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ON THE SCHRÖDINGER HEAT KERNEL IN HORN-SHAPED DOMAINS

BY

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Abstract. We prove pointwise lower bounds for the heat kernel of Schrödinger semigroups on Euclidean domains under Dirichlet boundary conditions. The bounds take into account non-Gaussian corrections for the kernel due to the geometry of the domain. The results are applied to prove a general lower bound for the Schrödinger heat kernel in horn-shaped domains without assuming intrinsic ultracontractivity for the free heat semigroup.

1. Introduction. The *Dirichlet Laplacian* Δ_D on a proper, open and connected domain $D \subset \mathbb{R}^n$ is the unique nonpositive, self-adjoint operator such that $-\Delta_D$ is associated to the closure of the quadratic form

$$Q_0(f) = \int\limits_D |\nabla f|^2 \, dx, \qquad f \in C_c^\infty(D).$$

The one-parameter semigroup $T_{0,t} = \exp[t\Delta_D]$ is called the *free heat semigroup* on D. We shall refer to the jointly continuous, strictly positive integral kernel $K_0(t,x,y)$ as the *free heat kernel* in D. Given a nonnegative measurable function $V : D \to \mathbb{R}^n$, the corresponding *Schrödinger operator* H with Dirichlet boundary conditions is the unique nonnegative, self-adjoint operator associated to the closure of the quadratic form

$$Q(f) = \int\limits_D (|\nabla f|^2 + V|f|^2) \, dx, \qquad f \in C_c^\infty(D).$$

The one-parameter semigroup T_t associated to the Schrödinger operator H, called the *Schrödinger semigroup* relative to the *potential* V, also admits a strictly positive, jointly continuous integral kernel, denoted by $K(t,x,y)$.

The aim of this note is to prove a general lower bound for K, by using a technique introduced in [10], and to apply it to the case in which D is a *planar horn-shaped* region. This means that D is of the form

$$D = D_f = \{x \equiv (x_1,x_2) \in \mathbb{R}^2 : x_1 > 1, \, |x_2| < f(x_1)\}, \tag{1.1}$$

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where f is a piecewise continuous, strictly positive bounded function on $[1,\infty)$. The term "horn-shaped" should be reserved to the case in which f is nonincreasing, but that assumption will not always be really necessary. Also, we shall *not* assume in general that $f(x_1) \to 0$ as $x_1 \to \infty$. The proof will not make any use of the fact that the domain is two-dimensional, but we keep on studying this case for the sake of notational simplicity.

The problem of obtaining bounds for the *eigenfunctions* of the *Dirichlet Laplacian* in horn-shaped regions has been dealt with in [2], [3], [12]. Our techniques here are sufficiently general to discuss Schrödinger operators as well.

To motivate further our results, we discuss first the behaviour of the free Dirichlet heat kernel K_0 in proper subdomains of $\mathbb{R}^n$. It should differ from the free Gaussian kernel on $\mathbb{R}^n$,

$$\widetilde{K}(t,x,y) = \frac{1}{\sqrt{4\pi t^n}}\, e^{-|x-y|^2/(4\pi t)}, \qquad x,y \in \mathbb{R}^n, t>0,$$

essentially in two aspects (see [5], [6]). First, the geometry of the domain should be taken into account so that, in particular $K_0(t,x,y) \to 0$ as any of the spatial variable approaches ∂D (at least in sufficiently regular domains). Second, the long-time behaviour of K_0 need to be related both to the spectral properties of Δ_D and, again, to the geometry of D.

In fact, let us suppose that the free heat semigroup $T_{0,t}$ is *intrinsically ultracontractive* (IU for short) in the sense of Davies and Simon ([7]). Then the bottom of the L^2 spectrum of $-\Delta_D$ is positive and is a simple eigenvalue, whose normalized eigenfunction is denoted by ψ_0. It is then known that its kernel K_0 satisfies the following bounds for any $\varepsilon > 0$ and for any $t \geq T$, provided $T = T(\varepsilon)$ is sufficiently large:

$$(1-\varepsilon)e^{-E_0 t}\psi_0(x)\psi_0(y) \leq K_0(t,x,y) \leq (1+\varepsilon)e^{-E_0 t}\psi_0(x)\psi_0(y).$$

A necessary condition for intrinsic ultracontractivity to hold is that the generator of the corresponding semigroup has compact resolvent, and hence, purely discrete spectrum. This rules out, in the case of the heat semigroup, all domains in which $d(x,\partial D)$ does not tend to zero as $|x| \to \infty$. But if this condition is instead satisfied, intrinsic ultracontractivity is satisfied under suitable regularity assumptions on ∂D (see [1], [3], [4], [8]), and in [3], even a geometric characterization of intrinsic ultracontractivity for the heat semigroup on planar horn-shaped domains is given. A general lower bound ([1]) for the ground state eigenfunction (valid *without* any assumption on the domain) then shows that, for t sufficiently large and D a domain with sufficiently regular boundary,

$$K_0(t,x,y) \geq C_1 e^{-E_0 t} e^{-C_2 H_D(x,x_0)} e^{-C_2 H_D(y,x_0)}, \tag{1.2}$$

where $x_0 \in D$ is fixed and H_D (see [18]) is the *quasi-hyperbolic metric* of D, an analogue of the usual hyperbolic metric on the half-plane or on the unit ball, defined by

$$H_D(x,y) := \inf_\gamma \int_\gamma \frac{1}{d(\gamma(s),\partial D)}\,ds,$$

where s is arclength and the infimum is taken along rectifiable curves joining x to y and whose support is in D. In fact this follows by combining inequalities (1.8) and (2.6) of [1].

To introduce our result, we should comment first that the *principle of not feeling the boundary* [17] shows that the short-time behaviour is of Gaussian nature [11], [13], [15], [16]. Moreover, a general upper bound under some regularity condition on ∂D has been given in [4], and some regularity has necessarily to be assumed for an *upper* bound similar to (1.2) to hold, since otherwise the heat kernel need not even approach zero as the spatial variables approach the boundary. We are therefore led to wonder whether a lower bound similar to (1.2) holds *without assuming intrinsic ultracontractivity*, or even any regularity condition on the boundary, at least for sufficiently large times (since the short-time behaviour is well understood). This cannot be true in general, because time exponential decay of the heat kernel does not occur, e.g., for domains with infinite inradius (i.e. such that $\sup d(x,\partial D)=\infty$). However our result will show that the heat kernel of Schrödinger operators in a large class of horn-shaped domains satisfies a lower bound which, when $V=0$, is entirely similar to (1.2). This will be proved under a very minimal assumption, certainly true when f is nonincreasing. It is perhaps surprising that essentially no additional regularity condition on the defining function f has to be assumed.

In our bounds only the numerical value of constants can be improved. In fact one should compare (1.2) with the formula (3.6) below: the latter gives, when $V=0$ and the assumptions of Theorem 3.1 hold, a bound of the form $K_0(t,x,y) \ge e^{-At}e^{-B(K_D(x_0,x)+K_D(x_0,y))}$ for suitable constants and x_0, x, y belonging to the x_1-axis.

2. A general lower bound. We shall prove here the following general result.

THEOREM 2.1. *Let V be a nonnegative locally bounded measurable function on a proper, open and connected domain $D \subset \mathbb{R}^n$ and define, for all $\varepsilon \in (0,1)$,*

$$V_\varepsilon(x) := \sup_{y\in B(x,\varepsilon d(x))} V(x),$$

where $d(x) := d(x,\partial D)$. Let moreover $\gamma : [0,t] \to D$ be a rectifiable path in D joining $x, y \in D$. Define

$$k := \sup_{s\in[0,t]} \frac{d(x)\wedge d(y)}{d(\gamma(s))}, \tag{2.1}$$

Set finally, for any fixed $p>1$,

$$W_\varepsilon(x) := \frac{c_1}{d(x)^2} + V_\varepsilon(x),$$

where $c_1 = \alpha_n p(1+k)^2/\varepsilon^2$, α_n *being the ground state eigenvalue of the Dirichlet Laplacian on the unit ball of* $\mathbb{R}^n$. *Then, whenever*

$$t \ge t_0(x,y) := A(d(x)\wedge d(y))(2|x-y| + \varepsilon(d(x)\wedge d(y))), \tag{2.2}$$

for some $A>0$ *one has*

$$K(t,x,y) \ge \frac{C(A)}{[d(x)\wedge d(y)]^n} \exp\Big[-\int_0^t W_\varepsilon(\gamma(u))\,du - c_2\int_0^t |\dot\gamma(u)|^2\,du\Big], \tag{2.3}$$

where $c_2 = 1/2 + p\varepsilon^2/[8(p-1)]$.

Proof. Recall that (cf. [14])

$$K_0(t,x,y) = P_{t,x,y}(X_s \in D\ \forall s\in[0,t]),$$

where $P_{t,x,y}$ is conditional Wiener measure and $X_s = X_s^{(t,x,y)}$ is the Brownian bridge starting from x at time $s=0$ and arriving at y at time $s=t$. The Feynman–Kac formula ([14]) then gives

$$K(t,x,y) = E^{(D)}_{t,x,y}(e^{-\int_0^t V(X_u)\,du}),$$

$E_{t,x,y}$ denoting conditional Wiener measure and the superscript indicating that the expectation is taken with respect to paths which stay in D for all times $s\in[0,t]$. To see this, start from the usual Feynman–Kac formula

$$(e^{-tH}f)(x) = E^{(D)}_x(e^{-\int_0^t V(Z_u)\,du} f(Z_t)),$$

where E_x denotes expectation with respect to Wiener measure and Z_u is a standard Brownian motion starting at x, and notice that, by taking into account the disintegration property of the Wiener measure in terms of conditional Wiener measures,

$$\begin{aligned} K(t,x,y) &= (e^{-tH}\delta_y)(x) = E^{(D)}_x(e^{-\int_0^t V(Z_u)\,du}\delta_y(Z_t)) \\ &= \int_{\Omega^{(D)}_x} e^{-\int_0^t V(Z_u)\,du}\delta_y(Z_t)\,dP_x \\ &= \int_{\Omega^{(D)}_{t,x,z}\times D} e^{-\int_0^t V(X_u^{(t,x,z)})\,du}\delta_y(X_t^{(t,x,z)})\,dP_{t,x,z}\,dz \\ &= \int_{\Omega^{(D)}_{t,x,y}} e^{-\int_0^t V(X_u)\,du}\,dP_{t,x,y}, \end{aligned}$$

where Ω_x and $\Omega_{t,x,y}$ denote the sets of Brownian and, respectively, conditional Brownian paths, while P_x and $P_{t,x,z}$ denote the corresponding Wiener and conditional Wiener measures. Again the superscript "(D)" above indicates that we are integrating over paths which stay in D for all times.

We next recall some results of [10] in order to prove a lower bound on the conditional Wiener measure of a suitable set of paths. Fix from now on an arbitrary $t > 0$, and denote by Γ_δ the set of those paths ω such that

$$|\omega(s) - \gamma(s)| \le \delta d(\gamma(s)) \quad \forall s \in [0, t].$$

Then, applying (2.9) of [10] and assuming without loss of generality that $d(y) \le d(x)$, we obtain

$$P_x(\Gamma_\delta) \le cm(B(y, \delta d(y)))P_{t,x,y}(\Gamma_\varepsilon), \tag{2.4}$$

where P^x is Wiener measure starting at x and ε, δ are related by the equation $\varepsilon = \delta + \delta' k$, δ' being a positive number such that $\varepsilon < 1$. Incidentally, we notice that it is necessary that $\delta' \ge \delta$ for (2.9) of [10] to hold. Next notice that by (2.3) and (2.6) of [10] one has

$$P_x(\Gamma_\delta) \ge \exp\left[-\frac{\alpha_n p}{\delta^2}\int_0^t \frac{ds}{d(\gamma(s))^2} - \left(\frac{1}{2} + \frac{\delta^2 p}{2(p-1)}\right)\int_0^t |\dot\gamma(s)|^2\, ds\right], \tag{2.5}$$

where we have also used the fact that $|\nabla d| \le 1$, and we also corrected here a misprint in (2.6) of [10] where an incorrect factor $(p-1)^2$ must be changed to $p-1$. Finally we fix $\varepsilon < 1$ in (2.4) and choose $\delta' = \delta$ so that $\delta = \varepsilon/(1+k)$ to conclude, by (2.5), that

$$\begin{aligned} P_{t,x,y}(\gamma_\varepsilon) &\ge \frac{Ck^n}{[d(x)\wedge d(y)]^n} \exp\left[-\frac{\alpha_n p(1+k)^2}{\varepsilon^2}\int_0^t \frac{ds}{d(\gamma(s))^2}\right.\\ &\qquad \left. - \left(\frac{1}{2} + \frac{\varepsilon^2 p}{2(1+k)^2(p-1)}\right)\int_0^t |\dot\gamma(s)|^2\, ds\right]\\ &\ge \frac{C}{[d(x)\wedge d(y)]^n} \exp\left[-\frac{\alpha_n p(1+k)^2}{\varepsilon^2}\int_0^t \frac{ds}{d(\gamma(s))^2}\right.\\ &\qquad \left. - \left(\frac{1}{2} + \frac{\varepsilon^2 p}{8(p-1)}\right)\int_0^t |\dot\gamma(s)|^2\, ds\right] \end{aligned} \tag{2.6}$$

for all positive t and for some positive $C = C(A)$, where we have used the fact that $k \ge 1$ in the last step.

In fact, it is remarked in [10] that the factor $[d(x)\wedge d(y)]^{-n}$ is irrelevant, in the sense that it can be omitted by possibly changing the values of c_1, c_2.

We use these facts as follows. For all $\varepsilon \in (0,1)$ define the following sets of conditional Wiener paths:

$$\Gamma_\varepsilon := \{X : |X_s - \gamma(s)| < \varepsilon d(\gamma(s))\ \forall s\},$$

and let V_ε be as in the statement. Then

$$\begin{aligned} K(t,x,y) &= E^{(D)}_{t,x,y}(e^{-\int_0^t V(X_u)\,du}) = \int_{\Omega^{(D)}_{t,x,y}} e^{-\int_0^t V(X_u)\,du}\,dP_{t,x,y} \\ &\ge \int_{\Gamma_\varepsilon} e^{-\int_0^t V(X_u)du}\,dP_{t,x,y} \ge e^{-\int_0^t V_\varepsilon(\gamma(u))\,du} P_{t,x,y}(\Gamma_\varepsilon) \\ &= \frac{C}{[d(x)\wedge d(y)]^n} \\ &\quad \times \exp\Big[-\int_0^t \Big(\frac{c_1}{d(\gamma(u))^2} + V_\varepsilon(\gamma(u))\Big)\,du - c_2\int_0^t |\dot\gamma(u)|^2\,du\Big] \\ &= \frac{C}{[d(x)\wedge d(y)]^n} \exp\Big[-\int_0^t W_\varepsilon(\gamma(u))\,du - c_2\int_0^t |\dot\gamma(u)|^2\,du\Big]. \ \blacksquare \end{aligned}$$

REMARK 2.2. If the domain is sufficiently regular, one can investigate the conclusion of the theorem as in [4, Sec. 3], and using the properties of the quasi-hyperbolic metric [18] on the domain. A path minimizing the exponent on the right hand side of (2.3) exists for all x, y, and the corresponding k in (2.1) can be chosen to be independent of x, y.

3. Horn-shaped regions. In this section we shall consider planar horn-shaped regions $D_f \subset \mathbb{R}^2$ of the following form:

$$D = D_f = \{x \equiv (x_1, x_2) \in \mathbb{R}^2 : x_1 > 1,\ |x_2| < f(x_1)\}, \tag{3.1}$$

where f is a piecewise continuous, strictly positive bounded function on $[1,\infty)$. We shall apply the results of the previous section to the present context. The theorem below will deal, for simplicity, with the heat kernel $K(t,x,y)$ when x, y belong to the x_1-axis, but similar results can be proved for all x, y. Again we mention that our results can be seen as a generalization of [3], [12] which investigate the behaviour of the eigenfunctions of the Dirichlet Laplacian on horn-shaped domains.

We shall need some further notation. First, we identify each point belonging to the x_1-axis with its nonvanishing component. Then we define

$$\widetilde{k} = \sup_{1\le x\le z\le y} \frac{d(x)\wedge d(y)}{d(z)} \tag{3.2}$$

provided the quantity above is finite, where x, z, y belong to the x_1-axis. Next we notice that the constant c_1 in Theorem 2.1 depends upon the constant k

defined in (2.1), and hence upon the choice of the path γ considered there. Therefore, the *effective potential* W_ε depends on γ as well. We shall mark by a tilde the values of quantities depending on k when k is replaced by the path-independent quantity $\widetilde{k}$ defined in (3.2). In particular $\widetilde{c}_1$ the value of the constant c_1 when $k = \widetilde{k}$, and $\widetilde{W}_\varepsilon$ the value of the effective potential W_ε when $c_1 = \widetilde{c}_1$. Notice in particular that $\widetilde{c}_1, \widetilde{W}_\varepsilon$ are path-independent quantities.

THEOREM 3.1. *Let D_f be a horn-shaped planar domain with f satisfying the previous assumptions and V be a nonnegative locally bounded measurable function. Assume that x, y belong to the x_1-axis with $x, y \geq 2$ (identifying such points with their nonvanishing components).*

• *If $f \in \mathrm{L}^1(1,\infty)$ and is nonincreasing and if $V_\varepsilon(\cdot, 0) \in \mathrm{L}^1(1,\infty)$ for some $\varepsilon \in (0,1)$, then for all $t \geq c$, where c is a suitable positive constant depending on f, V and on the choices of ε and p in Theorem 2.1,*

$$K(t,x,y) \geq c_3 \exp\Big[-c_4 t - c_5\Big(\int_2^x \sqrt{\widetilde{W}_\varepsilon(u,0)}\, du + \int_2^y \sqrt{\widetilde{W}_\varepsilon(u,0)}\, du\Big)\Big] \tag{3.3}$$

for suitable positive constants c_3, c_4, c_5. If f is not decreasing the same bound holds provided the constant $\widetilde{k}$ defined in (3.2) is finite, whenever $t \geq t_0(x,y) \vee c$, t_0 being as in Theorem 2.1 and c being as above.

• *If f and V_ε are not necessarily integrable but $\widetilde{k}$ is finite, then the bound (3.3) also holds, but for times $t \geq t_0(x,y) \vee (\varepsilon_1 S(2,x) + \varepsilon_2 S(2,y))$ for some fixed $\varepsilon_1, \varepsilon_2 > 0$, and where now c_3 depends upon $\varepsilon_1, \varepsilon_2$ and diverges as any of them approaches zero. Here we have defined, for any $a < b$ lying on the x_1-axis,*

$$S(a,b) = \sqrt{c_2} \int_a^b \frac{1}{\sqrt{\widetilde{W}_\varepsilon(u,0)}}\, du.$$

Proof. We shall use the bound (2.3) relative to suitable paths lying on the x_1-axis, replacing k with $\widetilde{k}$: this is possible since by construction the quantity k corresponding to the paths chosen is not larger than $\widetilde{k}$, and c_1 is increasing as a function of k, whereas c_2 is independent of k.

As concerns the first part we first notice that, since f is integrable, so is the function $d(t) = d((t,0), \partial D)$ by definition of distance from the boundary, and hence $\widetilde{W}_\varepsilon(\cdot, 0)$ is integrable as well, because of the numerical inequality

$$\frac{1}{\sqrt{\alpha+\beta}} \leq \frac{2}{\sqrt{\alpha}} + \frac{2}{\sqrt{\beta}}.$$

Given two points $a < b$ on the x_1-axis, consider a path γ going from a to b in time $T > 0$ and staying on the x_1-axis for all times. Consider the reparametrization defined by the condition

$$\dot{\gamma}(s) = \sqrt{\widetilde{W}_\varepsilon(\gamma(s))/c_2}, \tag{3.4}$$

where, with a slight abuse of notation, $\gamma(s)$ is one-dimensional and $\widetilde{W}_\varepsilon(\gamma(s)) = \widetilde{W}_\varepsilon(\gamma(s), 0)$.

Therefore, the reparametrized path $\widehat{\gamma}$ has a parameter running over the interval $[0, S(a,b)]$, where $S(a,b)$ is defined as in the second part of the statement. Given x, y as in the statement and an arbitrary point $z < x \wedge y$ on the x_1-axis, notice that the assumption on f implies that

$$a(x,y,z) := S(z,y) + S(z,x)$$

is bounded as a function of all its variables. Choose c larger than $2\sqrt{c_2}||f||_1 + 1$. We can consider a path γ on the x_1-axis running first from x to $z = 2$ and satisfying condition (3.4) with the opposite sign, then staying at z for a time

$$t_1 := t - \sqrt{c_2}\int_2^x \frac{1}{\sqrt{\widetilde{W}_\varepsilon(u,0)}}\,du - \sqrt{c_2}\int_2^y \frac{1}{\sqrt{\widetilde{W}_\varepsilon(u,0)}}\,du,$$

which can be taken to be comparable to t, uniformly in x, y, possibly by changing c, and finally running from z to y and satisfying (3.4) again. The resulting bound is (3.3). In fact the term appearing in the denominator of (2.3), with the choice $k = \widetilde{k}$, can be absorbed by the exponential factor, possibly by changing the values of the constants c_1, c_2 (as stated in Remark 2.2) but in the present situation it is certainly irrelevant since the inradius $\mathrm{Inr}(D_f)$ is finite because f is bounded. The exponential time decay is given by the contribution of the part of the path γ chosen above which stays at $z = 2$ for a time comparable with t.

To complete the proof of the first assertion we need to show that t_0 in Theorem 2.1 is a bounded function under the assumption that f is nonincreasing. In fact, it suffices to show that $xf(x)$ is bounded on $[1, \infty)$, by the definition of t_0 and since D_f has finite inradius. To show this, suppose by contradiction that for all $n \in \mathbb{N}$ there exists a sequence $\{x_{n,k}\}_k$ tending to ∞ as $k \to \infty$ and such that $f(x_{n,k}) > n/x_{n,k}$. Define a sequence $\{x'_n\}_n$ as follows: $x'_1 = x_{1,1}$, $x'_n = x_{n,k_n}$ where k_n is chosen so that $1 - x'_{n-1}/x'_n > 1/2$. Then, since f is nonincreasing,

$$\int_1^\infty f(x)\,dx \ge \sum_{n=2}^\infty f(x'_n)(x'_n - x'_{n-1}) \ge \sum_{n=2}^\infty n\left(1 - \frac{x'_{n-1}}{x'_n}\right) = \infty.$$

If f is not necessarily integrable, we first consider any path from z to w with $z < w$, lying on the x_1-axis and joining the two points in time t. Consider the path $\widehat{\gamma}$ which is obtained from γ by the reparametrization (3.4).

Then $\widehat{\gamma}$ joins z to w in time $S(z,w)$. The path $\check{\gamma}(s)=\check{\gamma}_{z,w}(s)$ defined by

$$\check{\gamma}(s):=\widehat{\gamma}\bigg(\frac{sS(z,w)}{t}\bigg)$$

thus joins z to w in time t. An explicit calculation shows that

$$\begin{aligned}(3.5)\qquad &\int_0^t (|\check{\gamma}'(s)|^2+c_2\widetilde{W}_\varepsilon(\check{\gamma}(s)))\,ds\\ &=\int_0^{S(z,w)}\bigg(\frac{S(z,w)}{t}\,|\widehat{\gamma}'(s)|^2+\frac{c_2t}{S(x,y)}\,\widetilde{W}_\varepsilon(\widehat{\gamma}(s),0)\bigg)\,ds\\ &=\bigg(\frac{S(z,w)}{\sqrt{c_2}\,t}+\frac{c_2^{3/2}t}{S(z,w)}\bigg)\int_0^{S(z,w)}\sqrt{\widetilde{W}_\varepsilon(\widehat{\gamma}(s),0)}\,|\widehat{\gamma}'(s)|\,ds\\ &=\bigg(\frac{S(z,w)}{\sqrt{c_2}\,t}+\frac{c_2^{3/2}t}{S(z,w)}\bigg)\int_z^w\sqrt{\widetilde{W}_\varepsilon(u,0)}\,du.\end{aligned}$$

Suppose that $t>\varepsilon_1S(2,x)+\varepsilon_2S(2,y)$ for some fixed constants $\varepsilon_1,\varepsilon_2$. Then consider a path γ which is obtained by first going from x to $z=2$ in time $\varepsilon_1S(2,x)$ along a path $\check{\gamma}_{2,x}^{(1)}$ defined as above, then staying at $z=2$ for time $t-(\varepsilon_1S(2,x)+\varepsilon_2S(2,y))$ and finally running from $z=2$ to y in time $\varepsilon_2S(2,y)$ along a path $\check{\gamma}_{2,y}^{(2)}$ defined as above. Theorem 2.1 and (3.5) thus imply the assertion. ∎

COROLLARY 3.2. *Let D_f be as in* (3.1), *with the assumptions on f made after that formula and assuming in addition f to be nonincreasing and integrable. Then there exist $c_3,c_4,c_5>0$ such that for any $x,y\ge 2$ belonging to the x_1-axis, the free Dirichlet heat kernel K_0 in D_f satisfies the bound*

$$(3.6)\qquad K_0(t,x,y)\ge c_3\exp[-c_4t-c_5(H_{D_f}(x,2)+H_{D_f}(y,2))]$$

for any $t\ge c$, where c is a constant depending on f and on the choice of ε and p in Theorem 2.1. *If f is integrable but not necessarily nonincreasing, the same bound holds for $t\ge t_0(x,y)\vee c$, t_0 being as in Theorem* 2.1, *provided the constant $\widetilde{k}$ defined in* (3.2) *is finite. If f is not necessarily integrable but $\widetilde{k}$ is finite then the bound is valid for $t\ge t_0(x,y)\vee(\varepsilon_1S_0(2,x)+\varepsilon_2S_0(2,y))$, where for $b<a$ we have defined*

$$S_0(b,a)=\int_b^a d(u)\,du.$$

Proof. It suffices to notice that the terms in the exponential factor appearing in Theorem 3.1 coincide with the quasi-hyperbolic distance

$$H_{D_f}(x,y) := \inf_\gamma \int_\gamma \frac{ds}{d(\gamma(s))}$$

between the corresponding points, s being arclength and the infimum being taken over all rectifiable paths joining x to y ([18]), because of the reparametrization made and by the structure of the domain D_f. ■

REMARK 3.3. • One could make the assumption that there exists $c > 0$ such that for all $x \geq 2$ belonging to the x_1-axis, $d(x) \geq cf(x)$. A sufficient condition for this to hold is that f is differentiable and $f'(x) \to 0$ as $x \to \infty$. All quantities involving $d(x)$, including the quasi-hyperbolic distance, could then be estimated, under that assumption, by replacing $d(x)$ with $f(x)$.

• The results have been stated for notational simplicity in dimension $d = 2$ but identical results also hold in higher dimensions.

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