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ON SQUARE VALUES OF THE PRODUCT OF THE EULER TOTIENT AND SUM OF DIVISORS FUNCTIONS

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Abstract. If n is a positive integer such that $\phi(n)\sigma(n) = m^2$ for some positive integer m, then $m \le n$. We put $m = n - a$ and we study the positive integers a arising in this way.

1. Introduction. It is known (e.g. [2] and [8]), and we will revisit this argument shortly, that there are infinitely many positive integers n such that $\phi(n)\sigma(n) = \square$ (1). Here, we look at such positive integers n. Clearly, $n = 1$ has the property. Suppose that $n > 1$ and write its prime factorization as

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}. \tag{1.1}$$

Then

$$\frac{\phi(n)\sigma(n)}{n^2} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^{\alpha_i+1}}\right). \tag{1.2}$$

Thus, if $n > 1$ and $\phi(n)\sigma(n) = m^2$ for some positive integer m, then $m < n$, so we can write $m = n - a$ for some positive integer a. In this paper, we look at the positive integers a arising in this way. First, we fix such a number a and study the set

$$\mathcal{N}_a := \{n : n > a \text{ and } \phi(n)\sigma(n) = (n-a)^2\}.$$

It is easy to see that each $n \in \mathcal{N}_a$ has the same parity as a. Our first result shows that $\mathcal{N}_a$ is a finite set.

THEOREM 1. *All elements n in $\mathcal{N}_a$ have $\omega(n) > 1$ and $n \le 2a^3$.*

We conjecture that Theorem 1 is best possible. Indeed, if p is prime and $2p^2 - 1$ is also prime, then for $n = p(2p^2 - 1)$, $\sigma(n)\phi(n) = (n - p)^2$ and

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(1) We use $\square$ to denote the square of a positive integer.

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$n \sim 2p^3$. It is conjectured that there are infinitely many such primes (this is a special case of Schinzel's Hypothesis H).

Next, we look at the set

$$\begin{aligned}\mathcal{A} &= \{a \geq 1 : \mathcal{N}_a \neq \emptyset\}\\ &= \{2, 3, 6, 7, 8, 9, 11, 13, 17, 19, 23, 24, 26, 28, 32, 35,\\ &\qquad 37, 40, 41, 43, 45, 47, 53, \ldots\}.\end{aligned}$$

Clearly, $\mathcal{A}$ is infinite because on the one hand there are infinitely many n such that $\phi(n)\sigma(n) = \square$, while on the other hand for each a the set $\mathcal{N}_a$ is finite by Theorem 1. Our next result gives a lower bound for $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$.

THEOREM 2. *The estimate* $\#\mathcal{A}(x) \geq x^{1/8+o(1)}$ *holds as* $x \to \infty$.

In light of the examples given above ($n = p(2p^2 - 1)$) and the Bateman–Horn conjectures [3], it is likely that $\mathcal{A}(x) \gg x/\log^2 x$.

Throughout the paper, we use the Landau symbols O and o and the Vinogradov symbols $\gg$, $\ll$ and $\asymp$ with their usual meaning. We recall that $A = O(B)$, $A \ll B$ and $B \gg A$ are all equivalent and mean that the inequality $|A| \leq cB$ holds with some positive constant c. Further, $A \asymp B$ means that both estimates $A \ll B$ and $B \ll A$ hold, while $A = o(B)$ means that $A/B \to 0$. The symbols p, q always represent primes.

2. Background on solutions of Pell-type equations. Let $d > 1$ be a positive integer which is not a square. For $k \geq 1$, let (X_k, Y_k) be the kth positive solution of the Pell equation $X^2 - dY^2 = 1$. Recall that

$$X_k + \sqrt{d}\,Y_k = (X_1 + \sqrt{d}\,Y_1)^k \quad \text{for all } k = 1, 2, \ldots.$$

We shall use some basic facts about the sequences $(X_k)_{k\geq 1}$, such as relations of the type

$$X_{m+n} = X_m X_n + dY_m Y_n \quad \text{for all positive integers } m,\ n,$$

as well as the fact that $X_m \mid X_n$ whenever $m \mid n$ and n/m is odd. We need the following easy result concerning the indices k such that X_k is an odd prime power.

LEMMA 3. *If* $X_k = p^\alpha$ *for some odd prime* p *and positive integer* α, *then* k *is a power of* 2.

Proof. Suppose that k is not a power of 2. Let $h \geq 3$ be an odd divisor of k and put $r = k/h$. Since $X_r \mid X_k$, we have $X_r = p^\beta$ for some integer $1 \leq \beta < \alpha$. From

$$X_k + \sqrt{d}\,Y_k = (X_r + \sqrt{d}\,Y_r)^h,$$

we get

$$X_k = \sum_{i=0}^{(h-1)/2} \binom{h}{2i+1} X_r^{2i+1}(X_r^2-1)^{(h-1)/2-i}. \tag{2.1}$$

In particular,

$$p^\alpha = X_k > X_r^h = (p^\beta)^h = p^{h\beta},$$

therefore $\beta < \alpha/h$. Let j be the largest integer with $p^{j\beta} \mid h$. If $j \le h-2$, we reduce equation (2.1) modulo $p^{(j+2)\beta}$. Upon observing that $j+2 \le h$, therefore $(j+2)\beta \le h\beta < \alpha$, we infer that $p^{(j+2)\beta} \mid X_k$. Thus,

$$0 \equiv \sum_{0\le i\le j/2} \binom{h}{2i+1} p^{(2i+1)\beta}(p^{2\beta}-1)^{(h-1)/2-i} \pmod{p^{(j+2)\beta}}. \tag{2.2}$$

We now show that $p^{(j+2)\beta} \mid \binom{h}{2i+1} p^{(2i+1)\beta}$ for all $1 \le i \le j/2$. Indeed, let $p^\lambda \parallel 2i+1$. Since $2i+1 \le p^{2i-1}$, it follows that $\lambda \le 2i-1$. Using Kummer's theorem concerning the power of a prime dividing a binomial coefficient and denoting by $\nu_p(m)$ the exponent of p in the factorization of m, we then have

$$\nu_p\left(\binom{h}{2i+1}\right) \ge \nu_p(h) - \nu_p(2i+1) \ge 2j\beta - \lambda,$$

so

$$\begin{aligned}(j+2)\beta &\le \nu_p\left(\binom{h}{2i+1}\right) + \lambda + 2\beta \le \nu_p\left(\binom{h}{2i+1}\right) + (2i-1) + 2\beta \\ &\le \nu_p\left(\binom{h}{2i+1} p^{(2i+1)\beta}\right).\end{aligned}$$

Thus, $p^{(j+2)\beta} \mid \binom{h}{2i+1} p^{(2i+1)\beta}$. The congruence (2.2) then implies

$$0 \equiv hp^\beta (p^{2\beta}-1)^{(h-1)/2} \pmod{p^{(j+2)\beta}},$$

which implies $p^{(j+1)\beta} \mid h$, a contradiction. Hence, $j \ge h-1$, so h is divisible by $p^{h-1} > h$, a contradiction. ∎

Let $a, b > 1$ be coprime square free integers such that the Diophantine equation

$$aU^2 - bV^2 = 1$$

has a positive integer solution (U, V). It is well-known that it then has infinitely many positive integer solutions (U, V). Further, writing (U_1, V_1) for the smallest such solution, all solutions of the above equation are of the form (U_{2j+1}, V_{2j+1}) for some $j \ge 0$, where

$$\sqrt{a}\, U_{2j+1} + \sqrt{b}\, V_{2j+1} = \gamma^{2j+1} \quad \text{where} \quad \gamma = \sqrt{a}\, U_1 + \sqrt{b}\, V_1.$$

Furthermore, if we put

$$\gamma^{2j} = U_{2j} + \sqrt{ab}\, V_{2j} \qquad \text{for } j \ge 1,$$

then the pairs $(X,Y) = (U_{2j}, V_{2j})$ for $j \geq 1$ form all the positive integer solutions of the Pell equation $X^2 - (ab)Y^2 = 1$. All these facts follow from Theorem 3 of [10].

We need the following result which is similar to Lemma 3.

LEMMA 4. *With the above notation, let $a = p$ be an odd prime and let h be an odd positive integer. If $U_h = p^\alpha$ for some $\alpha \geq 0$, then $h = 1$ or $(a,b,h) = (3,2,3)$.*

Proof. If $\alpha = 0$, then there is nothing to prove. So, assume that $\alpha > 0$ and $h > 1$. Write $h = rs$ with $1 \leq r < h$. Since $U_r \mid U_h$, it follows that $U_r = p^\beta$, where $0 \leq \beta < \alpha$. Write

$$p^\alpha = U_h = \sum_{i=0}^{(s-1)/2} \binom{s}{2i+1} U_r^{2i+1} p^i (bV_r^2)^{(s-1)/2-i}. \tag{2.3}$$

Let $p^j \parallel s$ and assume that $j < \alpha - \beta$. As in the previous proof, for $i \geq 1$ let $p^\lambda \parallel 2i+1$. Observe that $\lambda \leq i$ and in fact $\lambda \leq i-1$ except when $p = 3$ and $i = 1$. Then

$$\nu_p\left(\binom{s}{2i+1}\right) \geq \nu_p(s) - \nu_p(2i+1) = j - \lambda,$$

therefore

$$\nu_p\left(\binom{h}{2i+1} U_r^{2i+1} p^i\right) \geq j + (2i+1)\beta + i - \lambda.$$

If $\lambda \leq i-1$ or if $\beta > 0$, the right hand side above is at least $j+1+\beta$. Thus, in (2.3) all terms with $i \geq 1$ are divisible by $p^{j+1+\beta}$. This implies

$$0 \equiv sp^\beta (bV_1^2)^{(s-1)/2} \pmod{p^{j+1+\beta}},$$

so $p^{j+1} \mid s$, a contradiction. Thus, we have $j \geq \alpha - \beta$ and hence $U_h/U_r \mid s$. This is impossible, as (2.3) implies

$$\frac{U_h}{U_r} > p^{(s-1)/2} \geq s.$$

It remains to treat the exceptional case $i = 1$, $\beta = 0$, $p = 3$ for which $U_1 = 1$, $b = 2$, $V_1 = 1$. Note that in this case $U_3 = 9 = 3^2$. No other odd numbers h give $U_h = 3^\alpha$, however. To see this, apply (2.3) with $r = 1$, $s = h$ and deduce that $3 \mid h$. If $h > 3$, we apply the above argument with $r = 3$, $s = h/3$ and $\beta = 2$, and deduce a contradiction as before. ∎

The proofs of Lemmas 3 and 4 can be simplified by invoking the Primitive Divisor Theorem for Lucas and Lehmer sequences (see [5], [11] and [4]). We gave the current proofs in order to make the proof of Theorem 1 self-contained.

3. The proof of Theorem 1. Suppose that $n \in \mathcal{N}_a$, let $k = \omega(n)$ and factor n canonically as $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. If $k = 1$, then $n = p_1^{\alpha_1}$ and

$$\phi(n)\sigma(n) = p_1^{\alpha_1-1}(p_1^{\alpha_1+1} - 1) = \square.$$

Since the two factors $p_1^{\alpha_1+1} - 1$ and $p_1^{\alpha_1-1}$ are coprime and their product is a square, it follows that each of them is a square. So, $\alpha_1 - 1 = 2\beta_1$ is even, and $p_1^{\alpha_1+1} - 1 = p_1^{2\beta_1+2} - 1 = \square$, which is impossible because there are no two consecutive perfect squares. Hence, $k \geq 2$.

We apply the AGM-inequality to the right side of (1.2) to get

$$\left(1 - \frac{1}{k}\sum_{i=1}^{k}\frac{1}{p_i^{\alpha_i+1}}\right)^2 \geq \left(1 - \frac{1}{k}\sum_{i=1}^{k}\frac{1}{p_i^{\alpha_i+1}}\right)^k \geq \prod_{i=1}^{k}\left(1 - \frac{1}{p_i^{\alpha_i+1}}\right)$$
$$= \frac{\sigma(n)\phi(n)}{n^2} = \left(1 - \frac{a}{n}\right)^2.$$

Taking square roots and rearranging gives

$$ak \geq n\sum_{i=1}^{k}\frac{1}{p_i^{\alpha_i+1}}. \tag{3.1}$$

Applying again the AGM-inequality to the right-hand side of (3.1), we get

$$ak \geq kn\prod_{i=1}^{k} p_i^{-(\alpha_i+1)/k} = k\prod_{i=1}^{k} p_i^{\alpha_i-(\alpha_i+1)/k}.$$

If $k \geq 3$, then since $\alpha_i - (\alpha_i + 1)/k \geq \alpha_i - (\alpha_i + 1)/3 = (2\alpha_i - 1)/3 \geq \alpha_i/3$ for all $i = 1, \ldots, k$, we get

$$a \geq \prod_{i=1}^{k} p_i^{\alpha_i/3} = n^{1/3}.$$

Thus, if $k \geq 3$, then $n \leq a^3$.

Next, suppose $k = 2$ and rewrite (1.2) as

$$\prod_{i=1}^{2} p_i^{\alpha_i-1}(p_i^{\alpha_i+1} - 1) = \left(\prod_{i=1}^{2} p_i^{\alpha_i} - a\right)^2.$$

If $\alpha_i \geq 2$, then $p_i^{\alpha_i-1} \mid a^2$, therefore $p_i \mid a$, and then $p_i^{\alpha_i} \mid a^3$. In particular, if $\alpha_1, \alpha_2 > 1$, then $n = p_1^{\alpha_1}p_2^{\alpha_2} \mid a^3$, so that $n \leq a^3$. The next case is when $\alpha_1 = 1$ and $\alpha_2 \geq 2$. If $\alpha_2 = 2$, then $p_2 \mid a$, hence $p_1 < p_2 \leq a$ and $n = p_1p_2^2 < a^3$. If $\alpha_2 \geq 3$, (3.1) implies that $2a \geq n/p_1^2 \geq p_2^{\alpha_2-1} \geq n^{1/2}$, so that $n \leq 4a^2 \leq 2a^3$ (recall that $a = 1$ is not possible).

The final case is when $k = 2$ and $\alpha_2 = 1$. Assume first that $p_1 = 2$. Then $p_2^2 - 1 \equiv 0 \pmod 8$, therefore $2^{\alpha_1+2} \mid \phi(n)\sigma(n) = (2^{\alpha_1} p_2 - a)^2$, showing that $2^{\alpha_1+1} \mid a^2$. Thus, by (3.1), we get

$$n \le 2^{\alpha_1+1}(2a) \le 2a^3.$$

From now on, we suppose that p_1 is odd. We break the argument into two subcases depending on whether α_1 is odd or even. First, suppose α_1 is odd and write $\alpha_1 = 2\beta - 1$, where $\beta \ge 1$. Here we have $p_1^{\beta-1} \mid a$, so we may write $a = p_1^{\beta-1} b$ for a positive integer b. Then our equation becomes

$$(p_1^{2\beta} - 1)(p_2^2 - 1) = (p_1^\beta p_2 - b)^2.$$

Consequently, there exists a square free number d and integers u, v such that $p_1^{2\beta} - 1 = du^2$ and $p_2^2 - 1 = dv^2$. Let (X_1, Y_1) be the minimal positive solution to the Pell equation $X^2 - dY^2 = 1$ and let (X_j, Y_j) be its jth solution. Since $p_1^\beta = X_\ell$ and $p_2 = X_m$ for some positive integers ℓ, m, it follows by Lemma 3 that both ℓ and m are powers of 2. Further, since

$$(X_\ell X_m - b)^2 = (p_1^\beta p_2 - a)^2 = (p_1^{2\beta} - 1)(p_2^2 - 1) = (dY_\ell Y_m)^2,$$

it follows that

$$b = X_\ell X_m - dY_\ell Y_m = X_{|m-\ell|}.$$

Suppose $\beta \le 2$. If $m < \ell$ then $p_1^\beta = X_\ell = 2X_{\ell/2}^2 - 1 \ge 2p_2^2 - 1 > p_2^2$, a contradiction. Hence, $m \ge 2\ell$ and $p_1^\beta = X_\ell \le b$, which implies $a = p_1^{\beta-1} b \ge p_1^{2\beta-1}$. We also have $p_2 = X_m = 2X_{m/2}^2 - 1 < 2b^2 \le 2a^2$ and consequently

$$n = p_1^{2\beta-1} p_2 < 2a^3.$$

Now suppose $\beta \ge 3$. If $m \ge 2\ell$, then we get $b \ge X_\ell = p_1^\beta$ as before. Otherwise, $m \le \ell/2$, $2 \mid \ell$ (because both ℓ and m are powers of 2) and

$$b \ge X_{\ell/2} = \sqrt{\frac{X_\ell + 1}{2}} \ge \sqrt{\frac{p_1^\beta}{2}}.$$

In both cases,

$$a = p_1^{\beta-1} b \ge \frac{p_1^{\beta-1+(\beta/2)}}{\sqrt{2}},$$

hence $p_1 \le (a\sqrt{2})^{2/(3\beta-2)}$. Using (3.1), we get $p_2 \le 2ap_1 \le 2a(a\sqrt{2})^{2/(3\beta-2)}$ and we conclude that

$$n \le 2a(a\sqrt{2})^{\frac{4\beta}{3\beta-2}} = 2^{1+\frac{2\beta}{3\beta-2}} a^{\frac{7\beta-2}{3\beta-2}} < 4a^{19/7} \le 2a^3,$$

the final inequality holding for $a \ge 12$ (for $a \le 11$, a quick search yields no solutions in the interval $[2a^3, 4a^{19/7}]$). This concludes the proof when α_1 is odd.

Finally, suppose α_1 is even and write $\alpha_1 = 2\beta$. Then $p_1^\beta \mid a$ and $p_1 \mid p_2^2 - 1$. Writing $a = p_1^\beta a_1$, we get

$$(p_1^{2\beta+1} - 1)\left(\frac{p_2^2 - 1}{p_1}\right) = (p_1^\beta p_2 - a_1)^2.$$

In particular, there exists a square free number d and integers u and v such that

$$p_1^{2\beta+1} - 1 = du^2 \quad \text{and} \quad p_2^2 - 1 = p_1 dv^2.$$

If $d = 1$, then the first equation above becomes $p_1^{2\beta+1} - u^2 = 1$, which has no solutions by known results on Catalan's equation (this particular case of Catalan's equation was solved by Lebesgue [9] more than 160 years ago). Thus, $d > 1$. Putting $x = p_1^\beta$ and $y = p_2$, we get

$$p_1 x^2 - du^2 = 1, \qquad y^2 - (p_1 d)v^2 = 1.$$

With the notation from the previous section, let $\gamma = U_1\sqrt{p_1} + V_1\sqrt{d}$ and $\delta = U_1\sqrt{p_1} - V_1\sqrt{d}$. Then

$$p_1^\beta = U_\ell \quad \text{and} \quad p_2 = U_m$$

for some positive integers ℓ odd and m even. By Lemma 4, we have $\ell = 1$ or $(p, x) = (3, 9)$. In the latter case, using (3.1) gives $n = 3^4 p_2 \le 3^4(6a) \le 2a^3$ for $a \ge 16$ (for $a \le 15$, there are no solutions $n \in [2a^3, 486a]$). Now suppose $\ell = 1$. By Lemma 3, m is a power of 2 and we get

$$\begin{aligned} a_1 = p_1^\beta p_2 - duv &= \left(\frac{\gamma + \delta}{2\sqrt{p_1}}\right)\left(\frac{\gamma^m + \delta^m}{2}\right) - \left(\frac{\gamma - \delta}{2}\right)\left(\frac{\gamma^m - \delta^m}{2\sqrt{p_1}}\right) \\ &= \frac{\gamma^{m-1} + \delta^{m-1}}{2\sqrt{p_1}} = U_{m-1} \ge U_1 = p_1^\beta. \end{aligned}$$

Hence, $a \ge p_1^{2\beta}$ and we conclude that

$$n = p_1^{2\beta} p_2 \le a p_2 \le a(2ap_1) \le 2a^{2+1/(2\beta)} \le 2a^{5/2}.$$

4. The proof of Theorem 2

4.1. Preliminary results. For an integer m we use $P(m)$ for the largest prime factor of m with the convention that $P(0) = P(\pm 1) = 1$. If m satisfies $P(m) \le y$, then m is called *y-smooth.*

We follow [8]. Given a polynomial $F(X) \in \mathbb{Z}[X]$ put

$$\pi_F(x, y) = \#\{p \le x : P(F(p)) \le y\}.$$

The following result appears in [6].

Lemma 5. *Let g be the largest of the degrees of the irreducible factors of $F(X)$ and let k be the number of irreducible factors of $F(X)$ of degree g. Assume that $F(0) \neq 0$ if $g = k = 1$, and let ε be any positive number. Then the estimate*

$$\pi_F(x, y) \asymp \frac{x}{\log x}$$

holds for all sufficiently large x provided that $y \geq x^{g+\varepsilon-1/2k}$.

In the remainder of this section, G is a finite abelian group. Let $n(G)$ be length of the longest sequence of elements of G (not necessarily distinct) such that no nonempty subsequence of it has a zero sum. The following result is from [7].

Lemma 6. *If m is the maximal order of an element of G, then*

$$n(G) < m(1 + \log(\#G/m)).$$

The following result is from [1].

Lemma 7. *Assume that $r > k > n = n(G)$ are integers. Then any sequence of r elements of G contains at least $\binom{r}{k}/\binom{r}{n}$ distinct subsequences of length between $k - n$ and k having zero sum.*

4.2. The proof of Theorem 2. Let x be large, and let $\varepsilon \in (0, 1/5)$, $x_1 = x^{1/2-\varepsilon}$ and

$$y = \frac{\log x_1}{\log\log x_1}.$$

Let $t = \pi(y)$ and $G = (\mathbb{Z}/2\mathbb{Z})^t$, so by Lemma 6,

$$n(G) < 2(1 + (\pi(y) - 1)\log 2). \tag{4.1}$$

Let $u = (3/4 + \varepsilon)^{-1}$. Applying Lemma 5 to the polynomial $F(X) = X^2 - 1$ for which $g = 1$ and $k = 2$, we get

$$\pi_F(y^u, y) \gg \frac{y^u}{\log y^u}.$$

In particular, by the Prime Number Theorem, there exists $c_1 \in (0, 1)$ such that if we put

$$\mathcal{S}_1(y) = \{p : c_1 y^u < p \leq y^u,\ P(p^2 - 1) \leq y\},$$

then

$$\#\mathcal{S}_1(y) \gg \frac{y^u}{\log y^u} \qquad \text{for } x > x_0. \tag{4.2}$$

Applying the above argument with y replaced by $c_1 y$, we also see that if we put

$$\mathcal{S}_2(y) = S_1(c_1 y) = \{p : c_1^{u+1} y^u < p \leq c_1^u y^u,\ P(p^2 - 1) \leq c_1 y\},$$

then

$$\#\mathcal{S}_2(y) \gg \frac{(c_1 y)^u}{\log((c_1 y)^u)} \gg \frac{y^u}{\log y^u} \qquad \text{for } x > x_0. \tag{4.3}$$

We put

$$k = \left\lfloor \frac{\log x_1}{\log y^u} \right\rfloor.$$

The argument from the proof of Theorem 1.1 in [8] shows that if we put

$$\mathcal{F}(y) = \{\ell < x_1 : \phi(\ell)\sigma(\ell) = \square \text{ and } p \in \mathcal{S}_1(y) \text{ for all } p \mid \ell\},$$

then

$$T = \#\mathcal{F}(y) = x_1^{1-1/u+o(1)} > x^{1/8-\varepsilon}$$

for large x. Now take

$$M = \left\lfloor \frac{\log x_1}{\log(c_1^{u+1} y^u)} \right\rfloor + n(G) + 2. \tag{4.4}$$

Note that

$$M \ll \frac{\log x_1}{\log y} + 2\pi(y) \ll y,$$

so in particular $2M < \#\mathcal{S}_2(y)$ for large x by (4.3). Choose $q_1, \ldots, q_{2M}$ in $\mathcal{S}_2(y)$ and write $q_i^2 - 1 = a_i\square$, where a_i is square free and $P(a_i) \le y$ for $i = 1, \ldots, 2M$. We think of a_i as elements of G where in the location corresponding to a prime $p \le y$ we assign the value 1 or 0 according to whether p divides a_i or not. We apply Lemma 7 with $r = 2M$, $k = M$ to deduce the existence of at least $\binom{2M}{M}/\binom{2M}{n(G)} \ge 1$ subsequences of length at most M and at least $M - n(G)$ with a zero sum. Fix one such subsequence $\{q_i\}_{i\in I}$ and put

$$w = \prod_{i\in I} q_i.$$

Then $\phi(w)\sigma(w) = v^2$ for some integer v. Furthermore, since

$$\left\lfloor \frac{\log x_1}{\log(c_1^{u+1} y^u)} \right\rfloor + 2 \le \#I \le M \le \left\lfloor \frac{\log x_1}{\log(c_1^{u+1} y^u)} \right\rfloor + n(G) + 2,$$

we get

$$w \ge (c_1^{u+1} y^u)^{\#I} \ge (c_1^{u+1} y^u)^{\lfloor \frac{\log x_1}{\log(c_1^{u+1} y^u)} \rfloor + 2} > 2x_1 > 2\ell \tag{4.5}$$

for all $\ell \in \mathcal{F}(y)$ when $x > x_0$, and

$$w < (c_1^u y^u)^{\lfloor \frac{\log x_1}{\log(c_1^{u+1} y^u)} \rfloor + O(\pi(y))} = x_1^{1+o(1)} < x^{1/2+\varepsilon}$$

for all sufficiently large x, where we used the fact that (see (4.1))

$$n(G) \ll \pi(y) = o(y) = o\left(\frac{\log x}{\log(c_1^{u+1} y^u)}\right) \qquad (x \to \infty).$$

Now consider

$$\mathcal{N}(y) = \{w\ell : \ell \in \mathcal{F}(y)\}.$$

Clearly, $n < x_1 w < x$ for all $n \in \mathcal{N}(y)$. Let $\ell_1, \ldots, \ell_T$ be all the elements of $\mathcal{F}(y)$. Let $n_i = \ell_i w$ for $i = 1, \ldots, T$. Then

$$\sigma(n_i)\phi(n_i) = (n_i - a_i)^2.$$

Clearly, $a_i < n_i < x$. Let us show that these a_i's are distinct. Put $\phi(n_i)\sigma(n_i) = m_i^2$ for $i = 1, \ldots, T$. If $a_i = a_j$ $(= a)$ for some $i \neq j$, then

$$m_i = n_i - a \quad \text{and} \quad m_j = n_j - a,$$

so

$$m_i - m_j = n_i - n_j = (\ell_i - \ell_j)w. \tag{4.6}$$

Observe that w is built with primes $p \leq c_1^u y^u < c_1 y^u$ and the numbers ℓ_s are built with primes $p > c_1 y^u$ for $s = 1, \ldots, T$, so $\gcd(\ell_s, w) = 1$. Hence, m_s is a multiple of v for all $s = 1, \ldots, T$. Thus, the left-hand side in (4.6) is a multiple of v. Clearly,

$$\begin{aligned} v = \sqrt{\phi(w)\sigma(w)} = w \prod_{q \mid w} \left(1 - \frac{1}{q^2}\right)^{1/2} > \frac{w}{\sqrt{\zeta(2)}} > \frac{w}{2} \\ > \max\{\ell_i, \ell_j\} > |\ell_i - \ell_j|, \end{aligned}$$

by inequality (4.5). Furthermore, v is divisible only by primes $p < y$, whereas w is divisible only by primes $q > c_1^{u+1} y^u > y$ for x sufficiently large, so that $\gcd(v, w) = 1$. Now equation (4.6) implies that $v \mid (\ell_i - \ell_j)$, hence $\ell_i = \ell_j$. So, $a_1, \ldots, a_T$ are distinct, therefore

$$\#\mathcal{A}(x) \geq T = \#\mathcal{F}(y) \geq x^{1/8 - \varepsilon + o(1)}$$

as $x \to \infty$. Letting ε tend to zero, we obtain the desired estimate.

Remark. If, as widely believed, $\pi_F(x, x^\varepsilon) \gg x/\log x$ for any $\varepsilon > 0$, then the above argument implies that $\#\mathcal{A}(x) > x^{1/2 - o(1)}$ as $x \to \infty$.

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