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MONOTONICITY OF GENERALIZED WEIGHTED MEAN VALUES

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Abstract. The author gives a new simple proof of monotonicity of the generalized extended mean values

$$M(r,s)=\left(\frac{\int f^s\,d\mu}{\int f^r\,d\mu}\right)^{1/(s-r)}$$

introduced by F. Qi.

Means and inequalities for them have a long history and rich literature. The basic inequality between the geometric and arithmetic means has been proved in many ways. More than fifty proofs can be found in [1]. The generalizations go in different directions. The power (or Hölder) mean $M(r)=((x^r+y^r)/2)^{1/r}$, $r\neq 0$, $M(0)=\sqrt{xy}=G(x,y)$, has been extended to the weighted power means

$$\text{(1)}\qquad \begin{aligned} M(r)&=\Big(\sum_i p_i a_i^r\Big/\sum_i p_i\Big)^{1/r},\\ M(0)&=\exp\Big(\sum_i p_i a_i\log a_i\Big/\sum_i p_i\Big),\end{aligned}$$

and further to the weighted integral means where sums are replaced by integrals. The monotonicity of $M(r)$ has been proved in many ways (see [1, 3, 6, 10]).

Another family of means arises from the logarithmic mean $L(x,y)=(x-y)/(\log x-\log y)$ by putting

$$S_p(x,y)=\left(\frac{y^p-x^p}{p(y-x)}\right)^{1/(p-1)},$$

$$S_0(x,y)=L(x,y),\qquad S_1(x,y)=e^{-1}\left(\frac{y^y}{x^x}\right)^{1/(y-x)}$$

(see Galvani [2]).

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Stolarsky [9] extended this family to the two-parameter extended mean values defined by

$$
(2)\qquad E(r,s;x,y)=\begin{cases}\left(\dfrac{r}{s}\,\dfrac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)}, & sr(s-r)(x-y)\neq 0,\\[2ex] \left(\dfrac{1}{r}\,\dfrac{y^r-x^r}{\log y-\log x}\right)^{1/r}, & r(x-y)\neq 0,\ s=0,\\ e^{-1/r}(y^{y^r}/x^{x^r})^{1/(y^r-x^r)}, & r=s,\ r(x-y)\neq 0,\\ \sqrt{xy}, & r=s=0,\ x-y\neq 0,\\ x, & x=y.\end{cases}
$$

Leach and Sholander [4, 5] have shown that E is increasing in all variables. In 1998 Qi [7] extended these notions by defining the generalized weighted mean values M as follows:

$$
(3)\quad M(r,s)=M(r,s;x,y)=\begin{cases}\left(\dfrac{\int_x^y p(t)f^s(t)\,dt}{\int_x^y p(t)f^r(t)\,dt}\right)^{1/(s-r)}, & r\neq s,\\[2ex] \exp\left(\dfrac{\int_x^y p(t)f^r(t)\log f(t)\,dt}{\int_x^y p(t)f^r(t)\,dt}\right), & r=s,\end{cases}
$$

where p and f are positive, integrable functions. Obviously $M(r,0)$ is the weighted power mean (1) and $M(r-1,s-1)=E(r,s)$ for $p\equiv 1$ and $f(t)=t$.

Qi [8] proved that for continuous p and f, M is increasing in p and s. He has also shown in [7] that if f is monotone then M is of the same monotonicity in x and y.

In this note we extend and generalize these results by showing that the monotonicity of $M(r,s)$ is a straightforward consequence of the Cauchy–Schwarz inequality and holds also in case of integrable functions. We also show that monotonicity of f is a necessary and sufficient condition for M to be monotone in x and y. We believe that our proofs are also simpler than the original reasoning.

THEOREM 1. *Let $f : X\to\mathbb{R}$ be a measurable, positive function on a measure space (X,μ) and*

$$
(4)\qquad M(r,s)=\begin{cases}\left(\dfrac{\int_X f^s\,d\mu}{\int_X f^r\,d\mu}\right)^{1/(s-r)}, & r\neq s,\\[2ex] \exp\left(\dfrac{\int_X f^s\log f\,d\mu}{\int_X f^s\,d\mu}\right), & r=s.\end{cases}
$$

Then M is increasing in both r and s.

Proof. Let $I(r) = \int_X f^r\,d\mu$. The Cauchy–Schwarz inequality applied to $f^{s/2}$ and $f^{r/2}$ gives

$$I\left(\frac{r+s}{2}\right) \le \sqrt{I(r)}\sqrt{I(s)},$$

which shows that $\log I$ is convex in the sense of Jensen, hence being continuous, it is convex.

Let us now recall the following property of convex functions: If h is convex then the function $g(x,y) = h(x) - h(y)/(x-y),\ x \neq y$, is increasing in both variables. This property applied to $\log I$ shows that $\log M(r,s)$ is increasing for $s \neq r$. As M is continuous, the monotonicity extends to the whole plane of parameters (r,s). ∎

THEOREM 2. *If $p, f : [a,b] \to \mathbb{R}$ are continuous and positive then the following conditions are equivalent*:

(i) *the function f is increasing* (*decreasing*, *respectively*).

(ii) *for every r,s the function $M(r,s;x,y)$ is increasing* (*decreasing*, *respectively*) *in x and y.*

Proof. As in the proof of Theorem 1 it is easier to consider monotonicity of $\log M$. For $r \neq s$ we have

$$\begin{aligned}\frac{\partial \log M}{\partial x} &= (s-r)^{-1}\left(\frac{-p(x)f^s(x)}{\int_x^y p(t)f^s(t)\,dt} - \frac{-p(x)f^r(x)}{\int_x^y p(t)f^r(t)\,dt}\right)\\ &= H\,\frac{\int_x^y p(t)\Big(\big(\frac{f(t)}{f(x)}\big)^s - \big(\frac{f(t)}{f(x)}\big)^r\Big)\,dt}{s-r}\\ &= H\,\frac{\int_x^y p(t)\big(\frac{f(t)}{f(x)}\big)^r\Big(\big(\frac{f(t)}{f(x)}\big)^{s-r} - 1\Big)\,dt}{s-r},\end{aligned} \tag{5}$$

where

$$H = H(r,s;x,y) = \frac{p(x)f^{r+s}(x)}{\int_x^y p(t)f^r(t)\,dt \int_x^y p(t)f^s(t)\,dt}$$

is positive. Observe that

$$\text{if}\quad f(x) = \min_{t\in[x,y]} f(t) \quad\text{then}\quad \frac{\big(\frac{f(t)}{f(x)}\big)^{s-r} - 1}{s-r} \ge 0, \tag{6}$$

$$\text{if}\quad f(x) = \max_{t\in[x,y]} f(t) \quad\text{then}\quad \frac{\big(\frac{f(t)}{f(x)}\big)^{s-r} - 1}{s-r} \le 0. \tag{7}$$

From (5), (6) and (7) we conclude that if f is increasing then $\log M$ is increasing in x. Similar reasoning shows monotonicity in y, so the implication (i)⇒(ii) holds.

If f is not monotone then one can choose an interval $[z, y]$ and $x_1, x_2 \in [z, y]$ such that

$$f(x_1) = \min_{t\in[z,y]} f(t) < f(y) < \max_{t\in[z,y]} f(t) = f(x_2)$$

and from (6) and (7),

$$\frac{\partial \log M}{\partial x}(r, s; x_1, y) < 0 < \frac{\partial \log M}{\partial x}(r, s; x_2, y),$$

which completes the proof in case $r \neq s$.

To prove the case $s = r$ it is enough to replace $(s-r)^{-1}\big(\big(\frac{f(t)}{f(x)}\big)^{s-r} - 1\big)$ with $(\log f(t) - \log f(x))$ in (5)–(7). ■

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