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SEMICLASSICAL DISTRIBUTION OF EIGENVALUES FOR ELLIPTIC OPERATORS WITH HÖLDER CONTINUOUS COEFFICIENTS, PART I: NON-CRITICAL CASE

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Abstract. We consider a version of the Weyl formula describing the asymptotic behaviour of the counting function of eigenvalues in the semiclassical approximation for self-adjoint elliptic differential operators under weak regularity hypotheses. Our aim is to treat Hölder continuous coefficients and to investigate the case of critical energy values as well.

1. Introduction. Since the papers of J. Chazarain [2] and B. Helffer and D. Robert [4], the semiclassical spectral asymptotics has been investigated in numerous works; we refer to the monographs [3], [8], [10] and [13]. The main results have been obtained by using the tools of microlocal analysis based on the approach of L. Hörmander [6]. However this approach works only for smooth problems and the semiclassical framework is usually considered for a non-critical energy value. Our aim is to present a method of obtaining semiclassical estimates for more general classes of differential operators.

(A) *Formulation of the results.* Let $r \in]0;1]$ and denote by $\mathcal{B}^r$ the set of bounded, Hölder continuous functions of exponent r on $\mathbb{R}^d$, i.e. $a \in \mathcal{B}^r$ means that $a \in L^\infty(\mathbb{R}^d)$ and there is $C > 0$ such that

$$|a(x) - a(y)| \le C|x-y|^r \qquad (x, y \in \mathbb{R}^d). \tag{1.1}$$

Let $m \in \mathbb{N}^*$ and for $\nu, \overline{\nu} \in \mathbb{N}^d$ with $|\nu|, |\overline{\nu}| \le m$ consider real-valued $a_{\nu,\overline{\nu}} \in \mathcal{B}^r$ such that $a_{\nu,\overline{\nu}} = a_{\overline{\nu},\nu}$ and

$$\sum_{|\nu|=|\overline{\nu}|=m} a_{\nu,\overline{\nu}}(x)\xi^{\nu+\overline{\nu}} \ge c|\xi|^{2m} \qquad (x, \xi \in \mathbb{R}^d), \tag{1.2}$$

for some constant $c > 0$.

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For $h > 0$ let $\mathcal{A}_h$ be the quadratic form defined for $\varphi, \psi \in C_0^m(\mathbb{R}^d)$ by

$$\mathcal{A}_h[\varphi, \psi] = \sum_{|\nu|,|\overline{\nu}|\le m} (a_{\nu,\overline{\nu}}(hD)^\nu \varphi, (hD)^{\overline{\nu}}\psi), \tag{1.3}$$

where $(\cdot,\cdot)$ is the scalar product of $L^2(\mathbb{R}^d)$ and $(hD)^\nu = (-ih)^{|\nu|}\partial^\nu/\partial x^\nu$.

The ellipticity hypothesis (1.2) ensures the $\mathcal{A}_h$ is bounded from below and its closure defines a self-adjoint operator A_h. We introduce

$$a(x,\xi) = \sum_{|\nu|,|\overline{\nu}|\le m} a_{\nu,\overline{\nu}}(x)\xi^{\nu+\overline{\nu}}, \tag{1.4}$$

and for $E \in \mathbb{R}$ we set

$$\varGamma_E = a^{-1}(]-\infty; E[) = \{(x,\xi) \in \mathbb{R}^{2d} : a(x,\xi) < E\}. \tag{1.5}$$

We have

Proposition 1.1. *Let $E, E_0 \in \mathbb{R}$ be such that $E < E_0$ and $\varGamma_{E_0}$ is bounded. Then one can find $h_0 > 0$ such that for $h \in]0; h_0]$, the spectrum of A_h is discrete in $]-\infty; E]$.*

Further on, E_0, E, h_0 are as in Proposition 1.1 and $|\varGamma_E| = \int_{a(x,\xi)<E} dx\, d\xi$ is the Lebesgue measure of $\varGamma_E$. For $h \in]0; h_0]$ we define the counting function $\mathcal{N}(A_h, E)$ as the number of eigenvalues (counted with their multiplicities) smaller than E. Our principal result is:

Theorem 1.2. *Let A_h be as above with $a_{\nu,\overline{\nu}} \in \mathcal{B}^r$ for some $r \in]0; 1]$. If $\mu \in]0; 2r/(2 + r)[$, then*

$$\mathcal{N}(A_h, E) = |\varGamma_E|(2\pi h)^{-d} + (|\varGamma_{E+h^\mu}| - |\varGamma_{E-h^\mu}|)O(h^{-d}). \tag{1.6}$$

Similarly to [1] one can observe that some additional conditions on a are needed to obtain a good estimate of $|\varGamma_{E+h^\mu}| - |\varGamma_{E-h^\mu}|$ as $h \to 0$. In this paper we are interested in the following condition:

$$a(x,\xi) = E \;\Rightarrow\; \nabla_\xi a(x,\xi) \ne 0. \tag{1.7}$$

If (1.7) holds then E will be called a *non-critical energy value* and it is easy to see that this condition ensures $|\varGamma_{E+h^\mu}| - |\varGamma_{E-h^\mu}| = O(h^\mu)$. Moreover it is possible to obtain the following stronger estimates:

Theorem 1.3. *Assume moreover* (1.7). *If $\mu \in]0; r[$, then*

$$\mathcal{N}(A_h, E) = |\varGamma_E|(2\pi h)^{-d} + O(h^{\mu-d}). \tag{1.8}$$

(B) *Comments.* The proof of Theorem 1.3 is presented below and a suitable development, which allows us to prove Theorem 1.2, will be described in [18]. The basic idea is to replace irregular coefficients by smooth ones and to investigate the corresponding smooth problem following some ideas of our earlier papers [14–15]. In the case of a non-critical energy value it is also possible to investigate the smooth problems by adapting the theory developed in the book of V. Ivriĭ [8], deducing Theorem 1.3 according to [9] (and we also have (1.8) with the optimal value $\mu = 1$ if the first order derivatives of the coefficients are Hölder continuous, cf. [9] or [17]).

However the approach we present here is quite different from [9] or [17]. It seems to us that the most interesting feature of this approach is the possibility of investigating non-critical and critical energy values in a quite similar way and the fact that the analysis of the smooth problem works for suitable classes of pseudodifferential operators as well.

A general plan is the following. In Section 2 we define the regularized operators P_h; the Fourier transform allows us to express suitable functions $\widetilde{f}_h(P_h)$ by means of the evolution group $U_t = e^{itP_h/h^\mu}$. In Section 3 we describe an approximation of U_t giving a pseudodifferential approximation of $\widetilde{f}_h(P_h)$ with correct asymptotic properties. The correct asymptotic behaviour of the approximation is proved at the end of Section 3 by means of simple integrations by parts. In Section 4 we explain how to implement a similar idea to estimate the difference between $\widetilde{f}_h(P_h)$ and the approximation. The final computations justifying this idea are presented in Section 5 and some standard supplementary details are given in Section 6.

(C) *Developments.* 1. Let $\widetilde{\mathcal{A}}_h = \mathcal{A}_h + h\mathcal{A}_{h,1}$, where $\mathcal{A}_h$ is as above and

$$\exists C_0 > 0\ \forall \varphi \in C_0^m(\mathbb{R}^d), \qquad |\mathcal{A}_{h,1}[\varphi,\varphi]| \le \mathcal{A}_h[\varphi,\varphi] + C_0\|\varphi\|^2. \tag{1.9}$$

Then $(\widetilde{\mathcal{A}}_h[\varphi,\varphi] + C_0\|\varphi\|^2)^{1/2}$ and $(\mathcal{A}_h[\varphi,\varphi] + C_0\|\varphi\|^2)^{1/2}$ are equivalent norms if $h < h_0$ with h_0 small enough and we can define $\widetilde{A}_h$, the associated self-adjoint operator in $L^2(\mathbb{R}^d)$. Moreover the assertions of Proposition 1.1, Theorem 1.2 and 1.3 still hold with $\widetilde{A}_h$ instead of A_h.

2. Let M be a compact (boundaryless) manifold with a density dx of class C^m and let $\mathcal{A}_{M,h}$ be a quadratic form on $C^m(M) \times C^m(M)$ satisfying

$$\operatorname{supp}\widetilde{\varphi} \cap \operatorname{supp}\widetilde{\psi} = \emptyset \ \Rightarrow\ \mathcal{A}_{M,h}[\widetilde{\varphi},\widetilde{\psi}] = 0.$$

Assume that in local coordinates on $\mathcal{U} \subset \mathbb{R}^d$ the form $\mathcal{A}_{M,h}$ acts on $\varphi,\psi \in C_0^m(\mathcal{U})$ according to the formula (1.3) with all the hypotheses of Theorem 1.2 (or 1.3) satisfied. Then a standard reasoning can be applied to obtain analogous estimates for the counting function of $A_{M,h}$, the associated self-adjoint operator in $L^2(M, dx)$.

3. For an operator $A_{M,1}$ considered in item 2, we can deduce the classical Weyl formula considering a semiclassical problem $\widetilde{A}_{M,h}$ with $h = \lambda^{-1/(2m)}$. We need to assume the Hölder continuity of top order coefficients ($|\nu| = |\overline{\nu}| = m$) and we can consider the lower order coefficients belonging to L^∞. Indeed, reasoning as in item 1 we can modify lower order coefficients and since the principal symbol is ξ-homogeneous, the energy value 1 is not critical,

allowing us to adapt the proof of Theorem 1.3 to obtain

$$\mathcal{N}(A_{M,1},\lambda)=\mathcal{N}(\widetilde{A}_{M,\lambda^{-1/(2m)}},1)=c\lambda^{d/(2m)}+O(\lambda^{(d-\mu)/(2m)}) \tag{1.10}$$

for every $\mu\in[0;r[$. This result was described in [14–15] and we refer to [8–11] and [16] for results concerning boundary value problems.

4. The regularity hypotheses on the coefficients $a_{\nu,\overline{\nu}}$ are in fact essential only for x such that $(x,\xi)\in\Gamma_{E_0}$ with some $E_0>E$, while the behaviour of the coefficients for other values of x can be more general: the main requirement is the possibility of reducing the problem by adding an auxiliary cut-off supported in Γ_{E_0} as in Proposition 6.1. In particular we have assumed $a_{\nu,\overline{\nu}}\in L^\infty(\mathbb{R}^d)$ for the sake of simplicity, but it is possible to consider unbounded coefficients in the framework of tempered variation models on $T^*\mathbb{R}^d$ (cf. e.g. [5]).

2. Regularized problem

(A) *Definition of smooth operators.* Let $\gamma\in C_0^\infty(\mathbb{R}^d)$ satisfy $\int\gamma(x)\,dx=1$ and let $\gamma_\varepsilon(x)=\varepsilon^{-d}\gamma(x/\varepsilon)$ for $\varepsilon>0$.

We fix $\delta\in\,]0;1[$ and define

$$a_{\nu,\overline{\nu},h}(x)=(a_{\nu,\overline{\nu}}*\gamma_{h^\delta})(x)=\int a_{\nu,\overline{\nu}}(y)\gamma(h^{-\delta}(x-y))h^{-\delta d}\,dy. \tag{2.1}$$

As explained in Section 6, the hypothesis $a_{\nu,\overline{\nu}}\in\mathcal{B}^r$ ensures the estimates

$$|a_{\nu,\overline{\nu}}(x)-a_{\nu,\overline{\nu},h}(x)|\le Ch^{\delta r}, \tag{2.2}$$

$$|\partial_x^\alpha a_{\nu,\overline{\nu},h}(x)|\le C_\alpha(1+h^{\delta(r-|\alpha|)}) \tag{2.3}$$

(for every $\alpha\in\mathbb{N}^d$). We define

$$p_h(x,\xi)=\sum_{|\nu|,|\overline{\nu}|\le m}a_{\nu,\overline{\nu},h}(x)\xi^{\nu+\overline{\nu}} \tag{2.4}$$

and assume further on that $r\delta>\mu$, hence (2.2) yields

$$|\partial_\xi^\alpha(a-p_h)(x,\xi)|\le C_\alpha h^\mu(1+|\xi|)^{2m-|\alpha|}. \tag{2.5}$$

Moreover the operator

$$P_h^\circ=\sum_{|\nu|,|\overline{\nu}|\le m}(hD)^\nu a_{\nu,\overline{\nu},h}(x)(hD)^{\overline{\nu}} \tag{2.6}$$

satisfies $|((A_h-P_h^\circ)\varphi,\varphi)|\le Ch^\mu((I-h^2\Delta)^m\varphi,\varphi)$, and defining

$$P_h^\pm=P_h^\circ\pm Ch^\mu(I-h^2\Delta)^m \tag{2.7}$$

(with C large enough) we obtain $P_h^-\le A_h\le P_h^+$ (in the sense of quadratic forms). If $h\in\,]0;h_0]$ with h_0 as in Proposition 1.1, then the min-max principle (cf. [12]) yields

$$\mathcal{N}(P_h^+, E) \le \mathcal{N}(A_h, E) \le \mathcal{N}(P_h^-, E),$$

and it is clear that it suffices to prove

THEOREM 2.1. *The formula* (1.8) *holds with* $P_h^{\pm}$ *instead of* A_h.

(B) *Microlocal trace formula.* For $E', E \in \mathbb{R}$ let $\mathbb{1}_{[E';E]} : \mathbb{R} \to \{0,1\}$ be the characteristic function of $[E';E]$ and let $\mathbb{1}_{[E';E]}(P_h^{\pm})$ denote the spectral projector of $P_h^{\pm}$ on $[E';E]$. If b_h is a polynomially bounded smooth function of $(x,\xi) \in \mathbb{R}^{2d}$, then $B_h = b_h(x, hD)$ denotes the pseudodifferential operator acting on $\varphi \in \mathcal{S}(\mathbb{R}^d)$ according to the formula

$$(B_h\varphi)(x) = \int \frac{d\xi}{(2\pi h)^d}\, e^{ix\xi/h} b_h(x,\xi) \int dy\, e^{-iy\xi/h}\varphi(y).$$

Let $s \in \mathbb{R}$. We write $b \in S^s_{0,\delta}$ if $b = (b_h)_{h\in]0;1]}$ is a family of smooth functions satisfying the estimates

$$|\partial_\xi^\alpha \partial_x^\beta b_h(x,\xi)| \le C_{\alpha,\beta} h^{-s-|\beta|\delta} \tag{2.8}$$

for every $\alpha, \beta \in \mathbb{N}^d$. In Section 6 we show that Theorem 2.1 follows from

THEOREM 2.2. *Let* $\overline{\Gamma}$ *be a closed subset of* Γ_{E_0} *such that*

$$(x,\xi) \in \overline{\Gamma} \;\Rightarrow\; |\nabla_\xi p_h(x,\xi)| \ge c \tag{2.9}$$

for some constant $c > 0$. *Let* $l = (l_h)_{h\in]0;1]} \in S^0_{0,\delta}$ *be such that* l_h *is real-valued and* $\operatorname{supp} l_h \subset \overline{\Gamma}$ *for every* $h \in]0;1]$. *If* $L_h = l_h(x, hD)$ *and* L_h^* *denotes its adjoint in* $L^2(\mathbb{R}^d)$, *then*

$$\operatorname{tr}(L_h \mathbb{1}_{[E';E]}(P_h^{\pm}) L_h^*) = \int_{E'<p_h(x,\xi)<E} \frac{dx\, d\xi}{(2\pi h)^d}\, l_h(x,\xi)^2 + O(h^{\mu-d}). \tag{2.10}$$

(C) *Plan of the proof of Theorem 2.2.* Further on we drop the index h. In particular we write simply L, l, p instead of L_h, l_h, p_h and we abbreviate $P_h^{\pm} = P$. Let $\widetilde{\gamma} \in C_0^\infty(]-1/2;1/2[)$ be such that $\int \widetilde{\gamma} = 1$ and $\widetilde{\gamma} \ge 0$. Then the convolution with $\widetilde{\gamma}_{h^\mu}(\lambda) = h^{-\mu}\widetilde{\gamma}(\lambda/h^\mu)$ allows us to replace $\mathbb{1}_{[E';E]}$ by the approximations

$$\widetilde{f}_h^- = \mathbb{1}_{[E'+h^\mu/2;E-h^\mu/2]} * \widetilde{\gamma}_{h^\mu}, \qquad \widetilde{f}_h^+ = \mathbb{1}_{[E'-h^\mu/2;E+h^\mu/2]} * \widetilde{\gamma}_{h^\mu}, \tag{2.11}$$

satisfying $\mathbb{1}_{[E'+h^\mu;E-h^\mu]} \le \widetilde{f}_h^- \le \mathbb{1}_{[E';E]} \le \widetilde{f}_h^+ \le \mathbb{1}_{[E'-h^\mu;E+h^\mu]}$, hence

$$\operatorname{tr} L\widetilde{f}_h^-(P)L^* \le \operatorname{tr} L\mathbb{1}_{[E';E]}L^* \le \operatorname{tr} L\widetilde{f}_h^+(P)L^*. \tag{2.12}$$

Clearly it suffices to prove (2.10) with $\widetilde{f}_h^{\pm}(P)$ instead of $\mathbb{1}_{[E';E]}$. Further on we abbreviate $\widetilde{f}_h^{\pm} = \widetilde{f}_h$; observe the estimates of derivatives

$$|\widetilde{f}_h^{(k)}(\lambda)| \le C_k h^{-k\mu} \qquad (k \in \mathbb{N}). \tag{2.13}$$

Next we introduce the h^μ-Fourier transform of $\widetilde{f}_h$,

$$f_h(t) = (\mathcal{F}_{h^\mu}\widetilde{f}_h)(t) = \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda t/h^\mu}\widetilde{f}_h(\lambda), \tag{2.14}$$

and remark that for any $k \in \mathbb{N}$ we have

$$t^k f_h(t) = (-i)^k h^{k\mu}(\mathcal{F}_{h^\mu}\widetilde{f}_h^{(k)})(t) = O(1). \tag{2.15}$$

Since $\widetilde{f}_h(\lambda) = (\mathcal{F}_{h^\mu}^{-1} f_h)(\lambda) = \int_{-\infty}^{\infty} \frac{dt}{2\pi h^\mu} f_h(t)e^{it\lambda/h^\mu}$, we can write

$$\operatorname{tr} L\widetilde{f}_h(P)L^* = \int_{-\infty}^{\infty} \frac{dt}{2\pi h^\mu} f_h(t) \operatorname{tr} LU_tL^*, \tag{2.16}$$

where we have set $U_t = e^{itP/h^\mu}$. Then our principal task is to construct a sequence of operators $(Q_{\overline{N},t})_{\overline{N}\in\mathbb{N}}$ which is a suitable approximation of LU_t. More precisely: assuming $Q_{\overline{N},t|t=0} = L$ and defining

$$\widetilde{Q}_{\overline{N},t} = \frac{d}{dt}Q_{\overline{N},t} - iQ_{\overline{N},t}P/h^\mu, \tag{2.17}$$

we can write formally

$$LU_t - Q_{\overline{N},t} = \int_0^t d\tau\, \frac{d}{d\tau}(Q_{\overline{N},t-\tau}U_\tau) = -\int_0^t d\tau\, \widetilde{Q}_{\overline{N},t-\tau}U_\tau \tag{2.18}$$

and observe that due to (2.15–16) and (2.18), Theorem 2.2 follows from

PROPOSITION 2.3. *Let $\overline{N} \in \mathbb{N}$. Then there is $Q_{\overline{N},t} \in B(L^2(\mathbb{R}^d))$ satisfying*

$$\int_{-\infty}^{\infty} \frac{dt}{2\pi h^\mu} f_h(t) \operatorname{tr} Q_{\overline{N},t}L^* = \int_{E'<p<E} \frac{dx\, d\xi}{(2\pi h)^d}\, l^2 + O(h^{\mu-d}), \tag{2.19}$$

$$|\operatorname{tr} \widetilde{Q}_{\overline{N},t-\tau}U_\tau L^*| \le h^{(1-\delta)\overline{N}-5d}(2+|t|)^{C_{\overline{N}}}, \tag{2.20}$$

where $\widetilde{Q}_{\overline{N},t}$ is given by (2.17), $C_{\overline{N}} > 0$ *is a constant large enough*, $(t,\tau) \in \mathbb{R}^* \times \mathbb{R}$ *and* $\tau/t \in [0;1]$.

In Section 3 we describe the construction of

$$Q_{\overline{N},t} = \Big(e^{itp/h^\mu} \sum_{0\le n\le \overline{N}} t^n q^\circ_{\overline{N},n}\Big)(x, hD), \tag{2.21}$$

as suitable pseudodifferential operators. At the end of Section 3 we check that (2.19) follows via integrations by parts; in Section 4 we describe a similar strategy to obtain (2.20), completing the proof in Section 5.

3. Description of the approximation. The operators (2.7) can be written in a standard form

$$P_h^{\pm} = \sum_{|\nu|\le 2m} p_{\nu,h}^{\pm}(x)(hD)^{\nu} = p_h^{\pm}(x,hD) \tag{3.1}$$

and it is easy to check that (2.3) still holds with $p_{\nu,h}^{\pm}$ instead of $a_{\nu,\overline{\nu},h}$ and

$$p_h^{\pm}(x,\xi) = \sum_{|\nu|\le 2m} p_{\nu,h}^{\pm}(x)\xi^{\nu} = a(x,\xi) + O(h^{\mu})(1+|\xi|)^{2m}. \tag{3.2}$$

For a smooth function $(x,\xi)\mapsto b_t(x,\xi)\in\mathbb{C}$ we define

$$\widetilde{\mathcal{P}}_{\overline{N}} b_t = e^{-itp/h^{\mu}}\Big(\partial_t(b_t e^{itp/h^{\mu}}) - \sum_{|\alpha|\le\overline{N}} \frac{h^{|\alpha|-\mu}}{\alpha!\, i^{|\alpha|+1}}\,\partial_\xi^{\alpha}(b_t e^{itp/h^{\mu}}\,\overline{\partial_x^{\alpha}p^{\pm}})\Big). \tag{3.3}$$

Further on, $l\in S_{0,\delta}^{0}$ is as in Theorem 2.2.

LEMMA 3.1. *Let $\varrho=\mu-1+\delta$. Let $b\in S_{0,\delta}^{s}$ be independent of t and such that* $\operatorname{supp} b\subset\operatorname{supp} l$. *Then we can find $b_n\in S_{0,\delta}^{s+n\varrho}$ for $n\in\{0,\dots,\overline{N}\}$ such that* $\operatorname{supp} b_n\subset\operatorname{supp} l$ *and*

$$\widetilde{\mathcal{P}}_{\overline{N}} b = \sum_{0\le n\le\overline{N}} t^n b_n. \tag{3.4}$$

Proof. First of all we recall that $b\in S_{0,\delta}^{s}$, $\widetilde{b}\in S_{0,\delta}^{\widetilde{s}} \Rightarrow b\widetilde{b}\in S_{0,\delta}^{s+\widetilde{s}}$ and it is easy to see that $\bar{b}p$, $\bar{b}p^{\pm}$ belong to $S_{0,\delta}^{s}$. Since (2.5) still holds with $p_h^{\pm}$ instead of p_h, and $|\alpha|\ge 1 \Rightarrow h^{|\alpha|}\partial_x^{\alpha}a_{\nu,\overline{\nu},h} = O(h^{\delta r+(1-\delta)|\alpha|})$, we obtain $\bar{b}(p-p^{\pm})\in S_{0,\delta}^{s-\delta r-(1-\delta)}$ and $h^{|\alpha|}\bar{b}\partial_x^{\alpha}p^{\pm}\in S_{0,\delta}^{s-\delta r-(1-\delta)|\alpha|}$ if $|\alpha|\ge 1$. Using moreover $\delta r\ge\mu$, we obtain

$$b_0 = ih^{-\mu}(p-\overline{p^{\pm}})b + \sum_{1\le|\alpha|\le\overline{N}} \frac{h^{|\alpha|-\mu}}{\alpha! i^{|\alpha|+1}}\,\partial_\xi^{\alpha}(b\,\overline{\partial_x^{\alpha}p^{\pm}}) \in S_{0,\delta}^{s}. \tag{3.5}$$

Next for $n\in\{1,\dots,\overline{N}\}$ we obtain

$$b_n = \sum_{\substack{\alpha=\alpha_0+\dots+\alpha_n\\ |\alpha|\le\overline{N},\,\alpha_k\ne 0 \text{ if } k\ne 0}} c_{\alpha_0,\dots,\alpha_n} h^{|\alpha|-(n+1)\mu}\partial_\xi^{\alpha_0}(b\,\overline{\partial_x^{\alpha}p^{\pm}})\partial_\xi^{\alpha_1}p\cdots\partial_\xi^{\alpha_n}p \in S_{0,\delta}^{s+n\varrho}$$

by observing that $1\le n\le|\alpha| \Rightarrow (n+1)\mu-(1-\delta)|\alpha|-\delta r\le n\varrho$. ∎

PROPOSITION 3.2. *Assume that $N\in\{0,1,\dots,\overline{N}\}$ and $\varrho=\mu-1+\delta\ge 0$. Then we can find*

$$q_{\overline{N},N,t} = \sum_{0\le n\le N} t^n q_{\overline{N},n}^{\circ} \tag{3.6(N)}$$

such that $q_{\overline{N},N,t|t=0} = q^{\circ}_{\overline{N},0} = l$ *and*

$$\widetilde{\mathcal{P}}_{\overline{N}} q_{\overline{N},N,t} = \sum_{N \le n \le N+\overline{N}} t^n \widetilde{q}^{\circ}_{\overline{N},N,n} \tag{3.7(N)}$$

with

$$q^{\circ}_{\overline{N},n} \in S^{(n-1)\varrho}_{0,\delta}, \quad \operatorname{supp} q^{\circ}_{\overline{N},n} \subset \operatorname{supp} l \quad (n \in \{1,\ldots,N\}), \tag{3.8(N)}$$

$$\widetilde{q}^{\circ}_{\overline{N},N,n} \in S^{n\varrho}_{0,\delta}, \quad \operatorname{supp} \widetilde{q}^{\circ}_{\overline{N},N,n} \subset \operatorname{supp} l \quad (n \in \{N,\ldots,N+\overline{N}\}). \tag{3.9(N)}$$

Proof. If $N = 0$ then we take $q_{\overline{N},0,t} = q^{\circ}_{\overline{N},0} = l \in S^0_{0,\delta}$, and Lemma 3.1 with $b = l$ gives the statement of Proposition 3.2 for $N = 0$. Next we assume that the statement holds for a given $N \le \overline{N} - 1$ and we prove that it still holds for $N + 1$ in place of N.

Using the induction hypothesis (3.7(N)) to express $\widetilde{\mathcal{P}}_{\overline{N}} q_{\overline{N},N,t}$ we find

$$\begin{aligned}\widetilde{\mathcal{P}}_{\overline{N}} q_{\overline{N},N+1,t} &= \widetilde{\mathcal{P}}_{\overline{N}}(t^{N+1} q^{\circ}_{\overline{N},N+1}) + \widetilde{\mathcal{P}}_{\overline{N}} q_{\overline{N},N,t} \\ &= t^N((N+1)q^{\circ}_{\overline{N},N+1} + \widetilde{q}^{\circ}_{\overline{N},N,N}) + t^{N+1}\widetilde{\mathcal{P}}_{\overline{N}} q^{\circ}_{\overline{N},N+1} + \sum_{N+1 \le n \le N+\overline{N}} t^n \widetilde{q}^{\circ}_{\overline{N},N,n}.\end{aligned}$$

In order to obtain (3.7($N+1$)) it suffices to cancel the term with t^N taking

$$q^{\circ}_{\overline{N},N+1} = -\widetilde{q}^{\circ}_{\overline{N},N,N}/(N+1).$$

Since $\widetilde{q}^{\circ}_{\overline{N},N,N} \in S^{N\varrho}_{0,\delta}$ and $\operatorname{supp} \widetilde{q}^{\circ}_{\overline{N},N,N} \subset \operatorname{supp} l$ by the induction hypothesis, we obtain (3.8($N+1$)), and using Lemma 3.1 with $b = q^{\circ}_{\overline{N},N+1}$ we observe that $\varrho \ge 0 \Rightarrow S^{(N+n)\varrho}_{0,\delta} \subset S^{(N+1+n)\varrho}_{0,\delta}$ and (3.9($N+1$)) holds. ∎

LEMMA 3.3. *For* $b \in C_0^\infty(\mathbb{R}^{2d})$ *define*

$$J_t(b) = \int \frac{dx\,d\xi}{(2\pi h)^d}\, e^{itp(x,\xi)/h^\mu} b(x,\xi). \tag{3.10}$$

If $b \in S^s_{0,\delta}$ *satisfies* $\operatorname{supp} b \subset \operatorname{supp} l$, *then for every* $k \in \mathbb{N}$ *one can find* $b_k \in S^s_{0,\delta}$ *such that* $\operatorname{supp} b_k \subset \operatorname{supp} l$ *and*

$$h^{-k\mu} t^k J_t(b) = J_t(b_k). \tag{3.11}$$

Proof. The hypothesis (2.9) ensures the existence of $\widetilde{b}_j \in S^s_{0,\delta}$ such that $b = \sum_{j=1}^d \widetilde{b}_j \partial_{\xi_j} p$ and integrating by parts we find

$$h^{-\mu} t J_t(\widetilde{b}_j \partial_{\xi_j} p) = J_t(i\partial_{\xi_j} \widetilde{b}_j). \tag{3.12}$$

Thus the statement of Lemma 3.3 holds for $k = 1$; the proof is completed by induction on $k \in \mathbb{N}$. ∎

Proof of the estimate (2.19). Taking $Q_{\overline{N},t}$ given by (2.21) with $q_{\overline{N},\overline{N},t}$ defined in Proposition 3.2, we observe that $Q_{\overline{N},t}L^*$ has the integral kernel

$$\mathcal{K}_{\overline{N},t}(x,y)=\int\frac{d\xi}{(2\pi h)^d}\,e^{i(x-y)\xi/h+itp(x,\xi)/h^\mu}q_{\overline{N},\overline{N},t}(x,\xi)l(y,\xi),$$

hence

$$\operatorname{tr}Q_{\overline{N},t}L^*=\int dx\,\mathcal{K}_{\overline{N},t}(x,x)=\int\frac{dx\,d\xi}{(2\pi h)^d}\,e^{itp/h^\mu}lq_{\overline{N},\overline{N},t}. \tag{3.13}$$

Therefore using Lemma 3.3 with $k=n-1$ and $b=h^{(n-1)\mu}q^\circ_{\overline{N},n}l\in S^0_{0,\delta}$ for $n=2,\dots,\overline{N}$ we can find $\widetilde{q}_{\overline{N}}\in S^0_{0,\delta}$ such that $\operatorname{supp}\widetilde{q}_{\overline{N}}\subset\operatorname{supp}l$ and

$$\operatorname{tr}Q_{\overline{N},t}L^*=\sum_{0\le n\le\overline{N}}t^nJ_t(q^\circ_{\overline{N},n}l)=J_t(l^2)+tJ_t(\widetilde{q}_{\overline{N}}).$$

Changing the order of integrals we find

$$\int_{-\infty}^{\infty}\frac{dt}{2\pi h^\mu}\,f_h(t)J_t(l^2)=\int\frac{dx\,d\xi}{(2\pi h)^d}\,l(x,\xi)^2\widetilde{f}_h(p(x,\xi)). \tag{3.14}$$

Since $\operatorname{supp}(\widetilde{f}_h-\mathbb{1}_{]E';E[})\subset[E'-h^\mu;E'+h^\mu]\cup[E-h^\mu;E+h^\mu]$, we can write (3.14) as

$$\int_{E'<p(x,\xi)<E}\frac{dx\,d\xi}{(2\pi h)^d}\,l(x,\xi)^2+\mathcal{R}(l,p-E,h^\mu)+\mathcal{R}(l,p-E',h^\mu),$$

where

$$\mathcal{R}(l,p-E,h^\mu):=O(h^{-d})\int_{E-h^\mu\le p(x,\xi)\le E+h^\mu}dx\,d\xi\,l(x,\xi)^2 \tag{3.15}$$

can be estimated by $O(h^{\mu-d})$ due to the hypothesis (2.9), and the same is true with E' instead of E. Next we use $tf_h(t)=-ih^\mu(\mathcal{F}_{h^\mu}\widetilde{f}_h')(t)$ to write

$$\int_{-\infty}^{\infty}\frac{dt}{2\pi h^\mu}\,f_h(t)tJ_t(\widetilde{q}_{\overline{N}})=\int\frac{dx\,d\xi}{(2\pi h)^d}\,\widetilde{q}_{\overline{N}}(x,\xi)(-i)h^\mu\widetilde{f}_h'(p(x,\xi)) \tag{3.16}$$

and we complete the proof of (2.19) by observing that since $\widetilde{q}_{\overline{N}}=O(1)$, $\operatorname{supp}\widetilde{q}_{\overline{N}}\subset\operatorname{supp}l$, $h^\mu\widetilde{f}_h'=O(1)$ and $\operatorname{supp}\widetilde{f}_h'\subset[E'-h^\mu;E'+h^\mu]\cup[E-h^\mu;E+h^\mu]$, we can estimate (3.16) as before by

$$\mathcal{R}(\widetilde{q}_{\overline{N}},p-E,h^\mu)+\mathcal{R}(\widetilde{q}_{\overline{N}},p-E',h^\mu)=O(h^{\mu-d}).\ \blacksquare$$

4. Auxiliary notations and properties

(A) *Expression* $\widetilde{Q}_{\overline{N},t-\tau}$. Set $\mathcal{V}=\{(t,\tau)\in\mathbb{R}^*\times\mathbb{R}:\tau/t\in[0;1]\}$.

For $s\in\mathbb{R}$ we will write $b\in\widetilde{S}^s$ if $b=(b_{h,t,\tau})_{h\in]0;1[,\,(t,\tau)\in\mathcal{V}}$ satisfies

$$|\partial_\xi^\alpha\partial_{x,y}^\beta b_{h,t,\tau}(x,\xi,y)|\le C_{\alpha,\beta}h^{-s-|\beta|\delta}(1+|t|)^{C_0} \tag{4.1}$$

for every $(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^{2d}$ and $\operatorname{supp}b_{h,t,\tau}\subset(\operatorname{supp}l)\times\mathbb{R}^d$.

Further on, $\|\cdot\|$ is the norm in the space $B(L^2(\mathbb{R}^d))$ of bounded operators, and $\|B\|_{\mathrm{tr}} := \mathrm{tr}\,(B^*B)^{1/2}$ is the trace class norm. If $b \in \widetilde{S}^s$ then the formula

$$(4.2)\quad (\mathrm{Op}_{t,\tau}(b)\varphi)(x) = \int \frac{dy\,d\xi}{(2\pi h)^d}\, e^{i(x-y)\xi/h + i(t-\tau)p(x,\xi)/h^\mu} b_{h,t,\tau}(x,\xi,y)\varphi(y)$$

defines $\mathrm{Op}_{t,\tau}(b) \in B(L^2(\mathbb{R}^d))$ satisfying

$$(4.3)\qquad \|\mathrm{Op}_{t,\tau}(b)\| \le Ch^{-s}(2+|t|)^{C_0}(1+|t|/h^\mu)^{2d}$$

for some constant $C > 0$ due to the Calderón–Vaillancourt theorem (cf. e.g. [7, Section 18.6]). As proved at the end of Section 6, $b \in \widetilde{S}^s$ also satisfies

$$(4.4)\qquad \|\mathrm{Op}_{t,\tau}(b)\|_{\mathrm{tr}} \le h^{-s-5d}(2+|t|)^{C'}.$$

Using (2.21), (3.6($\overline{N}$)) and $P = P^* = p^\pm(x, hD)^*$, we find

$$Q_{\overline{N},t}P = \mathrm{Op}_{t,0}(q_{\overline{N},\overline{N},t}(x,\xi)\,\overline{p^\pm(y,\xi)}),$$

hence writing the Taylor development of $y \mapsto \overline{p^\pm(y,\xi)}$ at x and using

$$(y-x)^\alpha e^{i(x-y)\xi/h} = (ih)^{|\alpha|}\partial^\alpha_{\xi_j}(e^{i(x-y)\xi/h})$$

to perform standard integrations by parts in (4.2), we find

$$(4.5)\qquad \widetilde{Q}_{\overline{N},t} = \mathrm{Op}_{t,0}((\widetilde{\mathcal{P}}_{\overline{N}}q_{\overline{N},\overline{N},t})(x,\xi) + r_{\overline{N},t}),$$

with the remainder term of the Taylor development of order $\overline{N}$,

$$(4.6)\qquad r_{\overline{N},t}(x,\xi,y) = e^{-itp(x,\xi)/h^\mu}(\overline{N}+1)\int_0^1 d\sigma\,(1-\sigma)^{\overline{N}}\widetilde{r}_{\overline{N},\sigma,t}(x,\xi,y),$$

where

$$(4.7)\qquad \widetilde{r}_{\overline{N},\sigma,t}(x,\xi,y) = \sum_{|\alpha|=\overline{N}+1}\frac{h^{\overline{N}+1-\mu}}{i^{\overline{N}+2}\alpha!}\,\partial^\alpha_\xi\big((q_{\overline{N},\overline{N},t}e^{itp/h^\mu})(x,\xi)\,\overline{\partial^\alpha_x p^\pm(x+\sigma(y-x),\xi)}\big).$$

Let $\widetilde{q}^{\,\circ}_{\overline{N},\overline{N},n}$ be as in Proposition 3.2 and define

$$(4.8)\qquad \widetilde{q}^{\,\circ}_{\overline{N},n,t,\tau}(x,\xi,y) = (1-\tau/t)^n\,\widetilde{q}^{\,\circ}_{\overline{N},\overline{N},n}(x,\xi).$$

Then $\widetilde{q}^{\,\circ}_{\overline{N},n} \in \widetilde{S}^{n\varrho} \subset \widetilde{S}^{n\mu-\overline{N}(1-\delta)}$ and we can write

$$(4.9)\qquad (\widetilde{\mathcal{P}}_{\overline{N}}q_{\overline{N},\overline{N},t-\tau})(x,\xi) = \sum_{\overline{N}\le n\le 2\overline{N}} t^n\widetilde{q}^{\,\circ}_{\overline{N},n,t,\tau}(x,\xi,y).$$

Using the form of $q_{\overline{N},\overline{N},t}$ in (4.6–7), as in the proof of Lemma 3.1 we find $r^\circ_{\overline{N},n} \in \widetilde{S}^{n\mu-\overline{N}(1-\delta)}$ such that

$$(4.10)\qquad r_{\overline{N},t-\tau} = \sum_{0\le n\le 2\overline{N}} t^n r^\circ_{\overline{N},n,t,\tau}$$

and deduce

Corollary 4.1. *There exist* $\widetilde{q}_{\overline{N},n} \in \widetilde{S}^{n\mu-\overline{N}(1-\delta)}$ *such that*

$$\widetilde{Q}_{\overline{N},t-\tau} = \sum_{0\le n\le 2\overline{N}} t^n \operatorname{Op}_{t,\tau}(\widetilde{q}_{\overline{N},n}). \tag{4.11}$$

(B) *Auxiliary operator algebra* $\mathcal{Y}$. It is easy to see that $U_t : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ and for any $B : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ we can define $Y_\tau(B) := U_\tau^* B U_\tau$. We write $Y \in \mathcal{Y}_0$ if there is $N \in \mathbb{N}$ such that $Y = (Y_\tau)_{\tau\in\mathbb{R}}$ has the form

$$Y_\tau = \sum_{1\le k\le N} \int_{[0;1]^N} dw\, s'_{w,k,\tau} Y_{\tau s_{w,k,1}}(B_{k,1}) \cdots Y_{\tau s_{w,k,N}}(B_{k,N}), \tag{4.12}$$

where $B_{k,k'} = b_{k,k'}(x, hD)$ with

$$\begin{cases} b_{k,k'} \in S^0_{0,\delta} & \text{for } (k,k') \in \{1,\dots,N\}^2, \\ s_{w,k,k'} \in [-1;1] & \text{for } (w,k,k') \in [0;1]^N \times \{1,\dots,N\}^2, \\ s'_{w,k,\tau} \in \mathbb{C}, \quad |s'_{w,k,\tau}| \le (2+|\tau|)^N & \text{for } (w,k,\tau) \in [0;1]^N \times \{1,\dots,N\} \times \mathbb{R}. \end{cases}$$

We observe that taking $s_{w,k,k'} = s'_{w,k,\tau} = 1$ we obtain $Y_\tau(B_{k,1}) \cdots Y_\tau(B_{k,N}) \in \mathcal{Y}_0$ and it is not difficult to see that $\mathcal{Y}_0$ is an algebra with the property

$$Y \in \mathcal{Y}_0 \;\Rightarrow\; \exists C > 0,\ \|Y_\tau\| \le (2+|\tau|)^C. \tag{4.13}$$

We define $\widehat{S}^s_{0,\delta}$ as the class of symbols b such that $b_n(x,\xi) = b(x,\xi)(1+|\xi|^{2n})$ belongs to $S^s_{0,\delta}$ for any $n \in \mathbb{N}$. Finally, we define $\mathcal{Y}$ as the set of operators $Y \in \mathcal{Y}_0$ of the form (4.12), where $b_{k,k'}$, $s_{w,k,k'}$, $s'_{w,k,\tau}$ are as before and for every $k \in \{1,\dots,N\}$ there is $k' \in \{1,\dots,N\}$ such that $b_{k,k'} \in \widehat{S}_{0,\delta}$.

For $b \in \widetilde{S}^s$, $Y \in \mathcal{Y}$ we define

$$J_{t,\tau}(b, Y) := \operatorname{tr} \operatorname{Op}_{t,\tau}(b) U_\tau Y_\tau. \tag{4.14}$$

Using this notation and (4.11) we can write

$$\operatorname{tr} \widetilde{Q}_{\overline{N},t-\tau} U_\tau L^* = \sum_{0\le n\le 2\overline{N}} t^n J_{t,\tau}(\widetilde{q}_{\overline{N},n}, L^*). \tag{4.15}$$

In Section 5 we will prove

Proposition 4.2. *Let* $Y \in \mathcal{Y}$ *and* $b_0 \in \widetilde{S}^s$. *Then for every* $n \in \mathbb{N}$ *one can find* $K_n \in \mathbb{N}^*$ *and* $b_{n,k} \in \widetilde{S}^s$, $Y_{n,k} \in \mathcal{Y}$ *for* $k = 1,\dots,K_n$ *such that*

$$h^{-n\mu} t^n J_{t,\tau}(b_0, Y) = \sum_{1\le k\le K_n} J_{t,\tau}(b_{n,k}, Y_{n,k}). \tag{4.16}$$

It is easy to see that Proposition 4.2 implies the estimates (2.20). Indeed, Proposition 4.2 allows us to replace each term of (4.15) by $J_{t,\tau}(b_{\overline{N},n,k}, Y_{\overline{N},n,k})$ with $b_{\overline{N},n,k} \in \widetilde{S}^{-\overline{N}(1-\delta)}$, hence (2.20) follows from (4.13) and (4.4).

(C) *Commutators with* x_j/h. We denote by x_j the operator of multiplication by the jth coordinate.

LEMMA 4.3. *If $Y \in \mathcal{Y}$ and $\varrho = \mu - 1 + \delta$, then there exist $\widetilde{Y}^+, \widetilde{Y}^- \in \mathcal{Y}$ such that*

$$[Y_\tau, x_j/h] = \widetilde{Y}^+_\tau + h^{-\varrho}\tau\widetilde{Y}^-_\tau. \tag{4.17}$$

Proof. Let $B = b(x, hD)$ with $b \in \widehat{S}^0_{0,\delta}$. We will show that

$$[Y_\tau(B_{k,k'}), x_j/h] = Y^+_\tau + h^{-\varrho}\tau Y^-_\tau \tag{4.18}$$

with some $Y^+, Y^- \in \mathcal{Y}$. To start we write

$$[Y_\tau(B), x_j/h] = U^*_\tau[B, Y_{-\tau}(x_j/h)]U_\tau \tag{4.19}$$

and introduce

$$P_j := [iP, x_j/h] = \partial_{\xi_j} p^\pm(x, hD). \tag{4.20}$$

Then we can write the Taylor formula

$$\begin{aligned} Y_{-\tau}(x_j/h) &= x_j/h - \tau(\partial_\tau Y_\tau)|_{\tau=0}(x_j/h) + \tau^2 \int_0^1 d\sigma\,(1-\sigma)\partial^2_\tau Y_{-\sigma\tau}(x_j/h) \\ &= x_j/h + h^{-\mu}\tau P_j + h^{-2\mu}\tau U_\tau \overline{Y}_\tau([P_j, iP])U^*_\tau, \end{aligned} \tag{4.21}$$

where $\overline{Y}_\tau(\widetilde{B}) := \tau\int_0^1 d\sigma\,(1-\sigma)Y_{(1-\sigma)\tau}(\widetilde{B})$ for any $\widetilde{B} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. Using (4.21) we can express the commutator (4.19) in the form

$$\begin{aligned} &Y_\tau([B, x_j/h]) + h^{-\varrho}\tau Y_\tau(h^{\delta-1}[B, P_j]) \\ &\qquad + h^{-\varrho}\tau[Y_\tau(B), \overline{Y}_\tau(h^{\delta-1-\mu}[P_j, P])] \end{aligned} \tag{4.22}$$

and since $[B,\, x_j/h] = -i\partial_{\xi_j} b(x, hD)$, the first term of (4.22) belongs to $\mathcal{Y}$.

Then using Lemma 6.3 we find that $h^{\delta-1}[B, P_j] = b_j(x, hD)$ with $b_j \in \widehat{S}^0_{0,\delta}$ and $h^{\delta-1-\mu}[P, P_j] = p_j(x, hD)$ with $(1 + |\xi|^2)^{-2m} p_j(x, \xi)$ belonging to $S^0_{0,\delta}$. Thus choosing $E'_0 \in \mathbb{R}$ large enough we have $P' = P + E'_0 I \geq I$ and we can write

$$\overline{Y}_\tau(h^{\delta-1-\mu}[P_j, P])] = Y_\tau(BP'^2)\overline{Y}_\tau(h^{\delta-1-\mu}P'^{-2}[P_j, P])].$$

Then $BP'^2 = \widetilde{b}(x, hD)$ with $\widetilde{b} \in \widehat{S}^0_{0,\delta}$ and the standard parametrix construction (cf. e.g. [7, Theorem 8.1.9]) gives $P'^{-2}h^{\delta-1-\mu}[P, P_j] = \widetilde{p}_j(x, hD)$ with $\widetilde{p}_j \in S^0_{0,\delta}$, completing the proof of (4.18).

If we assume only $B = b(x, hD)$ with $b \in S^0_{0,\delta}$, then reasoning as above we can find $Y^+, Y^- \in \mathcal{Y}_0$ such that

$$[Y_\tau(B_{k,k'}),\, x_j/h] = Y^+_\tau + h^{-\varrho}\tau Y^-_\tau P'^2. \tag{4.18$'$}$$

It is clear that to obtain the general statement of Lemma 4.3 it remains to commute successively x_j/h with $Y_{\tau s_{w,k,k'}}(B_{k,k'})$, $k' \in \{1, \ldots, N\}$. If $b_{k,k'} \in S^0_{0,\delta}$, then we still have $P'^2 B_{k,k'} P'^{-2} = \widetilde{b}_{k,k'}(x, hD)$ with $b_{k,k'} \in S^0_{0,\delta}$, hence P'^2 from the last term of (4.18′) can be put near $b_{k,k'}(x,hD)$ with k' such that $b_{k,k'}\in\widehat{S}^0_{0,\delta}$, i.e. $Y\in\mathcal{Y}\Rightarrow YP'^2\in\mathcal{Y}$. ∎

LEMMA 4.4. *Let $Y\in\mathcal{Y}$ and $P_j=[iP,x_j/h]$. Then one can find $Y^+,Y^-\in\mathcal{Y}$ such that*

$$h^{-\mu}\tau P_jU_\tau Y_\tau=[U_\tau Y_\tau,\,x_j/h]+U_\tau(Y^+_\tau+h^{-\varrho}\tau Y^-_\tau). \tag{4.23}$$

Proof. Using (4.21) to express $[U_\tau,\,x_j/h]=(Y_{-\tau}(x_j/h)-x_j/h)U_\tau$ and applying Lemma 4.3, we find

$$[U_\tau Y_\tau,\,x_j/h]=[U_\tau,\,x_j/h]Y_\tau+U_\tau[Y_\tau,\,x_j/h]$$
$$h^{-\mu}\tau P_jU_\tau Y_\tau+h^{-\varrho}\tau U_\tau\overline{Y}_\tau(h^{\delta-1-\mu}[P_j,iP])Y_\tau+U_\tau(\widetilde{Y}^+_\tau+h^{-\varrho}\tau\widetilde{Y}^-_\tau),$$

i.e. (4.23) holds with $Y^-_\tau=-\overline{Y}_\tau(h^{\delta-1-\mu}[P_j,iP])Y_\tau-\widetilde{Y}^-_\tau$, $Y^+_\tau=-\widetilde{Y}^+_\tau$. ∎

5. End of proof of Theorem 2.2. It remains to prove Proposition 4.2. We start with

LEMMA 5.1. *Let $b\in\widetilde{S}^s$. Define $b\partial_{\xi_j}p\in\widetilde{S}^s$ by*

$$(b\partial_{\xi_j}p)(x,\xi,y)=b(x,\xi,y)\partial_{\xi_j}p(x,\xi). \tag{5.1}$$

Then

$$[\mathrm{Op}_{t,\tau}(b),\,x_j/h]=h^{-\mu}(t-\tau)\mathrm{Op}_{t,\tau}(b\partial_{\xi_j}p)-\mathrm{Op}_{t,\tau}(i\partial_{\xi_j}b). \tag{5.2}$$

Proof. Since the integral kernel of $[\mathrm{Op}_{t,\tau}(b),x_j/h]$ is

$$(x,y)\mapsto\int\frac{d\xi}{(2\pi h)^d}\,\frac{y_j-x_j}{h}\,e^{i(x-y)\xi/h+i(t-\tau)p(x,\xi)/h^\mu}b(x,\xi,y), \tag{5.3}$$

using $\frac{y_j-x_j}{h}\,e^{i(x-y)\xi/h}=i\partial_{\xi_j}e^{i(x-y)\xi/h}$ to integrate by parts, we can rewrite it as

$$\int\frac{d\xi}{(2\pi h)^d}\,e^{i(x-y)\xi/h+i(t-\tau)p(x,\xi)/h^\mu}(h^{-\mu}(t-\tau)\partial_{\xi_j}pb-i\partial_{\xi_j}b)(x,\xi,y).$$ ∎

PROPOSITION 5.2. *Let $b\in\widetilde{S}^s$, $Y\in\mathcal{Y}$, $j\in\{1,\dots,d\}$ and set*

$$p_{(j),t,\tau}(x,\xi,y)=\left(1-\frac{\tau}{t}\right)\partial_{\xi_j}p(x,\xi)+\frac{\tau}{t}\,\overline{\partial_{\xi_j}p^\pm(y,\xi)}. \tag{5.4}$$

Then there exist $b_k\in\widetilde{S}^s$ and $Y_k\in\mathcal{Y}$ such that

$$h^{-\mu}tJ_{t,\tau}(p_{(j)}b,Y)=\sum_{1\le k\le2}(J_{t,\tau}(b_k,Y_k)+h^{-\varrho}tJ_{t,\tau}(b_{-k},Y_{-k})). \tag{5.5}$$

Proof. Since $P_j=P_j^*=\partial_{\xi_j}p^\pm(x,hD)^*$ is a differential operator of order $2m-1$, the standard composition formula gives

$$\mathrm{Op}_{t,\tau}(b)P_j=\mathrm{Op}_{t,\tau}(b\partial_{\xi_j}p^\sharp+h^{1-\delta}\widetilde{b}), \tag{5.6}$$

where $(b\partial_{\xi_j}p^\sharp)(x,\xi,y) = b(x,\xi,y)\,\overline{\partial_{\xi_j}p^\pm(y,\xi)}$ and $\widetilde{b} \in \widetilde{S}^s$ has the form

$$\widetilde{b}(x,\xi,y) = \sum_{1\le|\alpha|\le 2m-1} \frac{h^{|\alpha|-1+\delta}}{i^{|\alpha|}\alpha!}\,\partial_y^\alpha b(x,\xi,y)\,\overline{\partial_\xi^\alpha\partial_{\xi_j}p^\pm(y,\xi)}.$$

By definition $h^{-\mu}t\,J_{t,\tau}(p_{(j)}b,Y)$ can be expressed as

$$h^{-\mu}(t-\tau)\operatorname{tr}\operatorname{Op}_{t,\tau}(b\partial_{\xi_j}p)U_\tau Y_\tau + h^{-\mu}\tau\operatorname{tr}\operatorname{Op}_{t,\tau}(b\partial_{\xi_j}p^\sharp)U_\tau Y_\tau. \tag{5.7}$$

Due to (5.2) the first term of (5.7) can be written as

$$\operatorname{tr}[\operatorname{Op}_{t,\tau}(b),\, x_j/h]U_\tau Y_\tau + J_{t,\tau}(i\partial_{\xi_j}b,Y) \tag{5.8}$$

and (5.6) allows us to write the second term of (5.7) in the form

$$h^{-\mu}\operatorname{tr}\operatorname{Op}_{t,\tau}(b)\tau P_j U_\tau Y_\tau \;+\; h^{-\varrho}\tau J_{t,\tau}(\widetilde{b},Y). \tag{5.9}$$

Then by (4.23), the first term of (5.9) can be written as

$$\operatorname{tr}\operatorname{Op}_{t,\tau}(b)[U_\tau Y_\tau, x_j/h] \;+ J_{t,\tau}(b,Y^+) + h^{-\varrho}\tau J_{t,\tau}(b,Y^-). \tag{5.10}$$

Taking $b_1 = i\partial_{\xi_j}b$, $Y_1 = Y$, $b_2 = b$, $Y_2 = Y^+$, $b_{-1} = (\tau/t)\widetilde{b}$, $Y_{-1} = Y$, $b_{-2} = (\tau/t)b$ and $Y_{-2} = Y^-$ we obtain (5.5) by observing that

$$\begin{aligned}\operatorname{tr}[\operatorname{Op}_{t,\tau}(b), x_j/h]U_\tau Y_\tau &+ \operatorname{tr}\operatorname{Op}_{t,\tau}(b)[U_\tau Y_\tau, x_j/h]\\ &= \operatorname{tr}[\operatorname{Op}_{t,\tau}(b)U_\tau Y_\tau,\, x_j/h] = 0. \ \blacksquare\end{aligned}$$

LEMMA 5.3. *Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi = 1$ on a neighbourhood of* 0. *Let $b \in \widetilde{S}^s$ and $b_\chi(x,\xi,y) = b(x,\xi,y)\chi(x-y)$. Then one can find $b' \in \widetilde{S}^{s-1}$ such that $\operatorname{Op}_{t,\tau}(b) = \operatorname{Op}_{t,\tau}(b_\chi + b')$.*

Proof. Let $\varepsilon = \min\{1-\delta, 1-\mu\}$. We are going to show that for every $n \in \mathbb{N}$ one can find $b_n \in \widetilde{S}^{s-n\varepsilon}$ such that $\operatorname{Op}_{t,\tau}(b-b_\chi) = \operatorname{Op}_{t,\tau}(b_n)$ and $b_n(x,\xi,y) = 0$ for $x-y$ in a small neighbourhood of 0. Reasoning by induction we assume the existence of such b_n for a given $n \in \mathbb{N}$. Then we can find $b_{n,j} \in \widetilde{S}^{s-n\varepsilon}$ such that $b_n = \sum_{1\le j\le d}(x_j-y_j)b_{n,j}$ and $b_{n,j}(x,\xi,y) = 0$ for $y-x$ in a neighbourhood of 0. Next we observe that decomposing $\operatorname{Op}_{t,\tau}((x_j-y_j)b_{n,j})$ as a sum

$$\operatorname{Op}_{t,\tau}((x_j-y_j+h^{1-\mu}(t-\tau)\partial_{\xi_j}p)b_{n,j}) - h^{1-\mu}(t-\tau)\operatorname{Op}_{t,\tau}(\partial_{\xi_j}p b_{n,j}), \tag{5.11}$$

we can write the first term of (5.11) as $\operatorname{Op}_{t,\tau}(ih\partial_{x_j}b_{n,j})$. Thus (5.11) can be written as $\operatorname{Op}_{t,\tau}(b_{n+1,j})$ with $b_{n+1,j} = ih\partial_{x_j}b_{n,j} - (t-\tau)h^{1-\mu}\partial_{\xi_j}p b_{n,j} \in \widetilde{S}^{s-(n+1)\varepsilon}$ and $b_{n+1,j}(x,\xi,y) = 0$ for $x-y$ in a small neighbourhood of 0, i.e. the statement holds for $n+1$. ∎

Proof of Proposition 4.2. Choosing $h_0 > 0$ small enough we obtain $\sum_{j=1}^d |p_{(j)}(x,\xi,x)| \ge c/2$ for $(x,\xi) \in \operatorname{supp} l$ and $h < h_0$. Then using Lemma 5.3 we can modify b_0 to get $\sum_{j=1}^d |p_{(j)}(x,\xi,y)| \ge c/3$ for $(x,\xi,y) \in$

$\operatorname{supp} b_0$. Therefore we can find $b_{(j)} \in \widetilde{S}^s$ such that $b_0 = \sum_{j=1}^d b_{(j)} p_{(j)}$ and applying Proposition 4.2 we can write

$$h^{-\mu} t J_{t,\tau}(b_0, Y) = \sum_{1 \le k \le K} (J_{t,\tau}(b_k, Y_k) + h^{1-\delta-\mu} t J_{t,\tau}(b_{-k}, Y_{-k})) \tag{5.12}$$

with some $b_k \in \widetilde{S}^s$, $Y_k \in \mathcal{Y}$. Applying the analogous reasoning to express $h^{-\mu} t\, J_{t,\tau}(b_{-k}, Y_{-k})$, $k = 1, \ldots, K$, we can write a formula similar to (5.12) with $h^{1-\delta-\mu}$ replaced by $h^{2(1-\delta)-\mu}$ and after N iterations we can replace $h^{1-\delta-\mu}$ by $h^{N(1-\delta)-\mu}$. Thus for $N \ge \mu/(1-\delta)$ we obtain (4.16) with $n = 1$, and the statement of Proposition 4.2 clearly follows by induction on $n \in \mathbb{N}$. ■

6. Appendix

(A) *Proof of (2.2) and (2.3).* Dropping the indices $\nu, \overline{\nu}$ we can write

$$\partial_x^\alpha a_h(x) = \int a(y) \gamma_{h^\delta}^{(\alpha)}(x-y) h^{-\delta|\alpha|}\, dy. \tag{6.1}$$

Further on we assume $|\alpha| \ge 1$, hence $\int \gamma_{h^\delta}^{(\alpha)}(x-y)\, dy = 0$ and (6.1) still holds if $a(y)$ is replaced by $a(y) - a(x)$. Therefore

$$\begin{aligned}|\partial_x^\alpha a_h(x)| &\le \int |a(y) - a(x)|\, |\gamma_{h^\delta}^{(\alpha)}(x-y)| h^{-\delta|\alpha|}\, dy \\ &\le C \int |y - x|^r |\gamma_{h^\delta}^{(\alpha)}(x-y)| h^{-\delta|\alpha|}\, dy = C h^{(r-|\alpha|)\delta} \int |y|^r |\gamma^{(\alpha)}(y)|\, dy\end{aligned}$$

completes the proof of (2.3). To obtain (2.2) we write

$$a_h(x) - a(x) = \int (a(y) - a(x)) \gamma_{h^\delta}(x-y)\, dy,$$

i.e. $|a_h(x) - a(x)| \le C \int |y-x|^r |\gamma_{h^\delta}(x-y)|\, dy = C_r h^{r\delta}$.

(B) We describe how to deduce Theorem 2.1 from Theorem 2.2. Let $E' \in \mathbb{R}$ be fixed such that $p(x,\xi) \ge E'$ and the spectrum $\sigma(P)$ is contained in $[E'; \infty[$. Next we fix $E_1 < E_0$ and consider $\widetilde{g} \in C_0^\infty(]-\infty; E_1[)$. We observe that (2.5) allows us to find $h_{\widetilde{g}} > 0$ such that $\operatorname{supp}(\widetilde{g} \circ p) \subset \overline{\Gamma}_{E_1} \subset \Gamma_{E_0}$ for $h \in]0; h_{\widetilde{g}}]$, and we recall our hypothesis that Γ_{E_0} is bounded. Then reasoning as in the proof of Lemma 3.1 of [17] we obtain

Proposition 6.1. *Let $\widetilde{g}$ be as above. Fix $\widetilde{l} \in C_0^\infty(\mathbb{R}^{2d})$ such that $\widetilde{l} = 1$ on a neighbourhood of $\overline{\Gamma}_{E_1}$ and set $\widetilde{L} = \widetilde{l}(x, hD)$. Then for every $N \in \mathbb{N}$,*

$$\|\widetilde{g}(P)(I - \widetilde{L})\|_{\mathrm{tr}} = O(h^N). \tag{6.2}$$

If $E < E_1 < E_0$ then we can find $\widetilde{g}$ as above satisfying $\widetilde{g} \ge \mathbb{1}_{[E';E]}$. Now

$$\|\mathbb{1}_{[E';E]}(P)\|_{\mathrm{tr}} \le \|\widetilde{g}(P)\|_{\mathrm{tr}} \le \|\widetilde{g}(P)\|\, \|\widetilde{L}\|_{\mathrm{tr}} + O(h^N) = O(h^{-d})$$

follows from (6.2) and from the well known estimate $\|\widetilde{l}(x, hD)\|_{\mathrm{tr}} = O(h^{-d})$.

PROPOSITION 6.2. *If* $\widetilde{g}$ *is as above*, *then*

$$\|\widetilde{g}(P)-(\widetilde{g}\circ p)(x,hD)\|_{\mathrm{tr}}=O(h^{\mu-d}). \tag{6.3}$$

Proof. Let $\widetilde{l}$ be as in Proposition 6.1 and $Q_t^\circ=(e^{itp}\widetilde{l})(x,hD)$. Then it is easy to see that computing $\widetilde{Q}_t^\circ=\frac{d}{dt}Q_t^\circ-iPQ_t^\circ$ we can apply (4.3) with $s=-\delta r$, $\mu=0$ to obtain

$$\|\widetilde{Q}_t^\circ\widetilde{L}^*\|_{\mathrm{tr}}\le\|\widetilde{Q}_t^\circ\|\,\|\widetilde{L}^*\|_{\mathrm{tr}}\le h^{\delta r-d}(2+|t|)^C.$$

Reasoning as in (2.18) we find $\|(\widetilde{L}e^{itP}-Q_t^\circ)L^*\|_{\mathrm{tr}}\le h^{\delta r-d}(2+|t|)^{C+1}$, and introducing the Fourier transform $\mathcal{F}\widetilde{g}=g\in\mathcal{S}(\mathbb{R})$ to write

$$(\widetilde{L}\widetilde{g}(P)-(\widetilde{g}\circ p)(x,hD))\widetilde{L}^*=\int_{-\infty}^{\infty}\frac{dt}{2\pi}\,g(t)(\widetilde{L}e^{itP}-Q_t^\circ)\widetilde{L}^*,$$

we obtain $\|(\widetilde{L}\widetilde{g}(P)-(\widetilde{g}\circ p)(x,hD))\widetilde{L}^*\|_{\mathrm{tr}}=O(h^{\mu-d})$, which implies (6.3) due to (6.2). ∎

From (1.7) we have $E\in\,]E_1';E_1[$ with some $E_1,E_1'\in\mathbb{R}$ satisfying

$$E_1'\le a(x,\xi)\le E_1\Rightarrow\nabla_\xi a(x,\xi)\ne0, \tag{6.4}$$

and by (2.5) we can find $c>0$ and $h_0>0$ such that

$$E_1'\le p(x,\xi)\le E_1\Rightarrow|\nabla_\xi p(x,\xi)|\ge c \tag{6.5}$$

for $h\in\,]0;h_0]$. Let $g_0\in C_0^\infty(]E_1';E_1[)$ and $g_1\in C_0^\infty(]-\infty;E[)$ be real-valued, $g_0=1$ in a neighbourhood of E and $g_1+g_0^2=1$ on $[E';E]$. Then

$$\mathcal{N}(P,E)=\operatorname{tr}\mathbb{1}_{[E';E]}(P)=\operatorname{tr}g_1(P)+\operatorname{tr}(g_0^2\mathbb{1}_{[E';E]})(P). \tag{6.6}$$

Using (6.3) with g_1, g_0 in place of $\widetilde{g}$ we find that (6.6) can be written as

$$\operatorname{tr}(g_1\circ p)(x,hD)+\operatorname{tr}L\mathbb{1}_{[E';E]}(P)L^*+O(h^{\mu-d}), \tag{6.7}$$

where $L=l(x,hD)$ with $l=g_0\circ p$. Since $(x,\xi)\in\operatorname{supp}l\Rightarrow E_1'\le p(x,\xi)\le E_1$, (6.5) yields (2.9) and we can use Theorem 2.2 to express (6.7) as

$$\int\frac{dx\,d\xi}{(2\pi h)^d}\,g_1(p(x,\xi))+\int\frac{dx\,d\xi}{(2\pi h)^d}\,(g_0^2\mathbb{1}_{[E';E]})(p(x,\xi))+O(h^{\mu-d}). \tag{6.8}$$

Since $g_1+g_0^2\mathbb{1}_{[E';E]}=\mathbb{1}_{[E';E]}$, the sum of the two integrals from (6.8) gives $(2\pi h)^{-d}\int_{p<E}dx\,d\xi$. Finally, (2.5) and (1.7) imply

$$\varGamma_E-C'h^\mu\le\varGamma_{E-Ch^\mu}\le\int_{p<E}dx\,d\xi\le\varGamma_{E+Ch^\mu}\le\varGamma_E+C'h^\mu. \ ∎$$

(C) Proposition 1.1 now follows as described in the Appendix of [17].

(D) *Some properties of pseudodifferential operators.* From well known composition formulas (cf. e.g. [7, Theorem 18.5.4 and 18.5.10]) we have

LEMMA 6.3. *Let* $b \in S^s_{0,\delta}$, $\widetilde{b} \in S^{\widetilde{s}}_{0,\delta}$. *Then there is* $b \diamond \widetilde{b} \in S^{s+\widetilde{s}}_{0,\delta}$ *such that*

$$b(x, hD)\widetilde{b}(x, hD) = (b \diamond \widetilde{b})(x, hD).$$

Let $s', s'', \widetilde{s}', \widetilde{s}'' \in \mathbb{R}$ *be such that the conditions*

$$\partial_{x_k} b \in S^{s'}_{0,\delta}, \quad \partial_{\xi_k} b \in S^{s''}_{0,\delta}, \quad \partial_{x_k} \widetilde{b} \in S^{\widetilde{s}'}_{0,\delta}, \quad \partial_{\xi_k} \widetilde{b} \in S^{\widetilde{s}''}_{0,\delta}$$

hold for $k \in \{1, \ldots, d\}$, *and let* $\overline{s} = \max\{s' + \widetilde{s}'', s'' + \widetilde{s}'\} - 1$. *Then*

$$[b(x, hD), \widetilde{b}(x, hD)] = b'(x, hD) \quad \text{with } b' \in S^{\overline{s}}_{0,\delta}.$$

Proof of (4.4). Let $l_d(x, \xi) = (1 + |x|^2)^d (1 + |\xi|^2)^d$. Then $l_d(x, hD)^{-1}$ is of trace class and

$$\|\mathrm{Op}_{t,\tau}(b)\|_{\mathrm{tr}} \le \|(l_d(x, hD)^*)^{-1}\|_{\mathrm{tr}} \|l_d(x, hD)^* \mathrm{Op}_{t,\tau}(b)\|,$$

where $l_d(x, hD)^* \mathrm{Op}_{t,\tau}(b) = \mathrm{Op}_{t,\tau}(b_d)$ with $b_d \in \widetilde{S}^{s+2d}$, i.e. (4.4) follows from (4.3). ∎

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