j+1)} & \text{for } 1 \le j \le i, \\ 0 & \text{for } i < j. \end{cases}$$

Let $r \in \mathbb{Z}^{d-1}$ specify an arbitrary member $W_r \in \mathcal{K}$, where $W_r = \sum_{i=1}^{d-1} r_i P_i$.

Suppose $r_{d-1} \ge 1$. Then the $(d-1)$th coordinate of $W_r - f$ satisfies

$$\begin{aligned}(W_r - f)_{d-1} &= \frac{r_{d-1}}{m} - \frac{(1+m^{d-1})(m-1)}{2m(m^d-1)} \\ &\ge \frac{1}{m} - \frac{(1+m^{d-1})(m-1)}{2m(m^d-1)} \\ &= \frac{(m^{d-1}-1)(1+m)}{2m(m^d-1)}.\end{aligned}$$

Using the $(d-1,d)$-form of the metric, we have

$$\|W_r - f\| \ge \frac{m^{d-1}(W_r-f)_{d-1}}{m^{d-1}+m^{d-2}} = \frac{m^{d-1}-1}{2(m^d-1)} =: s_d.$$

Next, suppose that $r_{d-1} < 0$. Since $f_{d-1} > 0$ and $r_{d-1} \le -1$,

$$(f - W_r)_{d-1} > 0 - \frac{r_{d-1}}{m} \ge \frac{1}{m}.$$

By using the $(d-1,d)$-form of the metric, we see that

$$\|W_r - f\| > \frac{m^{d-1}(1/m) - m^{d-2}\cdot 0}{m^{d-2}+m^{d-1}} = \frac{1}{m+1} > \frac{1}{2m} > s_d.$$

For an induction hypothesis, suppose that $r_s = 0$ for $i+1 \le s \le d-1$ and $i > 0$. First, suppose that $r_i \ge 1$. Then the ith coordinate of W_r is r_i/m, therefore,

$$(W_r - f)_i \ge \frac{1}{m} - \frac{(1+m^{d-1})(m^{d-i}-1)}{2m^{d-i}(m^d-1)}.$$

Also,

$$(f - W_r)_{i+1} = f_{i+1} = \frac{(1+m^{d-1})(m^{d-i-1}-1)}{2m^{d-i-1}(m^d-1)}.$$

A computation using the $(i,i+1)$-form of the metric gives

$$\begin{aligned}\|W_r - f\| &\ge \frac{m^i(W_r-f)_i - m^{i-1}(W_r-f)_{i+1}}{m^i+m^{i-1}} \ge \frac{1}{m+1} \\ &\quad + \frac{-m(1+m^{d-1})(m^{d-i}-1)/m + (1+m^{d-1})(m^{d-i-1}-1)}{(m+1)2m^{d-i-1}(m^d-1)} \\ &= \frac{m^{d-1}-1}{2(m^d-1)} = s_d.\end{aligned}$$

Next, suppose that $r_i \le -1$. As $f_i > 0$, $(f - W_r)_i > 0 - r_i/m \ge 1/m$. By using the (i,d)-form of the metric, we have

$$\|W_r - f\| > \frac{m^{d-1}(1/m)}{m^{i-1} + m^{d-1}} = \frac{m^{d-i-1}}{1 + m^{d-i}} \ge \frac{m^{d-i-1}}{2m^{d-i}} = \frac{1}{2m} > s_d.$$

Thus, we may assume $r_i = 0$. We proceed backward through i, until we have $r_s = 0$ for $1 \le s \le d-1$.

That leaves only $W_r = 0$ and for that element of $\mathcal{K}$ we will use the $(1,d)$-form of the metric. Since

$$f_1 = \frac{(1 + m^{d-1})(m^{d-1} - 1)}{2m^{d-1}(m^d - 1)}$$

we deduce that

$$\|f\| \ge \frac{m^{d-1}(1 + m^{d-1})(m^{d-1} - 1)}{(m^{d-1} + 1)2m^{d-1}(m^d - 1)} = s_d.$$

This shows that for every choice of r, and therefore every choice of $W_r \in \mathcal{K}$, $\|f - W_r\| \ge s_d$, and consequently $\alpha_S(f) \ge s_d$. ∎

Similar arguments show that the upper bound of Prop. 3 is also an equality.

PROPOSITION 6. *For integers $m > 1$ and $d \ge 1$ we have*

$$\alpha\{1, m, \dots, m^d, m^d(m+1)\} = \frac{(m+1)(1 + \cdots + m^{d-1})}{2(1 + \cdots + m^d + m^2 + \cdots + m^{d+1})} =: t_d.$$

Proof. Set $D = 1 + \cdots + m^d + m^2 + \ldots + m^{d+1}$. For $j = 0, \dots, d-1$ let $A_j = \sum_{t=0}^{d-j-1} m^t$ and let $A_d = 0$.

It was proved earlier that $\alpha\{1, \dots, m^d, m^d(m+1)\} \le t_d$, thus to show equality, it will be enough to establish that $f \in \mathbb{R}^{d+1}$, described below, is an example of a worst point to approximate:

$$f_j = \begin{cases} 0 & \text{for } j = d+1, \\ \dfrac{1}{2m} + \dfrac{-A_0 + m^{d-j} + 2A_j}{2m^{d+1-j}D} & \text{for integers } j \in [1, d]. \end{cases}$$

The argument will be similar to the proof of the previous theorem. A basis for the appropriate discrete subgroup $\mathcal{K}$ consists of (particular) functions $P^{(j)}$, $1 \le j \le d+1$, which have the property that $P^{(d+1)}_{d+1} = 1/(m+1)$ and $P^{(j)}_j = 1/m$ for integers $j \in [1, d]$.

Given $\mathbf{r} \in \mathbb{Z}^{d+1}$ let $W_\mathbf{r} = \sum_{j=1}^{d+1} r_j P^{(j)}$. As in the previous proof, an induction argument using the $(j, j+1)$- and $(j, d+2)$-forms of the norm can be given to show that if any r_j is non-zero, then $\|f - W_r\| \ge t_d$. On the other hand, if all r_j are zero, then the $(1, d+2)$-form of the norm shows that $\|f - W_r\| = \|f\| \ge t_d$. The calculations are left to the reader. ∎

Corollary 7. *If $m \geq 2$ is an integer, then*

$$\alpha\{1, m, m(m+1)\} = \alpha\{1, m, m^2\}.$$

4. Kronecker constants for infinite Hadamard sequences. One reason for the interest in Kronecker constants of finite sets is that the Kronecker constant of an infinite set is the supremum of the Kronecker constants of the finite subsets.

Proposition 8. *If $S = \bigcup_j F_j$ with $F_j \subset F_{j+1}$ for all positive integers j, then*

$$\alpha(S) = \lim_{j\to\infty} \alpha(F_j) \quad \textit{and} \quad \kappa(S) = \lim_{j\to\infty} \kappa(F_j).$$

Proof. Since the sets F_j are nested, it is clear that $\alpha(S) \geq \alpha(F_j)$ for all j and $\alpha(F_j)$ is increasing. As $\alpha(F_j) \leq 1/2$, it follows that $\lim_{j\to\infty} \alpha(F_j)$ exists and equals $\sup \alpha(F_j)$.

Consider $f : S \to \mathbb{T}$ and let $f_j = f|_{F_j}$. Fix $E > \sup \alpha(F_j)$. Then there exists $x_j \in G$ such that

$$|\gamma(x_j) - f_j(\gamma)| < E \quad \text{for all } \gamma \in F_j.$$

Since G is compact, the net $\{x_j\}$ has cluster point x_0. Without loss of generality, $x_j \to x_0$ and then, by continuity, $\gamma(x_j) \to \gamma(x_0)$ for all $\gamma \in \Gamma$. Given any $\gamma \in S$ there is an index J such that $\gamma \in F_j$ for all $j \geq J$. Thus $|\gamma(x_j) - f(\gamma)| < E$ for all $j \geq J$ and that implies $|\gamma(x_0) - f(\gamma)| \leq E$. Thus $\alpha(S) \leq E$ and as $E > \sup \alpha(F_j)$ was arbitrary, it follows that $\alpha(S) \leq \sup \alpha(F_j)$, as we desired to show.

The statement for $\kappa(S)$ holds since $\kappa(S) = |\exp(2\pi i \alpha(S)) - 1|$. ∎

With this it is easy to determine the Kronecker constants of the set of powers of an integer.

Corollary 9. *Let $m > 1$ be an integer and $S = \{m^j\}_{j=0}^{\infty}$. Then*

$$\alpha(S) = 1/(2m) \quad \textit{and} \quad \kappa(S) = \sqrt{2(1 - \cos(\pi/m))}.$$

Proof. Note that $S = \bigcup_{j=1}^{\infty} F_j$ where $F_j = \{1, m, m^2, \ldots, m^{j-1}\}$ and

$$\alpha(F_j) = \frac{m^{j-1} - 1}{2(m^j - 1)} = \frac{1 - m^{-j+1}}{2(m - m^{-j+1})} \to \frac{1}{2m}. \ ∎$$

5. Closely related multiplicative sets. We will say that a finite set S of positive integers is *multiplicative* if $S = \{n_1, \ldots, n_d\}$ with

(i) $1 \leq n_1 < \cdots < n_d$.
(ii) For integers $j \in [1, d)$, n_j divides n_{j+1}.

As noted in [HR1], the Kronecker constant of $S \subseteq \mathbb{N}$ is unchanged if one divides each element of S by the greatest common divisor of the set. So one may assume $n_1 = 1$ for computing Kronecker constants of multiplicative sets.

The Kronecker constants for multiplicative sequences have some surprising features that do not appear with Hadamard sequences with constant ratio, such as no obvious "monotonicity". For example, by Theorem 5, $\alpha\{1,4,16\} = 5/42$ but, according to [HR2], $\alpha\{1,4,24\} = 3/25 > 5/42$.

However, large lacunary ratios "should" contribute less to the Kronecker constant. The results of this section give some examples of this.

PROPOSITION 10. *Let $n_1, \dots, n_d$ be any positive integers. Then*

$$\alpha\{n_1, \dots, n_d, n\} \to \alpha\{n_1, \dots, n_d\} \quad \text{as } n \to \infty.$$

Proof. Assume $\alpha\{n_1, \dots, n_d\} = \alpha$ and let $\varepsilon > 0$. Take $n > n_d/\varepsilon$ where n_d is the largest of the n_j. Let $\theta \in \mathbb{R}^{d+1}$ be arbitrary and choose $x \in [-1/2, 1/2]$ such that $\langle \theta_j - x n_j \rangle \le \alpha$ for all $j = 1, \dots, d$. Then

$$\langle \theta_j - (x+y)n_j \rangle \le \langle \theta_j - x n_j \rangle + \langle y n_j \rangle \le \alpha + \varepsilon$$

for any y with $|y| \le \varepsilon/n_d$.

The choice of n ensures that there exists $z \in [x - \varepsilon/n_d, x + \varepsilon/n_d]$ such that $nz \equiv \theta_{d+1} \bmod 1$. That means $\|\langle \theta - z\mathbf{n} \rangle\| \le \alpha + \varepsilon$ for $\mathbf{n} = (n_1, \dots, n_d, n)$. ∎

PROPOSITION 11. *Let $S = \{n_1, \dots, n_d\}$ be any set of d nonzero integers, with $d \ge 2$. Suppose that* $\gcd(n_1, \dots, n_d) = 1$ *and that* $m = \gcd(n_1, \dots, n_{d-1})$. *Then*

$$\alpha(S) \le \max\left\{ \frac{1}{2m}, \alpha\{n_1, \dots, n_{d-1}\} \right\}.$$

Proof. Let $\beta = \alpha\{n_1, \dots, n_{d-1}\} = \alpha\{n_1/m, \dots, n_{d-1}/m\}$. Let $\theta \in \mathbb{R}^d$ be given. The definition of the angular Kronecker constant ensures there is some real x and integers k_j such that

$$|\theta_j - (n_j/m)x - k_j| \le \beta \quad \text{for } 1 \le j < d.$$

For θ_d there is some integer s such that $|\theta_d - n_d x/m - s/m| \le 1/(2m)$. Because n_d and m are relatively prime, we can write $s = a n_d + bm$ for some integers a and b. Let $x_s = (x+a)/m$. Then

$$\theta_d - n_d x_s - b = (\theta_d - n_d x/m - n_d a/m) - b = \theta_d - n_d x/m - s/m$$

and consequently we have an integer b such that $|\theta_d - n_d(x_s) - b| \le 1/(2m)$.

For $1 \le j \le d-1$, with the integers $k_j' = k_j - n_j a/m$,

$$\begin{aligned} |\theta_j - n_j x_s - k_j'| &= |\theta_j - n_j x_s + n_j a/m - k_j| \\ &= |\theta_j - (n_j/m)x - k_j| \le \beta. \end{aligned}$$

Thus $\|\theta - \mathbf{n} x_s\| \le \max(1/(2m), \beta)$ and hence $\alpha(S) \le \max\{1/(2m), \beta\}$. ∎

EXAMPLE 12. For any integers $j, k \ge 5$ one has $\alpha\{1, j, jk, 4jk, 24jk\} = \alpha\{1, 4, 24\} = 3/25$.

Corollary 13. *Let $a_j > 1$ be integers and $S_d = \{1, a_1, a_1a_2, \ldots, a_1 \ldots a_d\}$ with $d \geq 1$. Then*

$$\alpha(S_d) \leq \max[\{(2a_j)^{-1} : j < d\} \cup \{(2(a_d + 1))^{-1}\}]$$

Proof. This is clear for $d = 1$. Now assume the result holds for any set $\{1, b_1, b_1b_2, \ldots, b_1 \ldots b_{d-1}\}$ with integers $b_j > 1$. Consider S_d and let $S' = S_d \setminus \{1\}$. By the induction hypothesis,

$$\alpha(S') = \alpha(S'/a_1) \leq \max[\{(2a_j)^{-1} : 1 < j < d\} \cup \{(2(a_d + 1))^{-1}\}].$$

By Proposition 11, we have

$$\begin{aligned}\alpha(S_d) &\leq \max\{1/(2a_1), \alpha(S')\} \\ &\leq \max[\{(2a_j)^{-1} : j < d\} \cup \{(2(a_d + 1))^{-1}\}]. \ \blacksquare\end{aligned}$$

An immediate consequence of Corollary 13 is that, if the last multiplier is smaller than the rest, it determines the Kronecker constant.

Corollary 14. *Let $a_j > 1$ be integers. Let*

$$S = \{1, a_1, a_1a_2, \ldots, a_1a_2 \ldots a_d\}$$

with $d \geq 1$ and suppose that $a_j > a_d$ for $j < d$. Then $\alpha(S) = 1/(2(1 + a_d))$.

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