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WHEN DOES THE KATĚTOV ORDER IMPLY THAT ONE IDEAL EXTENDS THE OTHER?

BY

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Abstract. We consider the Katětov order between ideals of subsets of natural numbers ("$\leq_K$") and its stronger variant—containing an isomorphic ideal ("$\sqsubseteq$"). In particular, we are interested in ideals $\mathcal{I}$ for which

$$\mathcal{I} \leq_K \mathcal{J} \ \Rightarrow\ \mathcal{I} \sqsubseteq \mathcal{J}$$

for every ideal $\mathcal{J}$. We find examples of ideals with this property and show how this property can be used to reformulate some problems known from the literature in terms of the Katětov order instead of the order "$\sqsubseteq$" (and vice versa).

1. Introduction. Given two ideals $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \leq_K \mathcal{J}$ if there exists a function $f\colon \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}$ whenever $A \in \mathcal{I}$. This preorder is called the *Katětov order* and it was introduced by Katětov in [Kat68] and [Kat72a] to investigate ideal convergence of sequences of continuous functions. (For more on this order see e.g. [Hru11].)

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \sqsubseteq \mathcal{J}$ if there exists a bijection $f\colon \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}$ whenever $A \in \mathcal{I}$. If $\mathcal{I} \sqsubseteq \mathcal{J}$ then we say that $\mathcal{J}$ *contains an ideal isomorphic to* $\mathcal{I}$. This preorder was used by Debs–Saint Raymond [DSR09] and Laczkovich–Recław [LR09] to characterize the class of all ideal limits of sequences of continuous functions.

Obviously, if $\mathcal{I} \sqsubseteq \mathcal{J}$ then $\mathcal{I} \leq_K \mathcal{J}$. In this note we are interested in the reverse implication. (The relationship between the Katětov order and the order "$\sqsubseteq$" was already considered by Solecki [Sol00, Section 2].) Let us introduce the main definition of this paper. We say that an ideal $\mathcal{I}$ *has the property* Kat ($\mathcal{I} \in \mathsf{Kat}$ for short) if for every ideal $\mathcal{J}$ the following are equivalent:

(1) $\mathcal{I} \sqsubseteq \mathcal{J}$,
(2) $\mathcal{I} \leq_K \mathcal{J}$.

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It is easy to see that the ideal Fin of all finite subsets of ω has the property Kat. It was observed in [FS12] that the ideal $\mathsf{Fin}\times\mathsf{Fin} = \{A \subseteq \omega\times\omega : (\exists N\in\omega)(\forall n > N)\,\{k : (n,k)\in A\} \text{ is finite}\}$ also has the property Kat.

In Section 3 we prove some general facts about the property Kat. In Section 4 we provide examples of ideals with and without the property Kat. For instance, we show (Example 4.1) that the ideal Fin^α has the property Kat for every $\alpha < \omega_1$. In Section 5 we show that the ideal Fin_ω, defined by the inductive limit, has the property Kat.

In Section 6 we show how the property Kat can be used to reformulate a conjecture of Debs–Saint Raymond [DSR09] and a question of Hrušák [Hru11] in terms of the Katětov order instead of the order "$\sqsubseteq$".

2. Preliminaries. Our notation and terminology follows that used in the most recent set-theoretic literature. The cardinality of the set X is denoted by $|X|$. By ω we denote the set of all natural numbers, and by ω_1 the smallest uncountable ordinal number.

An *ideal on* X is a nonempty family of subsets of a set X closed under taking finite unions and subsets of its elements. By $\mathsf{Fin}(X)$ we denote the ideal of all finite subsets of X (in case of $X=\omega$ we just write Fin). If not explicitly stated, all ideals we consider in this paper contain all finite sets ($\mathsf{Fin}(X)\subseteq\mathcal{I}$) and are defined on a countable infinite set X (hence, they can be seen as ideals on ω by identifying X with ω via a fixed bijection). We use the symbol $\mathsf{0}$ for the ideal $\{\emptyset\}$ (this ideal does not contain Fin).

An ideal $\mathcal{I}$ on ω is called *dense* if every infinite $A\subseteq\omega$ contains an infinite subset that belongs to the ideal.

2.1. Sums of ideals. Let $\mathcal{I}$ be an ideal on I, $\mathcal{J}$ be an ideal on J and $\mathcal{J}_i$ be ideals on J_i ($i\in I$).

For any family of sets $\{A_i\colon i\in I\}$ let $\sum_{i\in I} A_i$ be its disjoint union, i.e. $\sum_{i\in I} A_i = \bigcup_{i\in I}\{i\}\times A_i$.

The family $\{\sum_{i\in I} A_i\colon A_i\in\mathcal{J}_i \text{ for every } i\in I\}$ is an ideal on $\sum_{i\in I} J_i$. It is called the *sum* of the ideals $\mathcal{J}_i$ and denoted by $\sum\{\mathcal{J}_i\colon i\in I\}$.

The family $\{\sum_{i\in I} A_i\colon \{i\in I\colon A_i\notin\mathcal{J}_i\}\in\mathcal{I}\}$ is an ideal on $\sum_{i\in I} J_i$. It is called the $\mathcal{I}$-*Fubini sum* of the ideals $\mathcal{J}_i$ and denoted by $\mathcal{I}\text{-}\sum\{\mathcal{J}_i\colon i\in I\}$. Note that using this notation we may say that $\sum\{\mathcal{J}_i : i\in I\} = \mathsf{0}\text{-}\sum\{\mathcal{J}_i : i\in I\}$.

If $\mathcal{J}_i=\mathcal{J}$ for every $i\in I$ then we write $\mathcal{I}\times\mathcal{J} = \mathcal{I}\text{-}\sum\{\mathcal{J}_i : i\in I\}$.

If $\mathcal{J}_i=\mathsf{0}$ for every $i\in I$ then we write $\mathcal{I}\times\mathsf{0} = \mathcal{I}\text{-}\sum\{\mathcal{J}_i : i\in I\}$.

For more on sums and Fubini sums of ideals see e.g. [Kat68] and [DSR09] (note that in both papers the authors use the dual notion of filters instead of ideals).

2.2. Fin^α**.** The ideals Fin^α on X_α ($\alpha < \omega_1$) are defined in the following way ([Kat72b, p. 240], where the author considers the dual filters and denotes them by $\mathcal{N}^\alpha$):

(1) $\mathsf{Fin}^0 = \mathsf{0} = \{\emptyset\}$, $X_0 = \{0\}$,
(2) $\mathsf{Fin}^{\alpha+1} = \mathsf{Fin} \times \mathsf{Fin}^\alpha$, $X_{\alpha+1} = \sum_{n\in\omega} X_\alpha$,
(3) $\mathsf{Fin}^\lambda = \mathcal{I}_\lambda\text{-}\sum\{\mathsf{Fin}^\alpha : \alpha < \lambda\}$, where λ is a limit ordinal and $\mathcal{I}_\lambda$ is the ideal on λ generated by the family $\{\alpha : \alpha < \lambda\}$, $X_\lambda = \sum_{\alpha<\lambda} X_\alpha$.

3. The characterization. We will need the following lemma, which lists some immediate properties of preorders "$\sqsubseteq$" and "$\leq_K$".

LEMMA 3.1. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω (we do not need to assume that $\mathsf{Fin} \subseteq \mathcal{J}$).*

(1) $\mathcal{I} \sqsubseteq \mathsf{Fin} \Leftrightarrow \mathcal{I} = \mathsf{Fin}$.
(2) $\mathcal{I} \leq_K \mathsf{Fin} \Leftrightarrow \mathcal{I}$ *is not dense.*
(3) $\mathcal{I} \leq_K \mathcal{I} \times \mathcal{J}$.

LEMMA 3.2. *Let $\mathcal{I}$ be an ideal with the property* Kat. *Then either $\mathcal{I} = \mathsf{Fin}$ or $\mathcal{I}$ is a dense ideal.*

Proof. If $\mathcal{I}$ is not dense then, by Lemma 3.1(2), $\mathcal{I} \leq_K \mathsf{Fin}$. Since $\mathcal{I} \in \mathsf{Kat}$, we have $\mathcal{I} \sqsubseteq \mathsf{Fin}$. Thus, by Lemma 3.1(1), $\mathcal{I} = \mathsf{Fin}$. ∎

LEMMA 3.3 (see also [BNF12]). *Let $\mathcal{I}$ be a dense ideal. Then $\mathcal{I} \sqsubseteq \mathcal{J} \Leftrightarrow$ there is a* 1-1 *function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$.*

Proof. "$\Rightarrow$". Obvious.

"$\Leftarrow$". Let $A \in \mathcal{I}$ be an infinite set with $A \subseteq f[\omega]$. Let $B = f^{-1}[A] \in \mathcal{J}$. Let $g\colon B \to A \cup (\omega \setminus f[\omega])$ be a bijection. Then $h = (f\restriction(\omega \setminus B)) \cup g$ shows that $\mathcal{I} \sqsubseteq \mathcal{J}$. ∎

THEOREM 3.4. *Let $\mathcal{I}$ be a dense ideal. Then the following conditions are equivalent:*

(i) $\mathcal{I} \in \mathsf{Kat}$;
(ii) $\mathcal{I} \sqsubseteq \mathcal{I} \times \mathsf{0}$.

Proof. (i)$\Rightarrow$(ii). Let $\mathcal{I}$ be an ideal with the property Kat. By Lemma 3.1(3), $\mathcal{I} \leq_K \mathcal{I} \times \mathsf{0}$, so $\mathcal{I} \sqsubseteq \mathcal{I} \times \mathsf{0}$.

(ii)$\Rightarrow$(i). Let $\mathcal{I}$ be a dense ideal such that $\mathcal{I} \sqsubseteq \mathcal{I} \times \mathsf{0}$. Let $\phi : \omega \times \omega \to \omega$ be a bijection such that $\phi^{-1}[A] \in \mathcal{I} \times \mathsf{0}$ for every $A \in \mathcal{I}$, i.e.

$$(\star) \qquad \{n \in \omega\colon \exists_{k\in\omega}\ \phi(n,k) \in A\} \in \mathcal{I} \quad \text{whenever } A \in \mathcal{I}.$$

Let $\mathcal{K}$ be an ideal such that $\mathcal{I} \leq_K \mathcal{K}$. Let $f : \omega \to \omega$ be such that

$$(\star\star) \qquad f^{-1}[A] \in \mathcal{K} \quad \text{for every } A \in \mathcal{I}.$$

We define $g : \omega \to \omega$ by

$$g(m) = \phi(n,k) \quad \text{iff} \quad m \text{ is the } k\text{th element of the set } f^{-1}(n).$$

From the above definition it follows that for each $A \subseteq \omega$,

$$\begin{aligned} g(m) \in A \ &\Rightarrow\ (\exists_{k\in\omega}\ \phi(n,k) \in A \text{ and } f(m) = n) \\ &\Leftrightarrow\ f(m) \in \{n \in \omega\colon \exists_{k\in\omega}\ \phi(n,k) \in A\}. \end{aligned}$$

Fix any $A \in \mathcal{I}$. Since $\phi^{-1}[A] \in \mathcal{I} \times \mathsf{0}$, from the above implication and conditions $(\star)$, $(\star\star)$, it follows that

$$g^{-1}[A] \subseteq f^{-1}[\{n \in \omega\colon \exists_{k\in\omega}\ \phi(n,k) \in A\}] \in \mathcal{K}.$$

Since g is 1-1 and $\mathcal{I}$ is dense, Lemma 3.3 yields $\mathcal{I} \sqsubseteq \mathcal{K}$. ∎

Remark. The assumption that $\mathcal{I}$ is dense, in the above proposition, is necessary. Indeed, let $\mathcal{I} = \mathsf{Fin} \times \mathsf{0}$. (Recall that $\mathsf{Fin} \times \mathsf{0} \neq \mathsf{Fin}^1$.) Since $\mathcal{I}$ is not dense and contains an infinite set, Lemma 3.2 implies that $\mathcal{I} \notin \mathsf{Kat}$. On the other hand, it is not difficult to check that $\mathcal{I} \sqsubseteq \mathcal{I} \times \mathsf{0}$.

Theorem 3.4 gives us a characterization of the property Kat; however, for some ideals $\mathcal{I}$ it can be difficult to prove that $\mathcal{I} \not\sqsubseteq \mathcal{I} \times \mathsf{0}$. Below we give some necessary and some sufficient conditions for ideals to have the Kat property.

3.1. Local Q-ideals. In [BTW82], the authors introduced the following definition. An ideal $\mathcal{I}$ on ω is called a *local Q-ideal* if for every partition $F_0, F_1, \dots$ of ω into finite sets there exists $S \notin \mathcal{I}$ with $|A_n \cap S| \leq 1$ for every $n \in \omega$. The next lemma follows immediately from the definition of a local Q-point.

Lemma 3.5.

(1) *If $\mathcal{I} \sqsubseteq \mathcal{J}$ and $\mathcal{J}$ is a local Q-ideal, then $\mathcal{I}$ is a local Q-ideal.*
(2) *$\mathcal{I} \times \mathsf{0}$ is a local Q-ideal for every ideal $\mathcal{I}$.*

Proposition 3.6. *Let $\mathcal{I}$ be an ideal with the property* Kat. *Then $\mathcal{I}$ is a local Q-ideal.*

Proof. By Theorem 3.4, $\mathcal{I} \sqsubseteq \mathcal{I} \times \mathsf{0}$. By Lemma 3.5(2), $\mathcal{I} \times \mathsf{0}$ is a local Q-ideal, so, by Lemma 3.5(1), $\mathcal{I}$ is a local Q-ideal as well. ∎

Remark. The condition from the above proposition is not sufficient. Indeed, under the Continuum Hypothesis there is a maximal local Q-ideal $\mathcal{I}$ (the dual ultrafilter $\mathcal{I}^\star$ is called a Q-point) and by Example 4.2 below, $\mathcal{I} \notin \mathsf{Kat}$.

3.2. Analytic P-ideals. An ideal $\mathcal{I}$ is a *P-ideal* if for every sequence $(A_n)_{n\in\omega}$ of sets from $\mathcal{I}$ there is an $A \in \mathcal{I}$ such that $A_n \setminus A \in \mathsf{Fin}$ for all n, i.e. A_n is almost contained in A for each n.

By identifying sets of naturals with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the Cantor-space topology and therefore we can assign topological complexity to ideals of sets of integers. In particular, an ideal $\mathcal{I}$ is F_σ (resp. analytic) if it is an F_σ subset of the Cantor space (resp. a continuous image of a G_δ subset of the Cantor space). For example the ideal $\mathcal{I}_d$ of asymptotic density 0 sets is an analytic P-ideal (for more examples see e.g. [Far00]).

PROPOSITION 3.7. *Assume that $\mathcal{I}$ is an analytic P-ideal. Then $\mathcal{I} \in \mathsf{Kat}$ if and only if $\mathcal{I} = \mathsf{Fin}$.*

Proof. Suppose that an analytic P-ideal $\mathcal{I}$ has the property Kat and $\mathcal{I} \neq \mathsf{Fin}$. By Lemma 3.2 and Proposition 3.6, $\mathcal{I}$ is a dense local Q-ideal. On the other hand, by [FMRS11, Lemma 3.7], an analytic P-ideal which is a local Q-ideal is not dense, a contradiction. ∎

3.3. Sums of ideals. In this subsection we assume that $\mathcal{I}$ is an ideal on I, $\mathcal{J}$ is an ideal on J, and $\mathcal{J}_i$ are ideals on J_i ($i \in I$). The following facts are easy to prove.

FACT 3.8. *Suppose that $\mathcal{I} \leq_K \mathcal{J}$ and $f\colon \omega \to \omega$ witnesses this fact. If $\mathcal{J}$ contains an infinite set then we can assume that f is onto.* (*Else take an infinite set $A \in \mathcal{J}$ and put $f' = (f{\restriction}(\omega \setminus A)) \cup g$ where $g\colon A \to \omega$ is onto.*)

FACT 3.9.

(1) *$\mathcal{I}$-$\sum\{\mathcal{J}_i\colon i \in I\}$ is a dense ideal for any ideal $\mathcal{I}$.*
(2) *If $\mathcal{J}_i$ are dense for all $i \in I$ then $\sum\{\mathcal{J}_i\colon i \in I\}$ is a dense ideal.*

PROPOSITION 3.10. *Assume that $\mathcal{J}_i \in \mathsf{Kat}$ for all $i \in I$.*

(1) *$\mathcal{I}$-$\sum\{\mathcal{J}_i\colon i \in I\} \in \mathsf{Kat}$ for any ideal $\mathcal{I}$.*
(2) *If $\mathcal{J}_i$ are dense for all $i \in I$ then $\sum\{\mathcal{J}_i\colon i \in I\} \in \mathsf{Kat}$.*

Proof. We will prove (1) and (2) simultaneously.

Let $\mathcal{J} = \mathcal{I}$-$\sum\{\mathcal{J}_i\colon i \in I\}$ ($\mathcal{J} = \sum\{\mathcal{J}_i\colon i \in I\}$ in case (2)) and $J = \sum_{i\in I} J_i$. Let $\mathcal{K}$ be an ideal on ω such that $\mathcal{J} \leq_K \mathcal{K}$. Take a function $f\colon \omega \to J$ such that $f^{-1}[A] \in \mathcal{K}$ for any $A \in \mathcal{J}$. By Fact 3.9, $\mathcal{J}$ is a dense ideal, so, by Lemma 3.1(2), $\mathcal{K} \neq \mathsf{Fin}$ (hence there are infinite sets in $\mathcal{K}$). Thus, by Fact 3.8 we can assume that f is onto.

Define $B_i = f^{-1}[\{i\} \times J_i]$ ($i \in I$). Since f is onto, the sets B_i are infinite. Let $\mathcal{J}_i^{-1}$ be an ideal on B_i generated by the family

$$\{f^{-1}[\{i\} \times A]\colon A \in \mathcal{J}_i\}.$$

Then the function $f_i\colon B_i \to J_i$ given by

$$f_i(k) = a \iff f(k) = (i, a)$$

shows that $\mathcal{J}_i \leq_K \mathcal{J}_i^{-1}$. Since $\mathcal{J}_i \in \mathsf{Kat}$, there exists a bijection $g_i \colon B_i \to J_i$ such that $g_i^{-1}[A] \in \mathcal{J}_i^{-1}$ for any $A \in \mathcal{J}_i$.

Let $g \colon \omega \to J$ be given by

$$g(k) = (i, g_i(k)) \iff k \in B_i.$$

Then g is a bijection. Take any set $A \in \mathcal{J}$. Then there exists a set $C \in \mathcal{I}$ ($C = \emptyset$ in case (2)) such that $A_{(i)} = \{a \in J_i \colon (i,a) \in A\} \in \mathcal{J}_i$ for every $i \in I \setminus C$. Then $g^{-1}[\{i\} \times A_{(i)}] \in \mathcal{J}_i^{-1}$ for every $i \in I \setminus C$. Hence for any $i \in I \setminus C$ there exists a set $D_i \in \mathcal{J}_i$ such that $g^{-1}[\{i\} \times A_{(i)}] \subseteq f^{-1}[\{i\} \times D_i]$. Let

$$D = \bigcup_{i \in I \setminus C} \{i\} \times D_i \cup \bigcup_{i \in C} \{i\} \times J_i.$$

Then $D \in \mathcal{J}$ and $g^{-1}[A] \subseteq f^{-1}[D]$. Hence $g^{-1}[A] \in \mathcal{K}$. So $\mathcal{J} \sqsubseteq \mathcal{K}$. ∎

Remark. The assumption that the $\mathcal{J}_i$ are dense, in (2) of the above proposition, is necessary. Indeed, let $\mathcal{I}_1 = \mathsf{Fin}$ and $\mathcal{I}_2$ be a dense ideal with the property Kat (see Sections 4 and 5 for examples). Then the ideal $\mathcal{J} = \sum_{i \in \{1,2\}} \mathcal{I}_i$ is not dense and contains an infinite set. So, by Lemma 3.2, $\mathcal{J} \notin \mathsf{Kat}$.

Corollary 3.11. *If $\mathcal{J} \in \mathsf{Kat}$ then $\mathcal{I} \times \mathcal{J} \in \mathsf{Kat}$ for any ideal $\mathcal{I}$.*

4. Simple examples

Example 4.1. $\mathsf{Fin}^\alpha \in \mathsf{Kat}$ for each $\alpha < \omega_1$.

Proof. By transfinite induction based on Proposition 3.10. ∎

Example 4.2. No maximal ideal has the property Kat.

Proof. Let $\mathcal{I}$ be a maximal ideal. If $\mathcal{I} \in \mathsf{Kat}$, then by Theorem 3.4, $\mathcal{I} \sqsubseteq \mathcal{I} \times 0$, and thus $\mathcal{I} \times 0$ would be a maximal ideal, a contradiction. ∎

By Proposition 3.7 no analytic P-ideal but Fin has the property Kat. The assumption that $\mathcal{I}$ is a P-ideal is necessary in Proposition 3.7, as shown by

Example 4.3. The ideal

$$\mathcal{ED} = \{A \subseteq \omega \times \omega \colon (\exists_{n,m \in \omega})(\forall_{k>n})\ |\{i \in \omega \colon (k,i) \in A\}| \leq m\}$$

is a dense F_σ ideal with the property Kat.

Proof. First of all it is not difficult to show that $\mathcal{ED}$ is a dense F_σ ideal (see e.g. [MA09, p. 12]).

Now we show that $\mathcal{ED} \in \mathsf{Kat}$. Let $\mathcal{I}$ be an ideal such that $\mathcal{ED} \leq_K \mathcal{I}$. In [MA09, p. 57], the author noticed that

$$\mathcal{ED} \leq_K \mathcal{I} \iff \mathcal{I} \text{ is not locally selective}$$

(recall that an ideal $\mathcal{I}$ is *locally selective* if for every partition $A_0, A_1, \ldots \in \mathcal{I}$ of ω there is a selector $S \notin \mathcal{I}$).

Let $A_0, A_1, \ldots \in \mathcal{I}$ be a partition of ω without a selector $S \notin \mathcal{I}$. Let $f : \omega \to \omega \times \omega$ be a one-to-one function such that $f[A_n] \subseteq \{n\} \times \omega$. Then $f^{-1}[A] \in \mathcal{I}$ for every $A \in \mathcal{ED}$. Thus, by Lemma 3.3, $\mathcal{ED} \sqsubseteq \mathcal{I}$. ∎

The proof of the above example gives

$$\mathcal{ED} \sqsubseteq \mathcal{I} \iff \mathcal{I} \text{ is not locally selective.}$$

We give more applications of the Kat property in Section 6.

By conv we denote the ideal of all subsets of $\mathbb{Q}_{\mathbb{I}} = \mathbb{Q} \cap [0,1]$ which have only finitely many cluster points, and by nwd the ideal of all nowhere dense subsets of $\mathbb{Q}_{\mathbb{I}}$ (with euclidean metric).

EXAMPLE 4.4. $\{\mathsf{conv}, \mathsf{nwd}\} \subseteq \mathsf{Kat}$.

Proof. Let $\mathcal{I} = \mathsf{conv}$ ($\mathcal{I} = \mathsf{nwd}$, respectively). By Theorem 3.4 it is enough to show that $\mathcal{I} \sqsubseteq \mathcal{I} \times \mathsf{0}$. Thus, by Lemma 3.3, it is enough to find a 1-1 function $f : \mathbb{Q}_{\mathbb{I}} \times \omega \to \mathbb{Q}_{\mathbb{I}}$ such that $f^{-1}[A] \in \mathcal{I} \times \mathsf{0}$ for every $A \in \mathcal{I}$.

Let $\mathbb{Q}_{\mathbb{I}} = \{q_n : n \in \omega\} = \bigcup_{k\in\omega} Q_k$, where $Q_k = \{q_n^k : n \in \omega\}$ are pairwise disjoint such that $|q_n^k - q_n| < 1/(n + k + 1)$ for each $n, k \in \omega$. We define an injection $f : \mathbb{Q}_{\mathbb{I}} \times \omega \to \mathbb{Q}_{\mathbb{I}}$ by

$$f(q, k) = q_n^k \iff q = q_n.$$

Let $A \in \mathcal{I}$. We will show that $f^{-1}[A] \in \mathcal{I} \times \mathsf{0}$, i.e.

$$A' = \{q_n \in \mathbb{Q}_{\mathbb{I}} : q_n^k \in A \text{ for some } k \in \omega\} \in \mathcal{I}.$$

CASE 1: $\mathcal{I} = \mathsf{conv}$. It is enough to prove that if x is a cluster point of A' then it is also a cluster point of A. Let $x = \lim_{i\to\infty} q_{n(i)}$ ($q_{n(i)} \in A'$ for each i). Without loss of generality we may assume that $n(i)$ is increasing to infinity. For each i there exists $k(i)$ such that $q_{n(i)}^{k(i)} \in A$. Since

$$|q_{n(i)}^{k(i)} - q_{n(i)}| < \frac{1}{n(i) + k(i) + 1} < \frac{1}{n(i) + 1} \xrightarrow{i\to\infty} 0,$$

we have $\lim_{i\to\infty} q_{n(i)}^{k(i)} = \lim_{i\to\infty} q_{n(i)} = x$. Thus x is also a cluster point of A.

CASE 2: $\mathcal{I} = \mathsf{nwd}$. It is enough to prove that for any nonempty open set $U \subseteq \mathbb{Q}_{\mathbb{I}} \setminus A$ there is a nonempty open set $V \subseteq U \setminus A'$. Let $B(x, \varepsilon) \subseteq U$ be an open ball with center x and radius ε.

If $B(x, \varepsilon/3) \cap A'$ is finite, then there is a nonempty open set $V \subseteq B(x, \varepsilon/3) \setminus A' \subseteq U \setminus A'$, and we are done.

Now, suppose that $B(x, \varepsilon/3) \cap A'$ is infinite. Then there is $n \in \omega$ such that $q_n \in B(x, \varepsilon/3) \cap A'$ and $1/n < \varepsilon/3$. Let $k \in \omega$ be such that $q_n^k \in A$. Then $|q_n^k - q_n| < 1/(n + k + 1) < \varepsilon/3$. Since $q_n \in B(x, \varepsilon/3)$, it follows that $q_n^k \in B(x, 2\varepsilon/3) \subseteq U$. Thus $A \cap U \neq \emptyset$, a contradiction. ∎

5. Inductive limits. In [DSR09], in Section 6 the authors consider some canonical filters (which are dual objects to ideals) $\mathcal{N}_\alpha$ for $\alpha < \omega_1$. We shall denote their dual ideals by Fin_α. In the following we recall some crucial definitions.

A *filter* is a nonempty family closed under taking finite intersections and supersets. We denote the domain of a filter $\mathcal{F}$ by $\mathrm{dom}(\mathcal{F}) = \bigcup \mathcal{F}$. Let $\mathcal{I}^*$ denote the dual filter of the ideal $\mathcal{I}$, i.e. $\mathcal{I}^* = \{(\bigcup \mathcal{I}) \setminus A\colon A \in \mathcal{I}\}$. Following [DSR09] we say that $\pi\colon \mathcal{F} \to \mathrm{dom}(\mathcal{G})$ is a *quasi-homomorphism* from the filter $\mathcal{F}$ to the filter $\mathcal{G}$ if $F \in \mathcal{F}$ and $\pi^{-1}[A] \in \mathcal{F}$ for all $A \in \mathcal{G}$. It is clear that $\mathcal{I} \leq_K \mathcal{J}$ iff there is a quasi-homomorphism from $\mathcal{J}^*$ to $\mathcal{I}^*$.

Let $(I, \leq)$ be a directed set. Following [DSR09] we say that $(\pi_{i,j})_{i,j\in I,\, i\leq j}$ is a *coherent system of quasi-homomorphisms* for a family $(\mathcal{F}_i)_{i\in I}$ of filters if for all $i \leq j \leq k$, $i, j, k \in I$, $\pi_{i,j}$ is a quasi-homomorphism from $\mathcal{F}_j$ to $\mathcal{F}_i$, and $\pi_{i,k}(a) = \pi_{i,j}(\pi_{j,k}(a))$ for all $a \in \mathrm{dom}(\pi_{i,k}) \cap \pi_{j,k}^{-1}[\mathrm{dom}(\pi_{i,j})]$. Then we say that $((\mathcal{F}_i)_{i\in I}, (\pi_{i,j})_{i,j\in I,\, i\leq j})$ is a *quasi-inductive system of filters.* The filter on $\sum_{i\in I} \mathrm{dom}(\mathcal{F}_i)$ generated by the family $\{\sum_{j\geq i} \pi_{i,j}^{-1}[A]\colon A \in \mathcal{F}_i, i \in I\}$ (as a subbase) is called the *inductive limit* of the system $((\mathcal{F}_i)_{i\in I}, (\pi_{i,j})_{i,j\in I,\, i\leq j})$ and is denoted by $\varinjlim((\mathcal{F}_i)_{i\in I}, (\pi_{i,j})_{i,j\in I,\, i\leq j})$.

It was noticed in [DSR09, p. 203] that since every Fin^α is the Fubini sum of the ideals Fin^β with $\beta < \alpha$, we can fix a coherent system of quasi-homomorphisms $(\pi_{\beta,\alpha})_{\beta\leq\alpha<\omega_1}$ for the family $\{\mathsf{Fin}^\alpha\colon \alpha < \omega_1\}$ using inductively [DSR09, Proposition 5.8] in the following way:

- $\pi_{\alpha,\alpha} = \mathrm{id}$, $\mathrm{dom}(\pi_{\alpha,\alpha}) = \mathrm{dom}(\mathsf{Fin}^\alpha)$;
- $\pi_{\beta,\lambda} = \sum_{\gamma\geq\beta} \pi_{\beta,\gamma}$, $\mathrm{dom}(\pi_{\beta,\lambda}) = \sum_{\gamma\geq\beta} \mathrm{dom}(\pi_{\beta,\gamma})$ for $\beta < \lambda$, λ a limit ordinal;
- $\pi_{\alpha,\alpha+1}(i, a) = a$ for $i \in \omega$, $a \in \mathrm{dom}(\mathsf{Fin}^\alpha)$, $\mathrm{dom}(\pi_{\alpha,\alpha+1}) = \omega \times \mathrm{dom}(\mathsf{Fin}^\alpha)$;
- $\pi_{\beta,\alpha+1} = \pi_{\beta,\alpha} \circ \pi_{\alpha,\alpha+1}$, $\mathrm{dom}(\pi_{\beta,\alpha+1}) = \pi_{\alpha,\alpha+1}^{-1}[\mathrm{dom}(\pi_{\beta,\alpha})]$ for $\beta < \alpha$.

Thus following [DSR09] we can define a family $\{\mathsf{Fin}_\alpha\colon \alpha < \omega_1\}$ as:

(1) $\mathsf{Fin}_0 = \mathsf{Fin}^0$,
(2) $\mathsf{Fin}_{\alpha+1} = \mathsf{Fin}^{1+\alpha}$,
(3) $\mathsf{Fin}_\lambda = \varinjlim((\mathsf{Fin}^\alpha)_{\alpha<\lambda}, (\pi_{\beta,\alpha})_{\beta\leq\alpha<\lambda})$ for limit ordinals λ.

It is clear that $\mathsf{Fin}_\alpha = \mathsf{Fin}^\alpha$ for $\alpha < \omega$ and $\mathsf{Fin}_{\alpha+1} = \mathsf{Fin}^\alpha$ for $\alpha \geq \omega$.

In the following proposition we give another example of the ideal with the property Kat.

PROPOSITION 5.1. $\mathsf{Fin}_\omega \in \mathsf{Kat}$.

Proof. $\mathsf{Fin}_\omega = \varinjlim((\mathsf{Fin}^n)_{n\in\omega}, (\pi_{i,j})_{i\leq j<\omega})$, where

$$\pi_{i,j}(n_1, n_2, \ldots, n_j) = (n_{j-i+1}, n_{j-i+2}, \ldots, n_j)$$

and $\mathrm{dom}(\pi_{i,j}) = \mathrm{dom}(\mathsf{Fin}^j) = \omega^j$ for $i \leq j < \omega$ and $\mathrm{dom}(\mathsf{Fin}_\omega) = \sum_{i\in\omega} \omega^i$.

We shall show that $\mathsf{Fin}_\omega \sqsubseteq \mathsf{Fin}_\omega \times \mathsf{0}$. Let $h\colon \omega^2 \to \omega$ be a fixed bijection. Define $f_i\colon \omega^{i+1} \to \omega^i$ as

$$f_i(n_1,\dots,n_{i-1},n_i,n_{i+1}) = (n_1,\dots,n_{i-1},h(n_i,n_{i+1}))$$

for $i \in \omega$. It is easy to see that f_i is a bijection.

Define $f\colon \sum_{i\in\omega}\omega^{i+1} \to \sum_{i\in\omega}\omega^i$ by $f(i,n) = (i, f_i(n))$ for $i \in \omega$, $n \in \omega^{i+1}$ and define $g\colon (\sum_{i\in\omega}\omega^i) \times \omega \to \sum_{i\in\omega}\omega^{i+1}$ by $g((i,k),n) = (i,(k,n))$ for $i,n \in \omega$, $k \in \omega^i$. Then f, g, and thus $f \circ g$ are bijections.

Take $B = \sum_{j\ge i}\pi_{i,j}^{-1}[A] = \sum_{j\ge i}\omega^{j-i} \times A$, an element of the subbase of the filter $(\mathsf{Fin}_\omega)^*$, where $A \in (\mathsf{Fin}^i)^*$ and $i \in \omega$. We shall show that $(f\circ g)^{-1}[B] \in (\mathsf{Fin}_\omega \times \mathsf{0})^*$. Firstly we have

$$f^{-1}[B] = \sum_{j\ge i} f_j^{-1}[\omega^{j-i}\times A] = \sum_{j\ge i}(\omega^{j-i}\times f_i^{-1}[A]),$$

and $f_i^{-1}[A] \in (\mathsf{Fin}^i \times \mathsf{0})^*$. Thus $f_i^{-1}[A] \supseteq C \times \omega$ for some $C \in (\mathsf{Fin}^i)^*$. Hence

$$f^{-1}[B] \supseteq \sum_{j\ge i}(\omega^{j-i}\times C\times\omega).$$

Thus

$$g^{-1}[f^{-1}[B]] \supseteq \Big(\sum_{j\ge i}\omega^{j-i}\times C\Big)\times\omega = \Big(\sum_{j\ge i}\pi_{i,j}^{-1}[C]\Big)\times\omega.$$

But $\sum_{j\ge i}\pi_{i,j}^{-1}[C] \in (\mathsf{Fin}_\omega)^*$, and hence $(\sum_{j\ge i}\pi_{i,j}^{-1}[C])\times\omega \in (\mathsf{Fin}_\omega\times\mathsf{0})^*$. Finally $g^{-1}[f^{-1}[B]] \in (\mathsf{Fin}_\omega\times\mathsf{0})^*$. As a result $\mathsf{Fin}_\omega \sqsubseteq \mathsf{Fin}_\omega\times\mathsf{0}$.

Now we shall show that Fin_ω is dense. Take an infinite set $X = \sum_{i\in\omega}X_i \subseteq \mathrm{dom}(\mathsf{Fin}_\omega)$. If $X \subseteq \sum_{i<n}\omega^i$ for some $n\in\omega$, then $X \in \mathsf{Fin}_\omega$. In the opposite case $N = \{n\colon X_n \ne \emptyset,\ n\ge 2\}$ is infinite. For each $n \in N$ fix any $x_n \in X_n$. Let $y_n = \pi_{2,n}(x_n)$ for $n\in N$. Let $M = \{y_n\colon n\in N\}$.

If M is finite, then $M \in \mathsf{Fin}^2$, and thus $\sum_{j\ge 2}\pi_{2,j}^{-1}[M] \in \mathsf{Fin}_\omega$, hence $\{(n,x_n)\colon n\in N\} \in \mathsf{Fin}_\omega$. If M is infinite, then there exists an infinite set $L \subseteq M$ such that $L \in \mathsf{Fin}^2$, since Fin^2 is dense. Consequently, $\sum_{j\ge 2}\pi_{2,j}^{-1}[L] \in \mathsf{Fin}_\omega$, hence $\{(n,x_n)\colon y_n \in L\} \in \mathsf{Fin}_\omega$. In both cases we arrived at an infinite subset of X in Fin_ω.

Finally, since Fin_ω is dense and $\mathsf{Fin}_\omega \sqsubseteq \mathsf{Fin}_\omega\times\mathsf{0}$, by Theorem 3.4 we have $\mathsf{Fin}_\omega \in \mathsf{Kat}$. ∎

Problem 5.2. *Do all ideals Fin_α, where $\alpha < \omega_1$, have the property* Kat*?*

6. Applications

6.1. The Borel separation rank of an ideal. Let X be a topological space. Let $\mathbf{\Sigma}^0_1(X)$ be the family of all open subsets of X, $\mathbf{\Pi}^0_\alpha(X)$ be the family of complements of sets from $\mathbf{\Sigma}^0_\alpha(X)$ $(1\le\alpha<\omega_1)$ and $\mathbf{\Sigma}^0_\alpha(X)$ be the family of countable unions of sets from $\bigcup_{\beta<\alpha}\mathbf{\Pi}^0_\beta(X)$ $(\alpha>1)$.

The *Borel separation rank* (rank, for short) of an ideal $\mathcal{I}$ was defined by Debs and Saint Raymond [DSR09] as the unique ordinal

$$\mathsf{rk}\,(\mathcal{I}) = \min\{\alpha < \omega_1 \colon \mathcal{I} \subseteq A \text{ and } \mathcal{I}^\star \cap A = \emptyset \text{ for some set } A \in \mathbf{\Sigma}^0_{1+\alpha}\}.$$

It is known that every analytic ideal has a countable rank ([DSR09, p. 197]). Moreover, if $\mathcal{I} \sqsubseteq \mathcal{J}$ then $\mathsf{rk}\,(\mathcal{I}) \leq \mathsf{rk}\,(\mathcal{J})$ ([DSR09, Lemma 7.2]). The same is true if we replace the order "$\sqsubseteq$" with the Katětov order "$\leq_K$" ([DSR09, Lemma 5.2]).

In [DSR09], the authors posed the following conjecture.

Conjecture 6.1 ([DSR09, Conjecture 7.8]). *Let $\mathcal{I}$ be an analytic ideal. Then $\mathsf{rk}\,(\mathcal{I}) \geq \alpha$ if and only if $\mathsf{Fin}_\alpha \sqsubseteq \mathcal{I}$.*

Note that by [DSR09, Thms. 6.5, 3.2 and Lem. 5.2], the implication "$\Leftarrow$" of the above conjecture is true. Using the fact that $\mathsf{Fin}^\alpha \in \mathsf{Kat}$ (Example 4.1) and $\mathsf{Fin}_\omega \in \mathsf{Kat}$ (Proposition 5.1) we can rewrite the above conjecture making the implication "$\Rightarrow$" easier to prove (see also Problem 5.2).

Conjecture 6.2. *Let $\mathcal{I}$ be an analytic ideal. Then $\mathsf{rk}\,(\mathcal{I}) \geq \alpha$ if and only if $\mathsf{Fin}_\alpha \leq_K \mathcal{I}$, for successor $\alpha < \omega_1$ and for $\alpha = \omega$.*

6.2. Ideal convergence. This section is motivated by the question of Hrušák.

Problem 6.3 ([Hru11, Q. 5.16]). *Let $\mathcal{J}$ be a Borel ideal. Are the following conditions equivalent?*

(i) $\mathsf{conv} \not\leq_K \mathcal{J}$*;*
(ii) *$\mathcal{J}$ can be extended to a proper F_σ ideal.*

We do not know the answer, but we are able to formulate some equivalent conditions and show that the answer is positive if $\mathcal{J}$ is a P-ideal.

We say that an ideal $\mathcal{I}$ has the *finite Bolzano–Weierstrass property* ($\mathcal{I} \in \mathsf{FinBW}$ for short) if for any bounded sequence $(x_n)_{n\in\omega}$ of reals there is $A \notin \mathcal{I}$ such that $(x_n)_{n\in A}$ is convergent. By the well-known Bolzano–Weierstrass theorem the ideal Fin has the FinBW property (for a discussion and applications of this property see [FMRS07]).

Proposition 6.4 ([MA09]). *For any ideal $\mathcal{J}$ the following conditions are equivalent:*

(i) $\mathsf{conv} \not\leq_K \mathcal{J}$*;*
(ii) $\mathsf{conv} \not\sqsubseteq \mathcal{J}$*;*
(iii) $\mathcal{J} \in \mathsf{FinBW}$*.*

Proof. Since $\mathsf{conv} \in \mathsf{Kat}$ (Example 4.4), (i) is equivalent to (ii). The equivalence (i)$\Leftrightarrow$(iii) was proved in [MA09, Section 2.7]. ■

In [FMRS07, Thm. 4.2], the authors proved that an analytic P-ideal has the property FinBW if and only if it can be extended to a proper F_σ ideal. Thus, if $\mathcal{J}$ is an analytic P-ideal then the answer to the question of Hrušák is positive.

PROPOSITION 6.5. *If $\mathcal{J}$ is an analytic P-ideal then the following conditions are equivalent:*

(i) $\mathsf{conv} \not\leq_K \mathcal{J}$;
(ii) $\mathsf{conv} \not\sqsubseteq \mathcal{J}$;
(iii) $\mathcal{J} \in \mathsf{FinBW}$;
(iv) *$\mathcal{J}$ can be extended to a proper F_σ ideal.*

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