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ARITHMETIC THEORY OF HARMONIC NUMBERS (II)

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Abstract. For $k = 1, 2, \ldots$ let H_k denote the harmonic number $\sum_{j=1}^{k} 1/j$. In this paper we establish some new congruences involving harmonic numbers. For example, we show that for any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv \frac{7}{24} pB_{p-3} \pmod{p^2}, \quad \sum_{k=1}^{p-1} \frac{H_{k,2}}{k2^k} \equiv -\frac{3}{8} B_{p-3} \pmod{p},$$

and

$$\sum_{k=1}^{p-1} \frac{H_{k,2n}^2}{k^{2n}} \equiv \frac{\binom{6n+1}{2n-1} + n}{6n+1} pB_{p-1-6n} \pmod{p^2}$$

for any positive integer $n < (p-1)/6$, where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers, and $H_{k,m} := \sum_{j=1}^{k} 1/j^m$.

1. Introduction. Recall that *harmonic numbers* are

$$H_n := \sum_{0<k\le n} \frac{1}{k} \quad (n \in \mathbb{N} = \{0, 1, 2, \ldots\}),$$

where $H_0 := 0$ since we consider the value of an empty sum as zero. They play important roles in mathematics. In 1862 J. Wolstenholme [W] showed the congruence $H_{p-1} \equiv 0 \pmod{p^2}$ for any prime $p > 3$. Throughout this paper, for a prime p and two rational p-adic integers A and B, we write $A \equiv B \pmod{p^n}$ (with $n \in \mathbb{N}$) to mean that $A - B$ is divisible by p^n in the ring of p-adic integers.

In [Su] the first author investigated arithmetic properties of harmonic numbers systematically. For example, he proved that for any prime $p > 5$ we have

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv \sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p}.$$

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For $m \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, *harmonic numbers of order* m are defined by

$$H_{n,m} := \sum_{0<k\le n} \frac{1}{k^m} \quad (n \in \mathbb{N}).$$

It is known that

$$\sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{\pi^2}{12} \quad \text{(S. W. Coffman [C], 1987)}$$

and

$$\sum_{k=1}^{\infty} \frac{H_{k,2}}{k2^k} = \frac{5}{8}\zeta(3) \quad \text{(B. Cloitre, 2004).}$$

Both identities can be found in [SW].

Our first theorem is as follows.

THEOREM 1.1. *For any prime $p > 3$, we have*

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv \frac{7}{24} pB_{p-3} \pmod{p^2} \tag{1.1}$$

and

$$\sum_{k=1}^{p-1} \frac{H_{k,2}}{k2^k} \equiv -\frac{3}{8} B_{p-3} \pmod{p}, \tag{1.2}$$

where $B_0, B_1, B_2, \dots$ are Bernoulli numbers.

REMARK 1.1. Formula (1.1) confirms the first part of [Su, Conjecture 1.1]. The second part of [Su, Conjecture 1.1] states that $\sum_{k=1}^{p-1} H_k^2/k^2 \equiv \frac{4}{5}pB_{p-5} \pmod{p^2}$ for any prime $p > 3$; this was confirmed by R. Meštrović [M] quite recently.

Our second theorem confirms the second conjecture of [Su].

THEOREM 1.2 ([Su, Conjecture 1.2]). *Let p be an odd prime and let n be a positive integer with $p - 1 \nmid 6n$. Then*

$$\sum_{k=1}^{p-1} \frac{H_{k,2n}^2}{k^{2n}} \equiv 0 \pmod{p}. \tag{1.3}$$

Furthermore, when $p > 6n + 1$ we have

$$\sum_{k=1}^{p-1} \frac{H_{k,2n}^2}{k^{2n}} \equiv \frac{s(n)}{6n+1} pB_{p-1-6n} \pmod{p^2}, \tag{1.4}$$

where

$$s(n) = \binom{6n+1}{2n-1} + n.$$

REMARK 1.2. We give here four initial values of the integer sequence $\{s(n)\}_{n\ge1}$:

$$s(1) = 8, \quad s(2) = 288, \quad s(3) = 11631, \quad s(4) = 480704.$$

We will show Theorems 1.1 and 1.2 in Sections 2 and 3 respectively.

2. Proof of Theorem 1.1

LEMMA 2.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\equiv\frac{p}{2}B_{p-3}\pmod{p^2},\qquad\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\equiv-\frac{B_{p-3}}{2}\pmod{p},\tag{2.1}$$

and

$$\sum_{k=1}^{p-1}\frac{H_k}{k}\equiv\frac{p}{3}B_{p-3}\pmod{p^2},\qquad\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}H_k\equiv-\frac{B_{p-3}}{4}\pmod{p}.\tag{2.2}$$

Proof. It is known that (cf. [S, Corollaries 5.1 and 5.2])

$$\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv\frac{2}{3}pB_{p-3}\pmod{p^2},\qquad\sum_{k=1}^{p-1}\frac{1}{k^3}\equiv\frac{3}{4}pB_{p-4}\equiv-p\delta_{p,5}\pmod{p^2},$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv\frac{7}{3}pB_{p-3}\pmod{p^2},\qquad\sum_{k=1}^{(p-1)/2}\frac{1}{k^3}\equiv-2B_{p-3}\pmod{p}.$$

Thus

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}&=\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k^2}-\sum_{k=1}^{p-1}\frac{1}{k^2}=\frac{1}{2}H_{(p-1)/2,2}-H_{p-1,2}\\&\equiv\frac{7}{6}pB_{p-3}-\frac{2}{3}pB_{p-3}=\frac{p}{2}B_{p-3}\pmod{p^2}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}&=\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k^3}-\sum_{k=1}^{p-1}\frac{1}{k^3}\\&=\frac{1}{4}H_{(p-1)/2,3}-H_{p-1,3}\equiv\frac{-2B_{p-3}}{4}\pmod{p}.\end{aligned}$$

Therefore (2.1) holds.

By the proof of [S, Theorem 6.1],

$$\sum_{1\le j<k\le p-1}\frac{1}{jk}\equiv-\frac{p}{3}B_{p-3}\pmod{p^2}.$$

So we have

$$\sum_{k=1}^{p-1}\frac{H_k}{k}=\sum_{k=1}^{p-1}\frac{1}{k^2}+\sum_{1\le j<k\le p-1}\frac{1}{jk}\equiv\frac{2}{3}pB_{p-3}-\frac{p}{3}B_{p-3}=\frac{p}{3}B_{p-3}\pmod{p^2}.$$

This proves the first congruence in (2.2).

Now we prove the second congruence in (2.2). Since

$$H_{p-k} = H_{p-1} - \sum_{j=1}^{k-1} \frac{1}{p-j} \equiv H_{k-1} = H_k - \frac{1}{k} \pmod p$$

for all $k = 1, \ldots, p-1$, we have

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} H_k = \sum_{k=1}^{p-1} \frac{(-1)^{p-k}}{(p-k)^2} H_{p-k} \equiv -\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2}\left(H_k - \frac{1}{k}\right) \pmod p$$

and hence

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} H_k \equiv \frac{1}{2}\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \equiv -\frac{B_{p-3}}{4} \pmod p. \blacksquare$$

LEMMA 2.2.

(i) *For any positive integers k and m we have*

$$\sum_{n=1}^{m} \binom{n-1}{k-1} = \binom{m}{k}. \tag{2.3}$$

(ii) *For each $n = 1, 2, \ldots$ we have*

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} H_k = H_{n,2}. \tag{2.4}$$

Proof. (2.3) is well known (cf. [G, (1.5)]) and it can be easily proved by induction on m.

(2.4) is also known (cf. [H]). Here we prove it by induction. Clearly (2.4) holds for $n = 1$. Assume that (2.4) holds for a fixed positive integer n. Then

$$\begin{aligned}\sum_{k=1}^{n+1} \binom{n+1}{k} \frac{(-1)^{k-1}}{k} H_k &= \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} H_k + \sum_{k=1}^{n+1} \binom{n}{k-1} \frac{(-1)^{k-1}}{k} H_k \\ &= H_{n,2} + \frac{1}{n+1}\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{k-1} H_k.\end{aligned}$$

Note that

$$\begin{aligned}&\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{k-1} H_k \\ &\quad= \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} H_k + \sum_{k=1}^{n+1} \binom{n}{k-1} (-1)^{k-1}\left(H_{k-1} + \frac{1}{k}\right) \\ &\quad= \sum_{k=1}^{n+1} \binom{n}{k-1} \frac{(-1)^{k-1}}{k} = -\frac{1}{n+1}\sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^k = \frac{1}{n+1}.\end{aligned}$$

So

$$\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^{k-1}}{k}H_k = H_{n,2}+\frac{1}{n+1}\cdot\frac{1}{n+1}=H_{n+1,2}$$

as desired. ∎

LEMMA 2.3. *Let $p>3$ be a prime. Then*

$$\sum_{1\le j\le k\le p-1}\frac{2^j(j+k)}{j^2k^2}\equiv\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\pmod p. \tag{2.5}$$

Proof. Observe that

$$\begin{aligned}\sum_{1\le i\le j\le k\le p-1}\frac{2^i}{ijk}-\sum_{1\le i<j<k\le p-1}\frac{2^i}{ijk}&\\ &=\sum_{1\le j\le k\le p-1}\frac{2^j}{j^2k}+\sum_{1\le i\le j\le p-1}\frac{2^i}{ij^2}-\sum_{k=1}^{p-1}\frac{2^k}{k^3}\\ &=\sum_{1\le j\le k\le p-1}\left(\frac{2^j}{j^2k}+\frac{2^j}{jk^2}\right)-\sum_{k=1}^{p-1}\frac{2^k}{k^3}.\end{aligned}$$

Similarly,

$$\begin{aligned}2&\sum_{1\le i\le j\le k\le p-1}\frac{(-1)^i}{ijk}-2\sum_{1\le i<j<k\le p-1}\frac{(-1)^i}{ijk}-2\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\\ &=2\sum_{1\le j<k\le p-1}\left(\frac{(-1)^j}{j^2k}+\frac{(-1)^j}{jk^2}\right)\\ &\equiv\sum_{1\le j<k\le p-1}\left(\frac{(-1)^j}{j^2k}+\frac{(-1)^j}{jk^2}+\frac{(-1)^{p-j}}{(p-j)^2(p-k)}+\frac{(-1)^{p-j}}{(p-j)(p-k)^2}\right)\\ &=\sum_{1\le j<k\le p-1}\frac{(-1)^j}{j^2k}+\sum_{1\le k<j\le p-1}\frac{(-1)^j}{j^2k}\\ &\quad+\sum_{1\le j<k\le p-1}\frac{(-1)^j}{jk^2}+\sum_{1\le k<j\le p-1}\frac{(-1)^j}{jk^2}\\ &=H_{p-1}\sum_{j=1}^{p-1}\frac{(-1)^j}{j^2}+H_{p-1,2}\sum_{j=1}^{p-1}\frac{(-1)^j}{j}-2\sum_{j=1}^{p-1}\frac{(-1)^j}{j^3}\pmod p.\end{aligned}$$

Thus, with the help of $H_{p-1}\equiv H_{p-1,2}\equiv 0\pmod p$, we have

$$\sum_{1\le i\le j\le k\le p-1}\frac{(-1)^i}{ijk}\equiv\sum_{1\le i<j<k\le p-1}\frac{(-1)^i}{ijk}\pmod p.$$

By [ZS, Theorem 1.2],

$$\sum_{1\le i<j<k\le p-1}\frac{(1-x)^i}{ijk}\equiv\sum_{1\le i<j<k\le p-1}\frac{x^i}{ijk}\pmod p.$$

So, in view of the above, we have

$$\begin{aligned}\sum_{1\le i\le j\le k\le p-1}\frac{(-1)^i}{ijk}&\equiv\sum_{1\le i<j<k\le p-1}\frac{2^i}{ijk}\\&\equiv\sum_{1\le i\le j\le k\le p-1}\frac{2^i}{ijk}+\sum_{k=1}^{p-1}\frac{2^k}{k^3}-\sum_{1\le j\le k\le p-1}\frac{2^j(j+k)}{j^2k^2}\pmod p.\end{aligned}$$

It remains to show that

$$\sum_{1\le i\le j\le k\le p-1}\frac{2^i-(-1)^i}{ijk}\equiv\sum_{k=1}^{p-1}\frac{(-1)^k-2^k}{k^3}\pmod p. \tag{2.6}$$

With the help of Lemma 2.2, we have

$$\begin{aligned}\sum_{1\le i\le j\le k\le p-1}\frac{2^i-(-1)^i}{ijk}&=\sum_{1\le i\le j\le k\le p-1}\frac{1}{ijk}\sum_{r=0}^{i}(1-(-2)^r)\binom{i}{r}\\&=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r}\sum_{1\le j\le k\le p-1}\frac{1}{jk}\sum_{i=1}^{j}\binom{i-1}{r-1}\\&=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r}\sum_{1\le j\le k\le p-1}\frac{1}{jk}\binom{j}{r}=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r^2}\sum_{k=1}^{p-1}\frac{1}{k}\sum_{j=1}^{k}\binom{j-1}{r-1}\\&=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r^2}\sum_{k=1}^{p-1}\frac{1}{k}\binom{k}{r}=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r^3}\sum_{k=1}^{p-1}\binom{k-1}{r-1}\\&=\sum_{r=1}^{p-1}\frac{1-(-2)^r}{r^3}\binom{p-1}{r}\equiv\sum_{r=1}^{p-1}\frac{(-1)^r-2^r}{r^3}\pmod p.\ \blacksquare\end{aligned}$$

Proof of Theorem 1.1. We prove (1.2) first. In view of (2.4), we have

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{H_{n,2}}{n2^n}&=\sum_{n=1}^{p-1}\frac{1}{n2^n}\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}H_k\\&=\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}H_k\sum_{n=k}^{p-1}\frac{1}{n2^n}\binom{n}{k}=\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^2 2^k}H_k\sum_{n=k}^{p-1}\binom{n-1}{k-1}\frac{1}{2^{n-k}}\end{aligned}$$

$$= \sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^2 2^k}H_k\sum_{j=0}^{p-1-k}\binom{k+j-1}{j}\frac{1}{2^j}$$

$$= \sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^2 2^k}H_k\sum_{j=0}^{p-1-k}\binom{-k}{j}\frac{1}{(-2)^j}$$

and hence

$$\sum_{n=1}^{p-1}\frac{H_{n,2}}{n2^n} \equiv \sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^2 2^k}H_k\sum_{j=0}^{p-1-k}\binom{p-k}{j}\frac{1}{(-2)^j}$$

$$= \sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^2 2^k}H_k\frac{1+(-1)^k}{2^{p-k}} \equiv -\frac{1}{2}\sum_{k=1}^{p-1}\frac{H_k}{k^2}(1+(-1)^k) \pmod p.$$

Note that

$$\sum_{k=1}^{p-1}\frac{H_k}{k^2} \equiv B_{p-3} \pmod p \quad \text{and} \quad \sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}H_k \equiv -\frac{B_{p-3}}{4} \pmod p$$

by [ST, (5.4)] and (2.2) respectively. So we get

$$\sum_{n=1}^{p-1}\frac{H_{n,2}}{n2^n} \equiv -\frac{1}{2}\left(B_{p-3} - \frac{B_{p-3}}{4}\right) = -\frac{3}{8}B_{p-3} \pmod p.$$

Now we show (1.1). Observe that

$$\sum_{k=1}^{p-1}\frac{H_k}{k2^k} = \sum_{1\le j\le k\le p-1}\frac{1}{jk2^k} = \sum_{1\le j\le k\le p-1}\frac{1}{(p-k)(p-j)2^{p-j}}$$

$$= \sum_{1\le j\le k\le p-1}\frac{2^{j-p}(p+j)(p+k)}{(p^2-j^2)(p^2-k^2)} \equiv \sum_{1\le j\le k\le p-1}\frac{2^{j-p}(jk+p(j+k))}{j^2k^2}$$

$$\equiv 2^{-p}\sum_{1\le j\le k\le p-1}\frac{2^j}{jk} + \frac{p}{2}\sum_{1\le j\le k\le p-1}\frac{2^j(j+k)}{j^2k^2} \pmod{p^2}.$$

In view of Lemmas 2.2 and 2.1,

$$\sum_{1\le j\le k\le p-1}\frac{2^j-1}{jk} = \sum_{1\le j\le k\le p-1}\frac{1}{jk}\sum_{i=1}^{j}\binom{j}{i} = \sum_{i=1}^{p-1}\frac{1}{i}\sum_{k=1}^{p-1}\frac{1}{k}\sum_{j=1}^{k}\binom{j-1}{i-1}$$

$$= \sum_{i=1}^{p-1}\frac{1}{i}\sum_{k=1}^{p-1}\frac{1}{k}\binom{k}{i} = \sum_{i=1}^{p-1}\frac{1}{i^2}\sum_{k=1}^{p-1}\binom{k-1}{i-1}$$

$$= \sum_{i=1}^{p-1}\frac{1}{i^2}\binom{p-1}{i} = \sum_{i=1}^{p-1}\frac{(-1)^i}{i^2}\prod_{r=1}^{i}\left(1-\frac{p}{r}\right)$$

$$\equiv \sum_{i=1}^{p-1}\frac{(-1)^i(1-pH_i)}{i^2} \equiv \frac{p}{2}B_{p-3} - p\left(-\frac{B_{p-3}}{4}\right) \pmod{p^2}.$$

Note that
$$\sum_{1\le j\le k\le p-1}\frac{1}{jk}=\sum_{k=1}^{p-1}\frac{H_k}{k}\equiv\frac{p}{3}B_{p-3}\ (\mathrm{mod}\,p^2)$$
by (2.2). Combining the above with (2.5), we finally obtain
$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_k}{k2^k}&\equiv 2^{-p}\left(\frac{3}{4}pB_{p-3}+\frac{p}{3}B_{p-3}\right)+\frac{p}{2}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\\&\equiv\frac{13}{24}pB_{p-3}+\frac{p}{2}\left(-\frac{B_{p-3}}{2}\right)=\frac{7}{24}pB_{p-3}\ (\mathrm{mod}\,p^2)\quad(\text{by (2.1)}).\ \blacksquare\end{aligned}$$

3. Proof of Theorem 1.2

LEMMA 3.1. *Let $p>3$ be a prime and let m be a positive integer with $p-1\nmid 3m$. Then*
$$\sum_{1\le j<k\le p-1}\left(\frac{1}{j^mk^{2m}}+\frac{1}{j^{2m}k^m}\right)\equiv 0\ (\mathrm{mod}\,p).\tag{3.1}$$
Moreover, if $p>3m+1$, then
$$\sum_{1\le j<k\le p-1}\left(\frac{1}{j^mk^{2m}}+\frac{1}{j^{2m}k^m}\right)\equiv -p\frac{3m}{3m+1}B_{p-1-3m}\ (\mathrm{mod}\,p^2).\tag{3.2}$$

Proof. It is well known that
$$\sum_{k=1}^{p-1}\frac{1}{k^n}\equiv 0\ (\mathrm{mod}\,p)\quad\text{for any integer } n\not\equiv 0\ (\mathrm{mod}\,p-1).$$
Also,
$$\sum_{k=1}^{p-1}\frac{1}{k^n}\equiv\frac{pn}{n+1}B_{p-1-n}\ (\mathrm{mod}\,p^2)\quad\text{for } n=1,\dots,p-2$$
(see, e.g., [S, Corollary 5.1]). Thus
$$\sum_{1\le j<k\le p-1}\left(\frac{1}{j^mk^{2m}}+\frac{1}{j^{2m}k^m}\right)=\sum_{j=1}^{p-1}\frac{1}{j^m}\sum_{k=1}^{p-1}\frac{1}{k^{2m}}-\sum_{k=1}^{p-1}\frac{1}{k^{3m}}\equiv 0\ (\mathrm{mod}\,p).$$
Moreover, we have (3.2) if $p>3m+1$. $\blacksquare$

LEMMA 3.2. *Let $p>3$ be a prime and let m be a positive even integer. Then*
$$\sum_{1\le j<k\le p-1}\left(\frac{1}{j^mk^{2m}}-\frac{1}{j^{2m}k^m}\right)\equiv 0\ (\mathrm{mod}\,p).\tag{3.3}$$
Moreover, if $p>3m+1$ then
$$\sum_{1\le j<k\le p-1}\left(\frac{1}{j^mk^{2m}}-\frac{1}{j^{2m}k^m}\right)\equiv\frac{pm\binom{3m}{m}B_{p-1-3m}}{(m+1)(2m+1)}\ (\mathrm{mod}\,p^2).\tag{3.4}$$

Proof. As m is even, we have

$$\sum_{1\le j<k\le p-1}\frac{1}{j^m k^{2m}}=\sum_{1\le j<k\le p-1}\frac{1}{(p-k)^m(p-j)^{2m}}\equiv\sum_{1\le j<k\le p-1}\frac{1}{j^{2m}k^m}\pmod p.$$

Now suppose that $p>3m+1$. Then

$$\begin{aligned}\sum_{1\le j<k\le p-1}\frac{1}{j^m k^{2m}}&=\sum_{1\le j<k\le p-1}\frac{(p+k)^m(p+j)^{2m}}{(p^2-k^2)^m(p^2-j^2)^{2m}}\\ &\equiv\sum_{1\le j<k\le p-1}\frac{(k^m+pmk^{m-1})(j^{2m}+p2mj^{2m-1})}{j^{4m}k^{2m}}\\ &\equiv\sum_{1\le j<k\le p-1}\frac{1}{j^{2m}k^m}+pm\sum_{1\le j<k\le p-1}\left(\frac{1}{j^{2m}k^{m+1}}+\frac{2}{j^{2m+1}k^m}\right)\pmod{p^2}.\end{aligned}$$

So, (3.4) is reduced to

$$\sum_{1\le j<k\le p-1}\left(\frac{1}{j^{2m}k^{m+1}}+\frac{2}{j^{2m+1}k^m}\right)\equiv\frac{\binom{3m}{m}B_{p-1-3m}}{(m+1)(2m+1)}\pmod p. \tag{3.5}$$

Recall that for any integer n we have

$$\sum_{k=1}^{p-1}k^n\equiv\begin{cases}p-1\pmod p & \text{if } p-1\mid n,\\ 0\pmod p & \text{if } p-1\nmid n\end{cases}$$

(see, e.g., [IR, p. 235]). Also,

$$\sum_{j=0}^{k-1}j^n=\frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}B_j k^{n+1-j}$$

for any $k=1,2,\dots$ and $n=0,1,\dots$ (see, e.g., [IR, p. 230]). Therefore

$$\begin{aligned}\sum_{1\le j<k\le p-1}\frac{1}{j^{2m}k^{m+1}}&\equiv\sum_{k=1}^{p-1}\frac{1}{k^{m+1}}\sum_{j=0}^{k-1}j^{p-1-2m}\\ &=\sum_{k=1}^{p-1}\frac{1}{k^{m+1}(p-2m)}\sum_{j=0}^{p-1-2m}\binom{p-2m}{j}B_jk^{p-2m-j}\\ &\equiv-\frac{1}{2m}\sum_{j=0}^{p-1-2m}\binom{p-2m}{j}B_j\sum_{k=1}^{p-1}k^{p-1-3m-j}\\ &\equiv\frac{1}{2m}\sum_{j=0,\,p-1\mid j+3m}^{p-1-2m}\binom{p-2m}{j}B_j=\frac{1}{2m}\binom{p-2m}{m+1}B_{p-1-3m}\\ &\equiv\frac{1}{2m}\binom{-2m}{m+1}B_{p-1-3m}=\frac{(-1)^{m+1}}{2m}\binom{3m}{m+1}B_{p-1-3m}\pmod p.\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{1\le j<k\le p-1}\frac{1}{j^{2m+1}k^m} &\equiv \sum_{k=1}^{p-1}\frac{1}{k^m}\sum_{j=0}^{k-1}j^{p-2-2m}\\
&=\sum_{k=1}^{p-1}\frac{1}{k^m(p-1-2m)}\sum_{j=0}^{p-2-2m}\binom{p-1-2m}{j}B_jk^{p-1-2m-j}\\
&\equiv -\frac{1}{2m+1}\sum_{j=0}^{p-2-2m}\binom{p-1-2m}{j}B_j\sum_{k=1}^{p-1}k^{p-1-3m-j}\\
&\equiv \frac{1}{2m+1}\sum_{j=0,\,p-1\mid j+3m}^{p-2-2m}\binom{p-1-2m}{j}B_j = \frac{1}{2m+1}\binom{p-1-2m}{m}B_{p-1-3m}\\
&\equiv \frac{1}{2m+1}\binom{-1-2m}{m}B_{p-1-3m} = \frac{(-1)^m}{2m+1}\binom{3m}{m}B_{p-1-3m} \pmod p.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{1\le j<k\le p-1}\left(\frac{1}{j^{2m}k^{m+1}}+\frac{2}{j^{2m+1}k^m}\right)\\
&\qquad\equiv\left(\frac{(-1)^{m+1}}{2m}\binom{3m}{m+1}+2\frac{(-1)^m}{2m+1}\binom{3m}{m}\right)B_{p-1-3m}\\
&\qquad=\frac{(-1)^m}{(m+1)(2m+1)}\binom{3m}{m}B_{p-1-3m} \pmod p.
\end{aligned}$$

So (3.5) holds as m is even. ∎

Proof of Theorem 1.2. Let $m=2n$. Clearly

$$\begin{aligned}
\sum_{k=1}^{p-1}\frac{H_{k,m}^2}{k^m} &= \sum_{k=1}^{p-1}\frac{1}{k^m}\left(\sum_{j=1}^{k}\frac{1}{j^m}\right)^2 = \sum_{k=1}^{p-1}\frac{1}{k^m}\left(\sum_{j=1}^{k}\frac{1}{j^{2m}}+2\sum_{1\le i<j\le k}\frac{1}{i^mj^m}\right)\\
&= H_{p-1,3m}+\sum_{1\le j<k\le p-1}\frac{1}{j^{2m}k^m}+2\sum_{1\le i<j\le p-1}\frac{1}{i^mj^{2m}}\\
&\quad +2\sum_{1\le i<j<k\le p-1}\frac{1}{i^mj^mk^m}
\end{aligned}$$

and

$$\begin{aligned}
H_{p-1,m}^3 &= \sum_{i=1}^{p-1}\frac{1}{i^m}\left(\sum_{k=1}^{p-1}\frac{1}{k^{2m}}+2\sum_{1\le j<k\le p-1}\frac{1}{j^mk^m}\right)\\
&= H_{p-1,3m}+3\sum_{1\le j<k\le p-1}\left(\frac{1}{j^{2m}k^m}+\frac{1}{j^mk^{2m}}\right)+6\sum_{1\le i<j<k\le p-1}\frac{1}{i^mj^mk^m}.
\end{aligned}$$

As $H_{p-1,m} \equiv 0 \pmod p$, from the above we obtain

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_{k,m}^2}{k^m} &\equiv H_{p-1,3m} + \sum_{1\le j<k\le p-1}\left(\frac{1}{j^{2m}k^m}+\frac{2}{j^m k^{2m}}\right)\\ &\quad - \frac{H_{p-1,3m}}{3} - \sum_{1\le j<k\le p-1}\left(\frac{1}{j^{2m}k^m}+\frac{1}{j^m k^{2m}}\right)\\ &= \frac{2}{3}H_{p-1,3m} + \sum_{1\le j<k\le p-1}\frac{1}{j^m k^{2m}} \pmod{p^2}.\end{aligned}$$

Thus, by (3.1), (3.3) and the congruence $H_{p-1,3m} \equiv 0 \pmod p$, we immediately get (1.3).

Now assume that $p > 3m+1$. Adding (3.2) and (3.4) we obtain

$$\begin{aligned}2\sum_{1\le j<k\le p-1}\frac{1}{j^m k^{2m}} &\equiv pmB_{p-1-3m}\left(-\frac{3}{3m+1}+\frac{\binom{3m}{m}}{(m+1)(2m+1)}\right)\\ &= \frac{pm}{3m+1}\left(\frac{\binom{3m+1}{m}}{m+1}-3\right)B_{p-1-3m} \pmod{p^2}.\end{aligned}$$

Note also that

$$H_{p-1-3m} \equiv p\frac{3m}{3m+1}B_{p-1-3m} \pmod{p^2}.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_{k,m}^2}{k^m} &\equiv \frac{2}{3}\cdot p\frac{3m}{3m+1}B_{p-1-3m} + \left(\frac{\binom{3m+1}{m}}{m+1}-3\right)\frac{pm/2}{3m+1}B_{p-1-3m}\\ &= \left(\frac{\binom{3m+1}{m}}{m+1}+1\right)\frac{pm/2}{3m+1}B_{p-1-3m}\\ &= \left(\binom{3m+1}{m-1}+\frac{m}{2}\right)\frac{pB_{p-1-3m}}{3m+1} \pmod{p^2}.\end{aligned}$$

This proves (1.4). ∎

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