COLLOQUIUM MATHEMATICUM

VOL. 99 2004 NO. 2

ORDINARY CONVERGENCE FOLLOWS FROM STATISTICAL SUMMABILITY $(C,1)$ IN THE CASE OF SLOWLY DECREASING OR OSCILLATING SEQUENCES

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Abstract. Schmidt's Tauberian theorem says that if a sequence (x_k) of real numbers is slowly decreasing and $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} x_k = L$, then $\lim_{k\to\infty} x_k = L$. The notion of slow decrease includes Hardy's two-sided as well as Landau's one-sided Tauberian conditions as special cases. We show that ordinary summability $(C,1)$ can be replaced by the weaker assumption of statistical summability $(C,1)$ in Schmidt's theorem. Two recent theorems of Fridy and Khan are also corollaries of our Theorems 1 and 2. In the Appendix, we present a new proof of Vijayaraghavan's lemma under less restrictive conditions, which may be useful in other contexts.

1. Introduction. We begin with some historical remarks. The term "statistical convergence" first appeared in [2] by Fast, where he attributed this concept to Hugo Steinhaus. More exactly, Henry Fast has recently explained to the referee of our paper in an e-mail message that actually he had heard about this concept from Steinhaus, but in fact it was Antoni Zygmund who proved theorems on the statistical convergence of Fourier series in the first edition of his book "Trigonometric Series" in 1935, where he used the term "almost convergence" in place of statistical convergence. (See [11, Vol. 2, pp. 181 and 188].)

A sequence $(x_k : k = 1, 2, \ldots)$ of complex numbers is said to be *statistically convergent* if there exists a complex number L such that for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} n^{-1}|\{k \le n : |x_k - L| > \varepsilon\}| = 0, \tag{1.1}$$

where by $k \le n$ we mean that $k = 1, \ldots, n$, and by $|\mathcal{S}|$ we denote the number

2000 *Mathematics Subject Classification*: Primary 40E05, 40G05.

Key words and phrases: statistical convergence, statistical summability $(C,1)$, Hardy's two-sided Tauberian condition, Landau's one-sided Tauberian condition, slow decrease, slow oscillation, Vijayaraghavan's lemma.

This research was supported by the Hungarian National Foundation for Scientific Research under Grant T 046 192.

of elements in the set $\mathcal{S}$ of positive integers. Clearly, L in (1.1) is uniquely determined. In symbols, we write st-lim $x_k = L$.

Basic results on statistical convergence may be found in [2, 3, 9].

A sequence (x_k) is said to be *statistically summable* $(C, 1)$ to L whenever st-lim $\sigma_n = L$, where

$$\sigma_n := \frac{1}{n} \sum_{k=1}^{n} x_k, \qquad n = 1, 2, \ldots, \tag{1.2}$$

is the first arithmetic mean, also called the Cesàro mean (of first order).

We recall that a sequence (x_k) of *real numbers* is said to be *slowly decreasing* according to Schmidt [8] if

$$\lim_{\lambda \to 1+} \liminf_{n \to \infty} \min_{n < k \le \lambda n} (x_k - x_n) \ge 0. \tag{1.3}$$

Since the function

$$f(\lambda) := \liminf_{n \to \infty} \min_{n < k \le \lambda n} (x_k - x_n), \qquad \lambda > 1, \tag{1.4}$$

is clearly decreasing in λ on the interval $(1, \infty)$, the right-hand limit in (1.3) exists and can be equivalently replaced by $\sup_{\lambda > 1}$.

It is easy to see that (1.3) is satisfied if and only if for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda_0 = \lambda_0(\varepsilon) > 1$, as close to 1 as we wish, such that

$$x_k - x_n \ge -\varepsilon \quad \text{whenever} \quad n_0 \le n < k \le \lambda_0 n. \tag{1.5}$$

We recall that Hardy [6, pp. 124–125] defined the notion of slow decrease by the requirement that

$$\liminf_{j \to \infty} (x_{k_j} - x_{n_j}) \ge 0 \tag{1.6}$$

whenever

$$n_j \to \infty, \qquad 1 < k_j / n_j \to 1 \qquad \text{as } j \to \infty. \tag{1.7}$$

We claim that definition (1.3) and (1.6) & (1.7) are equivalent. First, assume that the sequence (x_k) satisfies (1.3). If (1.7) holds for some sequences $\{k_j\}$ and $\{n_j\}$ of positive integers, then for every $\lambda > 1$, the inequalities $n_j < k_j \le \lambda n_j$ are satisfied for every large enough j. By (1.5), for every $\varepsilon > 0$ we have

$$\liminf_{j \to \infty} (x_{k_j} - x_{n_j}) \ge -\varepsilon,$$

which proves (1.6).

Second, assume that the sequence (x_k) satisfies (1.6) for all sequences $\{k_j\}$ and $\{n_j\}$ of positive integers as in (1.7). We prove (1.3) indirectly. Namely, if (1.3) is not satisfied, then there exists some $\varepsilon_0 > 0$ such that for all $\lambda > 1$ and $m \ge 1$ there exist integers k and n for which

$$m \le n < k \le \lambda n, \qquad x_k - x_n < -\varepsilon_0.$$

In particular, let $m_1 := 1$ and $\lambda_1 := 2$; then there exist k_1 and n_1 such that

$$m_1 \le n_1 < k_1 \le \lambda_1 n_1, \quad x_{k_1} - x_{n_1} < -\varepsilon_0.$$

We proceed by induction. If $1 \le n_1 < k_1 < \cdots < n_{j-1} < k_{j-1}$ have been defined, then let $m_j := k_{j-1} + 1$ and $\lambda_j := (j+1)/j$; then there exist k_j and n_j such that

$$m_j \le n_j < k_j \le \lambda_j n_j, \quad x_{k_j} - x_{n_j} < -\varepsilon_0, \quad j = 1, 2, \ldots.$$

Clearly, (1.7) is satisfied, while (1.6) is not. This contradiction proves (1.3).

We note that definitions (1.3) and (1.6) & (1.7) of slow decrease resemble the equivalent definitions of continuity of a function at a point of the definition domain, given by Cauchy (in terms of neighbourhoods with radii ε and δ) and by Heine (in terms of sequences tending to the given point and function value, respectively).

One more remark is appropriate here. A sequence (x_k) of *real numbers* may be said to be *slowly increasing* if

$$\lim_{\lambda \to 1+} \limsup_{n \to \infty} \max_{n < k \le \lambda n} (x_k - x_n) \le 0. \tag{1.8}$$

Clearly, (x_k) is slowly increasing if and only if $(-x_k : k = 1, 2, \ldots)$ is slowly decreasing. In particular, the right-hand limit in (1.8) can be equivalently replaced by $\inf_{\lambda > 1}$.

We recall that a sequence (x_k) of *complex numbers* is said to be *slowly oscillating* if

$$\lim_{\lambda \to 1+} \limsup_{n \to \infty} \max_{n < k \le \lambda n} |x_k - x_n| = 0. \tag{1.9}$$

Again, the right-hand limit in (1.9) can be equivalently replaced by $\inf_{\lambda > 1}$.

It is easy to see that (1.9) is satisfied if and only if for every $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon)$ and $\lambda_0 = \lambda_0(\varepsilon) > 1$, as close to 1 as we wish, such that

$$|x_k - x_n| \le \varepsilon \quad \text{whenever} \quad n_0 \le n < k \le \lambda_0 n. \tag{1.10}$$

We note that Hardy [6, pp. 124–125] defined the notion of slow oscillation by the requirement

$$\lim_{j \to \infty} (x_{k_j} - x_{n_j}) = 0 \tag{1.11}$$

whenever the conditions in (1.7) are satisfied. The equivalence of definitions (1.9) and (1.11) & (1.7) can be justified exactly in the same way as in the case of slow decrease.

It is plain that a sequence (x_k) of real numbers is slowly oscillating if and only if (x_k) is both slowly decreasing and slowly increasing.

It is well known that if a sequence (x_k) of *complex numbers* satisfies *Hardy's two-sided Tauberian condition* (see [5] and also [6, p. 121]):

$$k|x_k - x_{k-1}| \le H \quad \text{for some } H \text{ and every } k, \tag{1.12}$$

then (x_k) is slowly oscillating. Furthermore, if a sequence (x_k) of *real numbers* satisfies *Landau's one-sided Tauberian condition* (see [7] and also [6, p. 121]):

$$k(x_k - x_{k-1}) \geq -H \quad \text{for some } H > 0 \text{ and every } k, \tag{1.13}$$

then (x_k) is slowly decreasing.

2. Main results.

The main results of the present paper are summarized in the following two theorems.

THEOREM 1. *If a sequence (x_k) of real numbers is statistically summable $(C, 1)$ to some L and slowly decreasing, then (x_k) converges to L.*

Theorem 2.3 in [4] by Fridy and Khan (under Landau's one sided Tauberian condition) is a corollary of Theorem 1.

THEOREM 2. *If a sequence (x_k) of complex numbers is statistically summable $(C, 1)$ to some L and slowly oscillating, then (x_k) converges to L.*

Theorem 2.1 in [4] by Fridy and Khan (under Hardy's two-sided Tauberian condition) is a corollary of Theorem 2.

3. Auxiliary results

LEMMA 1. *Let (x_k) be a sequence of real numbers. Condition* (1.3) *of slow decrease is equivalent to*

$$\lim_{\lambda \to 1-} \liminf_{n\to\infty} \min_{\lambda n \leq k < n} (x_n - x_k) \geq 0. \tag{3.1}$$

Proof. We consider the following extension of the function $f(\lambda)$ defined in (1.4):

$$f(\lambda) := \liminf_{n\to\infty} \min_{\lambda n \leq k < n} (x_n - x_k), \quad 0 < \lambda < 1.$$

Given an arbitrary $\lambda > 1$, by (1.4) there exists an increasing sequence $(n_p : p = 1, 2, \ldots)$ of natural numbers such that

$$f(\lambda) = \lim_{p\to\infty} \min_{n_p < k \leq \lambda n_p} (x_k - x_{n_p}).$$

Let us choose a sequence $(k_p : p = 1, 2, \ldots)$ of integers such that

$$x_{k_p} - x_{n_p} = \min_{n_p < k \leq \lambda n_p} (x_k - x_{n_p}), \quad n_p < k_p \leq \lambda n_p, \quad p = 1, 2, \ldots.$$

Since

$$n_p < k_p \leq \lambda n_p \quad \text{is equivalent to} \quad (1/\lambda) k_p \leq n_p < k_p,$$

$k_p \to \infty$ as $p \to \infty$, and

$$\min_{(1/\lambda) k_p \leq n < k_p} (x_{k_p} - x_n) \leq x_{k_p} - x_{n_p},$$

it follows immediately that

$$\begin{aligned} f(1/\lambda) &\le \liminf_{p\to\infty} \min_{(1/\lambda)k_p \le n < k_p} (x_{k_p} - x_n) \\ &\le \lim_{p\to\infty} (x_{k_p} - x_{n_p}) = f(\lambda), \quad \lambda > 1. \end{aligned}$$

The converse inequality

$$f(1/\lambda) \le f(\lambda), \quad 0 < \lambda < 1,$$

can be deduced in an analogous way. Thus, we conclude that

$$f(1/\lambda) = f(\lambda) \quad \text{for every } 0 < \lambda < \infty,\ \lambda \ne 1. \tag{3.2}$$

Now, the equivalence of (1.3) and (3.1) is a trivial consequence of (3.2). ∎

LEMMA 2. *Let (x_k) be a sequence of complex numbers. Condition* (1.9) *of slow oscillation is equivalent to*

$$\lim_{\lambda\to 1-} \limsup_{n\to\infty} \max_{\lambda n \le k < n} |x_n - x_k| = 0. \tag{3.3}$$

The proof runs along the same lines as that of Lemma 1. We omit the details.

LEMMA 3 (see [1, Lemma 4]). *Let (x_k) be a sequence of real numbers. If there exist a positive integer m_0 and a real number $\lambda > 1$ such that*

$$x_n - x_k \ge -1 \quad \textit{for all } m_0 \le k < n \le \lambda k, \tag{3.4}$$

then the sequence

$$\frac{1}{n}\sum_{k=1}^{n}(x_n - x_k), \quad n = 1, 2, \ldots,$$

is bounded below.

We note that Armitage and Maddox [1] stated Lemma 3 above for slowly decreasing sequences, but in their proof they actually made use of condition (3.4), while relying on a key lemma of Vijayaraghavan (see [10, Lemma 6]). In Lemma 8 in the Appendix, we present a new proof of Vijayaraghavan's lemma under our less restrictive conditions.

LEMMA 4. *Let (x_k) be a sequence of complex numbers. If there exist a positive integer m_0 and a real number $\lambda > 1$ such that*

$$|x_n - x_k| \le 1 \quad \textit{for all } m_0 \le k < n \le \lambda k, \tag{3.5}$$

then the sequence

$$\frac{1}{n}\sum_{k=1}^{n}|x_n - x_k|, \quad n = 1, 2, \ldots, \tag{3.6}$$

is bounded.

Proof. The proof of Lemma 4 is modelled after that of [1, Lemma 4]. By Lemma 9 in the Appendix, there exists a constant B such that

$$|x_n - x_k| \le B\log(n/k) \quad \text{for all } 1 \le k \le n/\lambda. \tag{3.7}$$

By (3.5) and (3.7), for $n \ge \lambda m_0$ we can estimate as follows:

$$\begin{aligned}\sum_{k=1}^{n} |x_n - x_k| &= \Big\{\sum_{k=1}^{[n/\lambda]} + \sum_{k=1+[n/\lambda]}^{n}\Big\}|x_n - x_k| \\ &\le B\sum_{k=1}^{[n/\lambda]} \log(n/k) + (n - [n/\lambda]) \\ &\le B\sum_{k=1}^{n} \log(n/k) + n = B\Big\{n\log n - \sum_{k=2}^{n}\log k\Big\} + n \\ &\le B\Big\{n\log n - \int_1^n \log u\,du\Big\} + n = (B+1)n, \qquad n \ge \lambda m_0,\end{aligned}$$

where $[\cdot]$ means the integral part (of a real number) and where we used the elementary fact that

$$\log k > \int_{k-1}^{k} \log u\,du, \qquad k = 2, 3, \dots, n.$$

This proves the boundedness of sequence (3.6). ∎

We note that in the proofs of Theorems 1 and 2 in Section 4 we shall only use weaker versions of Lemmas 3 and 4. However, we think that Lemmas 3 and 4 in the above formulation may be useful in other contexts.

The next auxiliary result is the so-called decomposition theorem due to Fridy [3].

LEMMA 5 (see [3, Theorem 1]). *If (x_k) is statistically convergent to some L, then there exists a sequence (y_k) which is convergent* (*in the ordinary sense*) *to L and*

$$\lim_{n\to\infty} \frac{1}{n}\,|\{k \le n : y_k \ne x_k\}| = 0. \tag{3.8}$$

Finally, we sharpen [5, Theorem 2.2] by replacing Landau's one-sided Tauberian condition by the weaker condition of slow decrease.

LEMMA 6. *Let (x_k) be a sequence of real numbers. If (x_k) is statistically convergent to some L and slowly decreasing, then (x_k) is convergent to L.*

Proof. We start with the decomposition theorem (see Lemma 5). Let $1 \le l_1 < l_2 < \cdots$ be the subsequence of those indices k for which $y_k = x_k$.

Then setting $n := l_m$ in (3.8) gives

$$\frac{1}{l_m}\,|\{k \le l_m : y_k = x_k\}| = \frac{m}{l_m} \to 1 \quad \text{as } m \to \infty.$$

Consequently, it follows that

$$\lim_{m\to\infty} \frac{l_{m+1}}{l_m} = \lim_{m\to\infty} \left(\frac{l_{m+1}}{m+1} \cdot \frac{m+1}{m} \cdot \frac{m}{l_m}\right) = 1. \tag{3.9}$$

By the definition of the subsequence (l_m) (cf. (3.8)), we have

$$\lim_{m\to\infty} x_{l_m} = \lim_{m\to\infty} y_{l_m} = L. \tag{3.10}$$

By (1.3), for every $\varepsilon > 0$ there exists $\lambda = \lambda(\varepsilon) > 1$ such that

$$\liminf_{n\to\infty} \min_{n<k\le\lambda n} (x_k - x_n) \ge -\varepsilon. \tag{3.11}$$

By (3.9), we have $l_{m+1} < \lambda l_m$ for every large enough m, whence

$$\min_{l_m<k<l_{m+1}} (x_k - x_{l_m}) \ge \min_{l_m<k\le\lambda l_m} (x_k - x_{l_m}).$$

By (3.11), we find that

$$\liminf_{m\to\infty} \min_{l_m<k<l_{m+1}} (x_k - x_{l_m}) \ge -\varepsilon. \tag{3.12}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\liminf_{m\to\infty} \min_{l_m<k<l_{m+1}} (x_k - x_{l_m}) \ge 0.$$

Taking into account that

$$\min_{l_m<k<l_{m+1}} x_k = \min_{l_m<k<l_{m+1}} (x_k - x_{l_m}) + x_{l_m},$$

by (3.10) we conclude that

$$\liminf_{m\to\infty} \min_{l_m<k<l_{m+1}} x_k \ge L. \tag{3.13}$$

On the other hand, by Lemma 1, for every $\varepsilon > 0$ there exists $\lambda = \lambda(\varepsilon) < 1$ such that

$$\liminf_{n\to\infty} \min_{\lambda n\le k<n} (x_n - x_k) \ge -\varepsilon. \tag{3.14}$$

Since for every large enough m, we have

$$\min_{l_m<k<l_{m+1}} (x_{l_{m+1}} - x_k) \ge \min_{\lambda l_{m+1}\le k<l_{m+1}} (x_{l_{m+1}} - x_k),$$

by (3.14) we conclude that

$$\liminf_{m\to\infty} \min_{l_m<k<l_{m+1}} (x_{l_{m+1}} - x_k) \ge -\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, it follows that

$$\liminf_{m\to\infty} \min_{l_m<k<l_{m+1}} (x_{l_{m+1}} - x_k) \ge 0.$$

Taking into account that

$$\min_{l_m<k<l_{m+1}}(-x_k)=\min_{l_m<k<l_{m+1}}(x_{l_{m+1}}-x_k)-x_{l_{m+1}},$$

by (3.10) we find that

$$\liminf_{m\to\infty}\min_{l_m<k<l_{m+1}}(-x_k)\ge -L,$$

which is equivalent to

$$\limsup_{m\to\infty}\max_{l_m<k<l_{m+1}}x_k\le L. \tag{3.15}$$

Combining (3.13) and (3.15) yields

$$L\le\liminf_{m\to\infty}\min_{l_m<k<l_{m+1}}x_k\le\limsup_{m\to\infty}\max_{l_m<k<l_{m+1}}x_k\le L,$$

which together with (3.10) shows that the whole sequence (x_k) is convergent to L. ■

LEMMA 7. *Let (x_k) be a sequence of complex numbers. If (x_k) is statistically convergent to some L and slowly oscillating, then (x_k) is convergent to L.*

Proof. It is similar to (and even simpler than) the proof of Lemma 6. Again, we start with the decomposition theorem, consider the subsequence $1\le l_1<l_2<\cdots$ of those indices k for which $y_k=x_k$, and have (3.9) and (3.10).

This time, by (1.9), for every $\varepsilon>0$ there exists $\lambda=\lambda(\varepsilon)>1$ such that

$$\limsup_{n\to\infty}\max_{n<k\le\lambda n}|x_k-x_n|\le\varepsilon. \tag{3.16}$$

Analogously to (3.11) and (3.12), by (3.9) and (3.16), we conclude that

$$\limsup_{m\to\infty}\max_{l_m<k<l_{m+1}}|x_k-x_{l_m}|\le\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, it follows that

$$\lim_{m\to\infty}\max_{l_m<k<l_{m+1}}|x_k-x_{l_m}|=0. \tag{3.17}$$

Combining (3.10) and (3.17) implies that the whole sequence (x_k) is convergent to L. ■

We note that if Lemmas 6 and 7 were true under the weaker assumptions of Lemmas 3 and 4, respectively, then we could prove stronger versions of Theorems 1 and 2.

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. First, we prove that if the sequence (x_k) of real numbers is slowly decreasing, then so is the sequence (σ_n) of the first arithmetic means. To this end, let $\varepsilon>0$ be given. By the slow decrease of (x_k), there exist $n_0 = n_0(\varepsilon)$ and $\lambda_0 = \lambda_0(\varepsilon) > 1$, as close to 1 as we wish, such that (1.5) is satisfied.

Let $n_0 \le n < k \le \lambda_0 n$. Then by (1.2) we obtain

$$\begin{aligned}\sigma_k - \sigma_n &= -\frac{k-n}{kn}\sum_{j=1}^{n} x_j + \frac{1}{k}\sum_{j=n+1}^{k} x_j \\ &= \frac{k-n}{kn}\sum_{j=1}^{n}(x_n - x_j) + \frac{1}{k}\sum_{j=n+1}^{k}(x_j - x_n).\end{aligned} \tag{4.1}$$

By Lemma 3, there exists a positive constant B such that

$$\frac{1}{n}\sum_{j=1}^{n}(x_n - x_j) \ge -B, \quad n = 1, 2, \ldots.$$

Using this inequality and (1.5), we may estimate the right-hand side in (4.1) as follows:

$$\sigma_k - \sigma_n \ge \frac{k-n}{k}(-B) + \frac{1}{k}(k-n)(-\varepsilon) = -\left(1 - \frac{n}{k}\right)(B + \varepsilon). \tag{4.2}$$

Since for $n < k \le \lambda_0 n$ and $\lambda_0 > 1$, we have

$$1 - \frac{n}{k} \le 1 - \frac{1}{\lambda_0} < \lambda_0 - 1, \tag{4.3}$$

it follows from (4.2) that

$$\sigma_k - \sigma_n \ge -(\lambda_0 - 1)(B + \varepsilon) \ge -\varepsilon, \quad n_0 \le n < k < \lambda_0 n,$$

provided that

$$1 < \lambda_0 \le 1 + \varepsilon/(B + \varepsilon). \tag{4.4}$$

This proves that the sequence (σ_n) is also slowly decreasing.

By assumption, the sequence (σ_n) is statistically convergent to L. Consequently, by Lemma 6, (σ_n) is convergent to L in the ordinary sense. Applying Schmidt's classical Tauberian theorem (see [8]) yields the ordinary convergence of the sequence (x_k) itself. ∎

Proof of Theorem 2. First, we prove that if (x_k) is slowly oscillating, then so is (σ_n). Let $\varepsilon > 0$ be given. By the slow oscillation of (x_k), there exist $n_0 = n_0(\varepsilon)$ and $\lambda_0 = \lambda_0(\varepsilon) > 1$, as close to 1 as we wish, such that (1.10) is satisfied.

Let $n_0 \le n < k \le \lambda_0 n$. Then by (4.1) we have

$$|\sigma_k - \sigma_n| \le \frac{k-n}{kn}\sum_{j=1}^{n}|x_n - x_j| + \frac{1}{k}\sum_{j=n+1}^{k}|x_j - x_n|.$$

By Lemma 4, there exists a constant B such that

$$\frac{1}{n}\sum_{j=1}^{n}|x_n - x_j| \le B, \quad n = 1, 2, \dots .$$

Similarly to (4.2) and (4.3), this time we conclude that

$$|\sigma_k - \sigma_n| \le \left(1 - \frac{n}{k}\right)(B+\varepsilon) < (\lambda_0 - 1)(B+\varepsilon) < \varepsilon, \qquad n_0 \le n < k \le \lambda_0 n,$$

provided (4.4) is satisfied. This proves that (σ_n) is also slowly oscillating.

By assumption, the sequence (σ_n) is statistically convergent to L. Consequently, by Lemma 7, (σ_n) is convergent to L in the ordinary sense. Applying Schmidt's classical Tauberian theorem yields the ordinary convergence of the sequence (x_k) itself. ∎

5. Appendix. Our goal is to give a new, more constructive proof of Vijayaraghavan's lemma (see [10, Lemma 6]), which plays a crucial role, via Lemma 3, in the proof of our Theorem 1. In addition, we prove Vijayaraghavan's lemma under the less restrictive condition (3.4) instead of the condition of slow decrease.

LEMMA 8. *Let (x_k) be a sequence of real numbers. If there exist a positive integer m_0 and a real number $\lambda > 1$ such that condition* (3.4) *is satisfied, then there exists a positive constant B such that*

(5.1) $$x_n - x_k \ge -B\log(n/k) \quad \textit{for all } 1 \le k \le n/\lambda.$$

Proof. Without loss of generality, we may assume that

(5.2) $$m_0 \ge 2\lambda/(\lambda - 1).$$

Given $n > m_0$, set $n_0 := n$ and define

(5.3) $$n_p := 1 + [n_{p-1}/\lambda], \quad p = 1, \dots, q,$$

where q is determined by the condition

(5.4) $$n_{q+1} \le m_0 < n_q.$$

It follows from (5.2) and (5.3) that

$$n_p < n_{p-1} < \lambda n_p, \quad p = 1, \dots, q+1,$$

and that (3.4) applies for each difference $x_{n_{p-1}} - x_{n_p}$.

Fix k such that $1 \le k \le n/\lambda$. First, we consider the case $m_0 \le k \le n/\lambda$. Then

(5.5) $$n_{p+1} \le k < n_p \quad \text{for some } 1 \le p \le q.$$

By (3.4), we estimate as follows:

(5.6) $$\begin{aligned} x_n - x_k = (x_n - x_{n_1}) + (x_{n_1} - x_{n_2}) + \dots + (x_{n_{p-1}} - x_{n_p}) \\ + (x_{n_p} - x_k) \ge -p - 1. \end{aligned}$$

By (5.3), we have

$$n_1 \le 1 + \frac{n}{\lambda}, \quad n_2 \le 1 + \frac{n_1}{\lambda} \le 1 + \frac{1}{\lambda} + \frac{n}{\lambda^2}, \; \ldots,$$
$$n_p \le 1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^{p-1}} + \frac{n}{\lambda^p} < \frac{\lambda}{\lambda - 1} + \frac{n}{\lambda^p}.$$

By this, (5.2) and (5.5), we conclude that

$$\frac{1}{2}\,\lambda^p \le \lambda^p\left(1 - \frac{\lambda}{(\lambda-1)m_0}\right) \le \lambda^p\left(1 - \frac{\lambda}{(\lambda-1)n_p}\right) < \frac{n}{n_p} < \frac{n}{k},$$

whence it follows that

$$p \le \frac{1}{\log\lambda}\log\frac{2n}{k}, \qquad n_{p+1} \le k < n_p,\; 1 \le p \le q. \tag{5.7}$$

Combining (5.6) and (5.7) gives

$$x_n - x_k \ge -1 - \frac{1}{\log\lambda}\log\frac{2n}{k}, \qquad m_0 \le k \le n/\lambda. \tag{5.8}$$

Second, we consider the case $1 \le k < m_0$. Again, by (3.4), we have (cf. (5.6))

$$\begin{aligned} x_n - x_k = (x_n - x_{n_1}) + (x_{n_1} - x_{n_2}) + \cdots + (x_{n_q} - x_{m_0}) \\ + (x_{m_0} - x_k) \ge -q - 1 + c, \end{aligned} \tag{5.9}$$

where

$$c := \min\{0, \min_{1\le k<m_0}(x_{m_0} - x_k)\}. \tag{5.10}$$

Similarly to (5.7), this time we find that

$$q \le \frac{1}{\log\lambda}\log\frac{2n}{k}, \qquad 1 \le k < m_0. \tag{5.11}$$

Combining (5.9) and (5.11) gives

$$x_n - x_k \ge -1 + c - \frac{1}{\log\lambda}\log\frac{2n}{k}, \qquad 1 \le k < m_0. \tag{5.12}$$

Since $c \le 0$ in (5.10), it follows from (5.8) and (5.12) that in either case we have

$$\begin{aligned} x_n - x_k &\ge -1 + c - \frac{\log 2}{\log\lambda} - \frac{\log(n/k)}{\log\lambda} \\ &\ge -B\log(n/k), \quad 1 \le k \le n/\lambda, \end{aligned} \tag{5.13}$$

provided

$$-1 + c - \frac{\log 2}{\log\lambda} \ge -\left(B - \frac{1}{\log\lambda}\right)\log\frac{n}{k}. \tag{5.14}$$

Now, this is the case if we define B as follows:

$$B := \frac{1}{\log\lambda}\left(2 - c + \frac{\log 2}{\log\lambda}\right). \tag{5.15}$$

In fact, by this choice of B we have

$$-1 + c - \frac{\log 2}{\log\lambda} = -\left(B - \frac{1}{\log\lambda}\right)\log\lambda \geq -\left(B - \frac{1}{\log\lambda}\right)\log\frac{n}{k},$$

where the equality is due to (5.15), while the inequality is due to the fact that $\lambda \leq n/k$ for all $1 \leq k \leq n/\lambda$.

To sum up, (5.13) holds with the choice (5.15) for B (which is clearly positive). This completes the proof of Lemma 8. ∎

The next lemma is new. It plays a crucial role in the proof of Lemma 4, which is of vital importance in the proof of our Theorem 2.

LEMMA 9. *Let (x_k) be a sequence of complex numbers. If there exist a positive integer m_0 and a real number $\lambda > 1$ such that condition* (3.5) *is satisfied*, *then there exists a constant B such that*

$$|x_n - x_k| \leq B\log(n/k) \quad \textit{for all } 1 \leq k \leq n/\lambda. \tag{5.16}$$

Proof. Again, we may assume that m_0 is large enough to satisfy (5.2). Given $n > m_0$, by (5.3) we define $n_0 := n > n_1 > n_2 > \cdots > n_q > m_0 \geq n_{q+1}$.

Fix k such that $1 \leq k \leq n/\lambda$. In case $m_0 \leq k \leq n/\lambda$, let p be defined by (5.5). Taking into account (3.5), this time (5.6) is of the form

$$\begin{aligned} |x_n - x_k| &\leq |x_n - x_{n_1}| + |x_{n_1} - x_{n_2}| + \cdots + |x_{n_{p-1}} - x_{n_p}| + |x_{n_p} - x_k| \\ &\leq p + 1, \end{aligned}$$

where p is estimated in (5.7), while (5.8) is of the form

$$|x_n - x_k| \leq 1 + \frac{1}{\log\lambda}\log\frac{2n}{k}, \quad m_0 \leq k \leq n/\lambda. \tag{5.17}$$

In case $1 \leq k < m_0$, (5.9) takes the form

$$\begin{aligned} |x_n - x_k| &\leq |x_n - x_{n_1}| + |x_{n_1} - x_{n_2}| + \cdots + |x_{n_q} - x_{m_0}| + |x_{m_0} - x_k| \\ &\leq q + 1 + c, \end{aligned}$$

again due to (3.5), where this time

$$c := \max_{1 \leq k < m_0} |x_{m_0} - x_k|$$

and q is estimated in (5.11). In place of (5.12), now we have

$$|x_n - x_k| \leq 1 + c + \frac{1}{\log\lambda}\log\frac{2n}{k}, \quad 1 \leq k < m_0. \tag{5.18}$$

Finally, we define (cf. (5.15))

$$B := \frac{1}{\log \lambda}\left(2 + c + \frac{\log 2}{\log \lambda}\right)$$

and (5.16) follows from (5.17) and (5.18) similarly to the way (5.1) followed from (5.8) and (5.12) in the proof of Lemma 8. ∎

Acknowledgments. We express our thanks to the referee for providing us with the historical remarks in the first paragraph of the introduction.

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Received 3 June 2003;
revised 15 January 2004 (4352)