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REPRESENTATION-TAME LOCALLY HEREDITARY ALGEBRAS

BY

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Abstract. Let A be a finite-dimensional algebra over an algebraically closed field. The algebra A is called *locally hereditary* if any local left ideal of A is projective. We give criteria, in terms of the Tits quadratic form, for a locally hereditary algebra to be of tame representation type. Moreover, the description of all representation-tame locally hereditary algebras is completed.

1. Introduction. Throughout the paper, we assume that K is an algebraically closed field. By an *algebra* we mean a finite-dimensional K-algebra (associative, with an identity), which we moreover assume to be basic and connected. An algebra A can be written as a bound quiver algebra $A = KQ/I$, where $Q = Q_A$ is the Gabriel quiver of A and I is an admissible ideal in the path algebra KQ of Q. We denote by $\operatorname{mod} A$ the category of all finite-dimensional right A-modules. The algebra A is said to be of *tame representation type* (or *representation-tame*) if, for each dimension d, the isomorphism classes of indecomposable modules in $\operatorname{mod} A$ of dimension d form at most finitely many one-parameter families. An algebra A is called of *wild representation type* if there exists an exact representation embedding $K\langle x, y\rangle$-mod $\to A$-mod [5], [14], [17]. From [5] we know that any algebra is either of tame or of wild representation type, and these two classes of algebras are disjoint. The reader is referred to [1] for a representation theory background, and to [5], [16, Sections 14.2–14.4] and [17] for precise definitions of representation-tame and wild algebras.

In [2], Bautista studies the algebras A for which every non-zero homomorphism between indecomposable projective A-modules is a monomorphism (see also [15]). Following [10], we call these algebras *locally hereditary*, or *L-hereditary* for short, because it was observed in [10] that this property of A holds if and only if every local left (and right) ideal of A is projective. We recall that a left ideal L of A is called *local* if L has a unique maximal left submodule.

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The class of locally hereditary algebras is rather large, interesting and plays an important role in the representation theory of algebras.

We recall that the hereditary algebras and the incidence algebras of partially ordered sets are locally hereditary.

The aim of the paper is to give criteria for locally hereditary algebras to be of tame representation type, and to describe the class of all tame locally hereditary algebras.

Following [3], by the *Tits quadratic form* of an arbitrary finite-dimensional bound quiver K-algebra $A = KQ/I$, where Q is a finite quiver without oriented cycle and I is an admissible ideal of KQ, we mean the integral quadratic form $q_A: \mathbb{Z}^n \to \mathbb{Z}$ defined by the formula

$$q_A(x) = \sum_{i\in Q_0} x_i^2 - \sum_{\alpha\in Q_1} x_{s(\alpha)}x_{t(\alpha)} + \sum_{i,j\in Q_0} r_{ij}x_ix_j, \tag{1.1}$$

where $n = |Q_0|$, $s(\alpha)$ and $t(\alpha)$ are the source and target of an arrow $\alpha \in Q_1$, r_{ij} is the cardinality of $R \cap KQ(i,j)$, R is a minimal set of relations which generate the ideal I, and $KQ(i,j)$ is the vector space spanned by the paths from i to j. We also recall that if $A = KQ/I$ is tame, then q_A is *weakly non-negative*, that is, $q_A(x) \geq 0$ for any $x \in \mathbb{Z}^n$ with non-negative coordinates (see [13]).

We know (see Section 2) that every locally hereditary algebra $A = KQ/I$ admits a universal Galois covering

$$\widetilde{A} \to \widetilde{A}/G \cong A, \tag{1.2}$$

where $\widetilde{A}$ is a simply connected locally bounded K-category and $G = \pi_1(Q,I)$ is the fundamental group of the bound quiver (Q,I), which is moreover a finitely generated free group (or is trivial, when A is simply connected).

Our main result is the following theorem proved in Section 3.

THEOREM 1.1. *Let $A = KQ/I$ be a locally hereditary algebra and $\widetilde{A}$ the simply connected universal Galois covering of A. The following conditions are equivalent*:

(i) *The algebra A is of tame representation type.*

(ii) *The Tits form q_B of any finite convex subcategory B of $\widetilde{A}$ is weakly non-negative.*

This completes the results from [8] and [9] to arbitrary locally hereditary algebras; in those papers a theorem similar to the above one is proved for the incidence algebras of finite partially ordered sets.

2. Preliminaries. The following characterization of locally hereditary algebras was stated in [10].

THEOREM 2.1. *Let A be a finite-dimensional algebra over a field. The following conditions are equivalent*:

(i) *A is right locally hereditary.*
(ii) *Any right local ideal of A is projective.*
(iii) *Any local submodule of a right projective module is projective.*
(iv) *A is left locally hereditary.*

The definition of a locally hereditary algebra was introduced in 1978 by D. Simson at the Toruń representation theory seminar. The theorem in the above form was proved in [7], and a variation of it is contained in [15].

Now we list important large classes of locally hereditary algebras.

EXAMPLE 2.2. The *hereditary algebras*, that is, the path algebras KQ of quivers Q without oriented cycles, are locally hereditary.

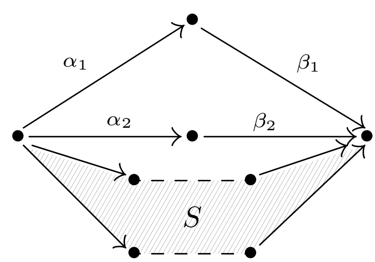
EXAMPLE 2.3. By a *finite poset* (partially ordered set) we mean a bounded quiver (Q, I) with no oriented cycles in Q and with no arrow parallel to another path, where the ideal I in the path algebra KQ is generated by all possible commutativity relations. Then $A = KQ/I$ is called the *incidence algebra* (of the poset Q).

It follows from Theorem 2.1 that the incidence algebra KQ/I of a finite poset Q is locally hereditary. If $I \neq 0$, then KQ/I is not hereditary.

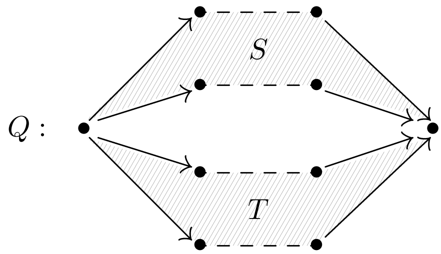
EXAMPLE 2.4. The canonical algebras (in the sense of Ringel [14]) are locally hereditary.

EXAMPLE 2.5. Consider two families of algebras given by the quivers (a) and (b) shown below:

(a) Q :

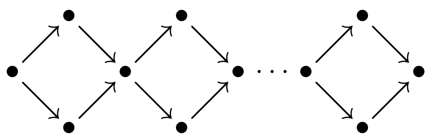


where the subquiver S is a poset and the ideal I is generated by the commutativity relations from S and $\beta_1\alpha_1 + \beta_2\alpha_2 + \sigma$ for some path σ in S from the minimal vertex to the maximal vertex in S;

(b) 

with the relations only in the subquivers S, T.

In (a) and (b) the subquivers S, T are finite posets different from vertices and each of them is a subposet of some garland



We note that the first algebra is an example of a single braid generalized canonical algebra and the second one is a double braid generalized canonical algebra in the sense of [11]. It is easy to see that both algebras are locally hereditary.

The following proposition is a direct consequence of the main result of [12]:

PROPOSITION 2.3. *For any locally hereditary algebra $A = KQ/I$ there is a universal simply connected Galois covering $F\colon \widetilde{A} \to A$ with finitely generated free* (*or trivial*) *fundamental group.*

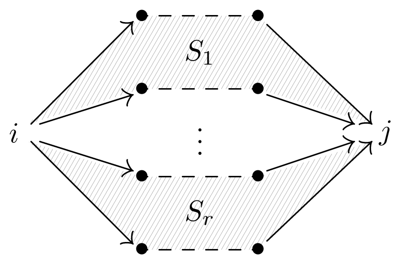
3. Proof of Theorem 1.1. The aim of this section is to prove our main result, Theorem 1.1.

(i)⇒(ii). If a locally hereditary algebra A is of tame representation type then, according to [4], so is $\widetilde{A}$, and hence any finite convex subcategory B of $\widetilde{A}$ is tame. Hence (ii) follows from [13].

(ii)⇒(i). For the incidence algebras of finite posets the assertion is already proved in [8], [9]. Assume that A is not the incidence algebra of a poset. Hence, there are vertices $i, j \in Q_0$ such that $\dim_K(P_j, P_i) \geq 2$, where P_k denotes the indecomposable projective module corresponding to the vertex k. Equivalently, there are two paths from i to j which are linearly independent in A.

Take a minimal convex subquiver M of Q having a unique source i, unique target j, and $\dim_K \operatorname{Hom}_A(P_j, P_i) \geq 2$. The minimality implies that

M is of the shape

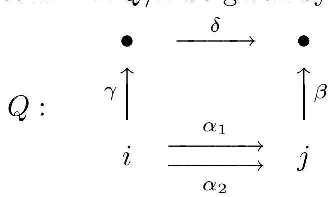
(3.1) 

with $r \geq 2$, where each of the subquivers $S_1, \ldots, S_r$ is a partially ordered set (different from a vertex) with unique minimal vertex i and unique maximal vertex j.

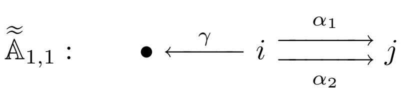
It is possible that some S_k is an arrow. Let t denote the number of such S_k's. Then $t \leq r$.

Our aim is to prove that $M = Q$. We illustrate the idea of the proof (for $t \geq 2$) on the following example.

EXAMPLE 3.1. Let $A = KQ/I$ be given by the quiver

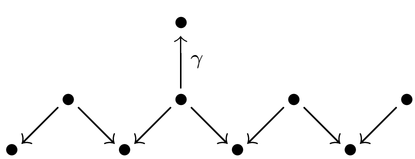


and the ideal $I = (\beta\alpha_1 + \beta\alpha_2 + \delta\gamma)$. Then A is simply connected of wild representation type and q_A is not weakly non-negative, since Q contains a convex subquiver



which is the quiver of a wild hereditary algebra.

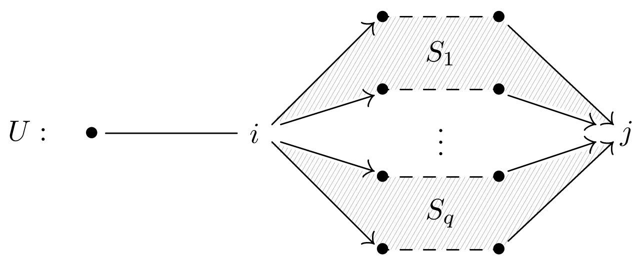
If we take another ideal $I' \subseteq KQ$ such that $A = KQ/I'$ is not simply connected and the quiver $i \rightrightarrows j$ is not a subquiver of $\widetilde{A}$, then $\widetilde{A}$ contains a convex subquiver R of the form

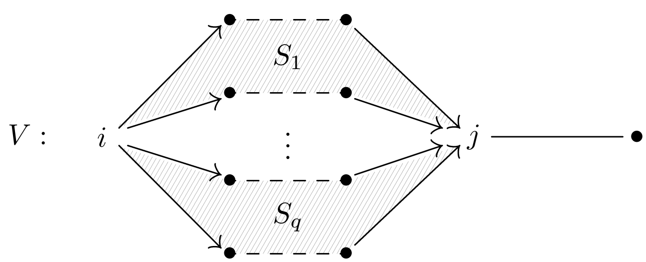


which is the quiver of a wild hereditary algebra of type $\widetilde{\widetilde{\mathbb{E}}}_7$ and again q_R is not weakly non-negative [13].

Lemma 3.2. *If $t \geq 2$, then $t = r = 2$ and $Q = \widetilde{A}_{1,1}$. Hence A is the Kronecker algebra.*

Proof. Suppose that the Galois covering $\widetilde{A}$ contains a subquiver $i \rightrightarrows j$ and there is an arrow in Q_1 connected to the vertex i (or j), which is not from M_1, and $\widetilde{A}$ contains a convex subcategory with quiver of one of the forms





where $2 \leq q \leq r$ and $S_1 = S_2$ is an arrow.

Take a dimension vector $\mathbf{x}$ with $x_p = 0$ for all coordinates corresponding to the vertices from $S_3 \cup \cdots \cup S_q \setminus \{i, j\}$ (there is such a dimension vector, since no arrow is a summand in a relation). Then $q_U(\mathbf{x})$ (and $q_V(\mathbf{x})$) is equal to the Tits form of a wild hereditary algebra with quiver of one of the forms

with at least two arrows from i to j. Hence q_U and q_V are not weakly non-negative [13].

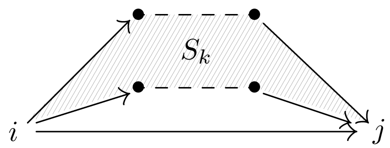
Suppose that $\widetilde{A}$ does not contain $i \rightrightarrows j$. Then there is an arrow in Q connected to i (or j), which is not from M_1, and $\widetilde{A}$ contains a convex subcategory W which has a factor isomorphic to a wild hereditary algebra of type $\widetilde{\widetilde{\mathbb{E}}}_7$. As above one can show that q_W is not weakly non-negative.

Assume now that the only arrows connected to i and j are in M_1. Then either $\widetilde{A}$ contains a convex subcategory of type $\widetilde{\widetilde{\mathbb{E}}}_7$, or $M = \widetilde{\widetilde{\mathbb{A}}}_{1,1}$, since no arrow appears in a relation (as a summand). ∎

Lemma 3.3. *Assume that* $t = 1$. *Then* $r = 2$ *and* $M = Q$. *Moreover*, A *is a double braid generalized canonical algebra from Example* 2.5(b) (*with one of the posets* S, T *being an arrow*).

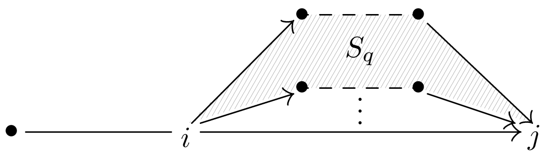
Proof. If the Galois covering $\widetilde{A}$ does not contain a convex subcategory with quiver

R :

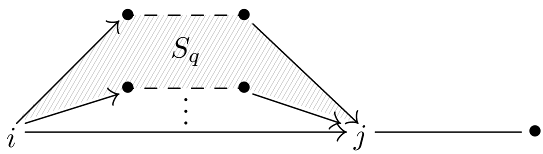


for any $k \leq r$, then as in the preceding example or lemma one may conclude that there is no arrow connected to i or j, $r = 2$, and there is no other arrow. Hence, $M = Q$ and A is the algebra from Example 2.5(b). Assume that $\widetilde{A}$ contains a full subcategory with quiver R (for some $k \leq r$). Then $\widetilde{A}$ contains a convex subcategory with one of the quivers

U :



V :



with $2 \leq q \leq r$ and with the only possible relations in the posets $S_1, \ldots, S_q$. Each S_l is different from an arrow, for $l \geq 2$ ($S_1 = i \to j$).

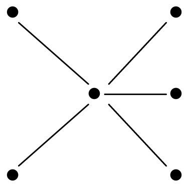
Let B be the path algebra of U. Take $\mathbf{x} = (x_m) \in \mathbb{Z}^n$, where $n = |U_0|$, such that

- $x_m = 0$ for $m \in S_l \setminus \{i, j\}$ for $l \geq 3$,
- $x_m = x_j$ for $m \in S_2 \setminus \{i\}$.

Then $q_B(\mathbf{x})$ is equal to the Tits form of a wild hereditary algebra with quiver $\widetilde{\widetilde{\mathbb{A}}}_{1,1}$. Therefore q_B is not weakly non-negative, a contradiction. The same can be proved for V. ∎

Now, assume that each S_k is different from an arrow (and from a vertex) and $r \leq 4$, because for $r \geq 5$ the quivers M, Q and $\widetilde{A}$ contain a convex subquiver of type

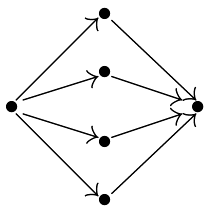
T_5 :



which is the quiver of a wild hereditary algebra, where •—• means either • ⟶ • or • ⟵ •.

LEMMA 3.4. *Assume that $r = 4$. Then $M = Q$ and Q is of the shape*

(3.2)



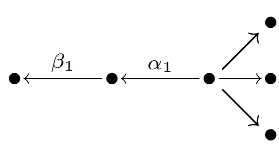
Moreover, the algebra A is either double braid generalized canonical, single braid generalized canonical, or tubular canonical of type $C(2,2,2,2,\lambda)$ (in the sense of [14]*).*

Proof. Denote by σ_k the (unique up to commutativity relations and linear independence) path in each poset S_k, for $k = 1, 2, 3, 4$.

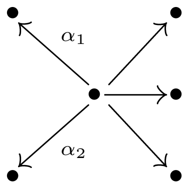
The relations in KM (an induced subcategory of KQ) are generated by linear combinations of the paths σ_1, σ_2, σ_3, σ_4 and the commutativity relations in each S_k. Since $\dim_K \mathrm{Hom}(P_j, P_i) \geq 2$, there are at most two relations which are generators of the ideal $I(i,j)$ in KM. Assume that at least one of the above paths, say σ_1, does not appear in any relation (as a summand). Then if the beginning part of S_1 is of the form

$$\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\beta_1} \bullet$$

then the Galois covering $\widetilde{A}$ contains a convex subquiver



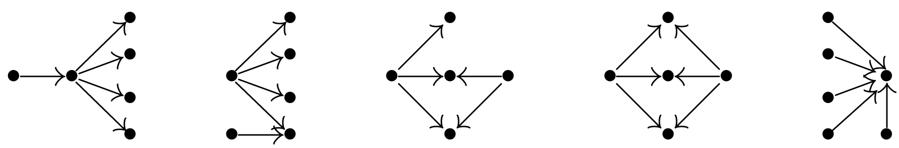
which is the quiver of a minimal wild hereditary algebra of type $\widetilde{\mathbb{D}}_4$; while if at least two arrows start from the source of S_1 then $\widetilde{A}$ contains a convex subquiver



of type T_5. The Tits form of a hereditary wild algebra is not weakly non-negative [13]. Therefore, each of the paths $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ appears in some relation. Moreover, each S_k is equal to

$$\mathbb{A}_3 : \bullet \longrightarrow \bullet \longrightarrow \bullet$$

so M is of the form (3.2). If we take a locally hereditary algebra which is a one-point extension (or coextension) of the algebra KM/I_M, then its quiver is a convex subquiver in $\widetilde{A}$ (and in Q) and has a convex subquiver of one of the forms



which are the quivers of minimal wild hereditary algebras. Hence $M = Q$.

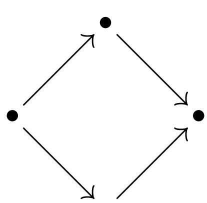
Suppose that there is only one relation generating the ideal I in KQ. We may assume that it is $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$. Then the algebra A is concealed of type T_5 [18]. The Tits form of a concealed wild hereditary algebra is not weakly non-negative [6], a contradiction.

Assume now that there are two linearly independent relations in the ideal I. It is easy to check that (up to permutation of $\{1, 2, 3, 4\}$) we may choose one of the following three sets as a set of generators of I:

(i) $\{\sigma_1 + \sigma_2, \sigma_3 + \sigma_4\}$,
(ii) $\{\sigma_1 + \sigma_2 + \sigma_3, \sigma_3 + \sigma_4\}$,
(iii) $\{\sigma_1 + \sigma_2 + \sigma_3, \lambda\sigma_2 + \sigma_3 + \sigma_4\}$ with $0 \neq \lambda \neq 1$.

Hence, A is isomorphic either to a double braid generalized canonical algebra with $S = T$ (in the notation of Example 2.5(b)) of the form

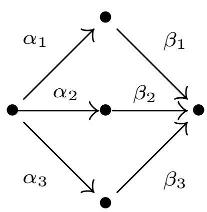
(3.3)



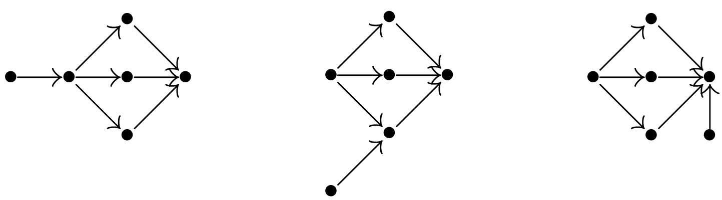
(in case (i)), to a single braid generalized canonical algebra with S (see Example 2.5(a)) of the form (3.2) (in case (ii)), or to a tubular canonical algebra of type $C(2,2,2,2,\lambda)$ (in case (iii)). ■

LEMMA 3.5. *Assume that $r = 3$. Then $M = Q$ and the algebra A is simply connected and isomorphic either to a canonical algebra or to a single braid generalized canonical algebra.*

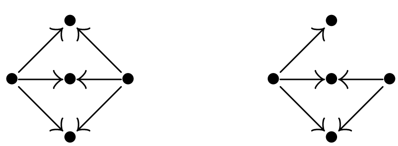
Proof. Suppose M is of the form (3.1). By the *width* of a poset S, denoted by $\omega(S)$, we mean the maximal number of incomparable vertices of S. If the width of S_1, S_2, or S_3 is 1, then M is the quiver of a canonical algebra. Then a short argument shows that any locally hereditary algebra which is a one-point extension (or coextension) of KM/I_M contains as a convex subcategory a concealed algebra of a minimal wild hereditary algebra. For example, consider M of the form



Then $I_M = (\beta_1\alpha_1 + \beta_2\alpha_2 + \beta_3\alpha_3)$ and KM/I_M is a canonical algebra of type $(2,2,2)$. The locally hereditary algebra which is a one-point extension of KM/I_M has either one of the quivers



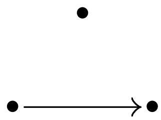
(of concealed minimal wild hereditary algebras, listed in [18]), or its quiver contains one of



which are the quivers of minimal wild hereditary algebras [18]. Hence $M = Q$ and the algebra $A = \widetilde{A}$ is canonical. We can show the same for any quiver M of a canonical algebra of type (p,q,r).

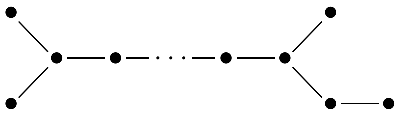
Assume now that M is of the form (3.1) and the width of S_3 is at least 2. If S_3 is not a subposet of a garland, then KM/I_M (hence A and $\widetilde{A}$) contains, as a convex subcategory, a minimal wild hereditary algebra. Indeed, S_3 not being a subposet of a garland is equivalent to the fact that either the width of S_3 is at least 3, or S_3 contains a disconnected subposet of the form

(3.4)



If $\omega(S_3) \geq 3$, then M (hence Q and $\widetilde{A}$) contains a convex subquiver of type T_5 or

$\widetilde{\widetilde{\mathbb{D}}}_n$:



for some natural n. It is easy to see that if S_3 contains a disconnected subposet of the shape (3.4), then M (hence Q and $\widetilde{A}$) contains a convex subquiver of type $\widetilde{\widetilde{\mathbb{D}}}_n$ for some n.

We obtain the same conclusion if for $\omega(S_3) = 2$ one of the posets S_1, S_2 is not equal to $\mathbb{A}_3$: $\bullet \to \bullet \to \bullet$. Hence M is a poset of the form 2.5(a).

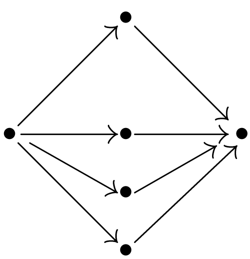
Inspecting possible one-point extensions and coextensions of KM/I_M we conclude that any locally hereditary algebra which is such an extension or coextension contains (hence A and $\widetilde{A}$ contain) a minimal wild hereditary algebra as a convex subcategory. Therefore $M = Q$ and A is single braid generalized canonical. ∎

Lemma 3.6. *Assume that $r = 2$. Then $M = Q$ and A is a double braid generalized canonical algebra.*

Proof. Assume that M is of the form (3.1) with $r = 2$. As in the above proof, we may observe that any locally hereditary algebra which is a one-point extension or coextension of KM/I_M (hence A and $\widetilde{A}$) contains some minimal wild hereditary algebra as a convex subcategory. Therefore $M = Q$.

Assume now that S_2 is not a subposet of a garland. Suppose that $\omega(S_2) \geq 3$. Then either Q is of the form

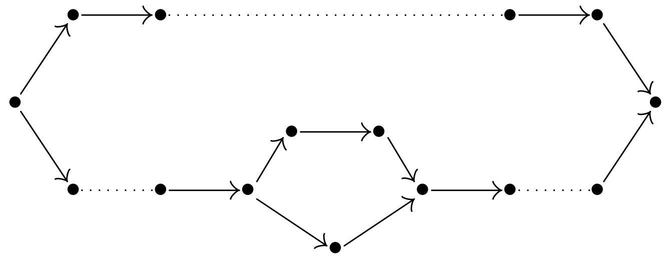
(3.5)



(with $S_1 = A_3$) or A (hence $\widetilde{A}$) contains an algebra of type T_5 or $\widetilde{\widetilde{\mathbb{D}}}_4$ as a convex subcategory.

If now Q is of the form (3.5), then the Galois covering $\widetilde{A}$ of A contains the algebra $K\widetilde{\widetilde{\mathbb{D}}}_4$ as a convex subcategory.

Assume that $\omega(S_k) \leq 2$ for $k = 1, 2$. Suppose S_1 contains a subposet of the shape (3.4). Then either A (and hence $\widetilde{A}$) contains a subalgebra of type $\widetilde{\widetilde{\mathbb{D}}}_n$ (for some n) as a convex subcategory, or Q is of the form



and then $\widetilde{A}$ contains, as a convex subcategory, some wild hereditary algebra with quiver of type $\widetilde{\widetilde{\mathbb{D}}}_n$ for some n. Hence Q is of the form 2.5(b), and A is a double braid generalized canonical algebra. ∎

The following proposition summarizes the above investigations.

PROPOSITION 3.7. *Suppose that $A = KQ/I$ is locally hereditary but is not the incidence algebra of a poset. Assume that the Tits form q_B of any finite convex subcategory B of the universal Galois covering $\widetilde{A}$ of A is weakly non-negative. Then A belongs to one of the following classes of algebras*:

(a) *canonical algebras*,
(b) *single braid generalized canonical algebras*,
(c) *double braid generalized canonical algebras.*

For the canonical algebras the implication (ii)⇒(i) of Theorem 1.1 follows from [14]. The proof of Theorem 1.1 yields the following proposition, proved in [11].

PROPOSITION 3.8. *The single and double braid generalized canonical algebras are representation-tame.*

Finally, we note that any representation-finite locally hereditary algebra is the incidence algebra of a poset. Moreover, any non-simply connected locally hereditary algebra of tame representation type is either the incidence algebra of a poset, or belongs to class (c) in Proposition 3.7.

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