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ON SOME PROPERTIES OF SQUARES OF SIERPIŃSKI SETS

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Abstract. We investigate some geometrical properties of squares of special Sierpiński sets. In particular, we prove that (under CH) there exists a Sierpiński set S and a function $p: S \to S$ such that the images of the graph of this function under $\pi'(\langle x, y\rangle) = x - y$ and $\pi''(\langle x, y\rangle) = x + y$ are both Lusin sets.

1. Notations and definitions. Let $\pi_x, \pi_y: \mathbb{R}^2 \to \mathbb{R}$ denote the projections on the first and second axis, respectively.

Define $\pi', \pi'': \mathbb{R}^2 \to \mathbb{R}$ and $\tau: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\pi'(\langle x, y\rangle) = x - y, \quad \pi''(\langle x, y\rangle) = x + y, \quad \tau(\langle x, y\rangle) = \langle x - y, x + y\rangle.$$

Notice that

$$\tau \circ \tau(\langle x, y\rangle) = \langle -2y, 2x\rangle, \quad \pi' \circ \tau(\langle x, y\rangle) = -2y, \quad \pi'' \circ \tau(\langle x, y\rangle) = 2x.$$

For $A \subseteq \mathbb{R}$ set

$$\mathcal{X}(A) = (\mathbb{R} \times A) \cup (A \times \mathbb{R}),$$

and for $X \subseteq \mathbb{R} \times \mathbb{R}$ and $a \in \mathbb{R}$ define

$$(X)_a = \{y \in \mathbb{R} : \langle a, y\rangle \in X\}, \quad (X)^a = \{x \in \mathbb{R} : \langle x, a\rangle \in X\}.$$

Let $\mathcal{N}$ and $\mathcal{MGR}$ denote the σ-ideals of measure zero sets and meager sets, respectively. We say that an uncountable X is a *Lusin set* (*Sierpiński set*, respectively) if $\forall_{Y \in \mathcal{MGR}}$ (resp. $\forall_{Y \in \mathcal{N}}$) $|X \cap Y| \leq \omega_0$.

A set $X \subseteq \mathbb{R}^n$ is said to be a *universal measure zero set* ($X \in \mathrm{UMZ}$) if every continuous (i.e. vanishing on singletons) finite, non-negative, countably additive measure defined on Borel subsets of $\mathbb{R}^n$ assigns X outer measure 0.

We will use the following terminology and notation from the theory of small subsets of the real line. For every $X \subseteq \mathbb{R}$ we write $X \in R^{\mathcal{N}}$ iff for each Borel set $B \subseteq \mathbb{R}^2$ such that $\forall_{x \in \mathbb{R}} (B)_x \in \mathcal{N}$ we have $\bigcup_{x \in X} (B)_x \neq \mathbb{R}$. Notice that every Sierpiński set belongs to the class $R^{\mathcal{N}}$; this result is due to J. Pawlikowski (see [P]).

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2. Introduction. From now on we shall identify a function with its graph. In [C], Cox showed, answering a question in [G], that (under CH) there exists a Sierpiński set $S \subseteq \mathbb{R}$ such that there exists a bijection $p: S \to S$ which is a universal measure zero set (on the plane). Moreover, $p^{-1} = p$. It is natural to ask whether we can find a Sierpiński set S and a bijection $p: S \to S$ such that the image of p under the rotation τ of the plane is a subset of a product of two UMZ sets, or a subset of a product of two strong measure zero sets, or two Lusin sets. And if the answer is yes, is this property of the Sierpiński set S stronger than the property considered by Cox?

The aim of this paper is to give an answer to these and other questions.

3. Squares of Sierpiński sets. Let us start with the following lemma due to I. Recław ([1]):

LEMMA 3.1 (Recław, private communication). *Suppose that $N \in \mathcal{N}$ and $M \in \mathcal{MGR}$. Then $\mathcal{X}(N) \cup \tau[\mathcal{X}(M)] \neq \mathbb{R}^2$.*

Proof. Without loss of generality we can assume that $M = \bigcup_{m\in\omega} F_m$, where the F_m are closed nowhere dense. It suffices to find

$$\langle x_0, y_0\rangle \notin (N \times \mathbb{R}) \cup \tau[\mathbb{R} \times M] \cup \tau[M \times \mathbb{R}] \cup (\mathbb{R} \times N).$$

Since $\langle x_0, y_0\rangle \notin \tau[M \times \mathbb{R}]$ is equivalent to $x_0 + y_0 \notin 2M$, and $\langle x_0, y_0\rangle \notin \tau[\mathbb{R} \times M]$ is equivalent to $y_0 - x_0 \notin 2M$, we have to find $x_0, y_0 \in \mathbb{R}$ such that $x_0, y_0 \notin N$ and $x_0 + y_0, x_0 - y_0 \notin M$. This condition can be written in the following form:

$$y_0 \notin N, \qquad x_0 \notin N \cup (M - y_0) \cup (M + y_0).$$

We will prove that the set $\{y \in \mathbb{R} : N \cup (M - y) \cup (M + y) = \mathbb{R}\}$ has measure zero. Choose an arbitrary perfect set $D \subseteq \mathbb{R} \setminus N$ such that for every open set $W \subseteq \mathbb{R}$ with $W \cap D \neq \emptyset$ we have $W \cap D \notin \mathcal{N}$. Let $(D_n)_{n\in\omega}$ be an enumeration of open basic subsets of D. We will use the following claim:

CLAIM 3.2. *Let E be a meager set and G be a Borel set such that $G \notin \mathcal{N}$. Then the set $\{t : G + t \subseteq E\}$ has measure zero.*

Proof. Set $Z = \{t : G + t \subseteq E\}$. Suppose on the contrary that $Z \notin \mathcal{N}$. Then by the classical theorem of Steinhaus the set $Z + G$ would contain an interval (a, b) and hence $(a, b) \subseteq Z + G \subseteq E$; therefore E would not be meager, which is the desired contradiction. ■

([1]) I would like to thank Professor Ireneusz Recław for his kind permission to include his result.

Hence we obtain

$$\{y : D \cap [(M-y) \cup (M+y)] \in \mathcal{MGR}(D)\} \\ = \bigcap_{n,m\in\omega} \{y : D_n \not\subseteq (F_m - y)\} \cap \{y : D_n \not\subseteq (F_m + y)\},$$

and this last set is of full measure. This proves that $\{y : D \subseteq (M-y) \cup (M+y)\}$ has measure zero. Since $D \cap N = \emptyset$, this finishes the proof. ■

Note that Lemma 3.1 can be easily strengthened to the following:

LEMMA 3.3. *Suppose that $M \in \mathcal{MGR}(\mathbb{R})$ and $N \in \mathcal{N}(\mathbb{R})$ and let $Y \notin \mathcal{N}$. Then there exist $x_0, y_0 \in \mathbb{R}$ such that $x_0 + y_0 \notin M$, $x_0 - y_0 \notin M$, $x_0 \notin N$ and $y_0 \in Y$.*

Proof. We only sketch the proof. As in the previous proof, the set $\{y : N \cup (M-y) \cup (M+y) = \mathbb{R}\}$ is of measure zero. Choose $y_0 \in Y$ such that $N \cup (M - y_0) \cup (M + y_0) \neq \mathbb{R}$. Next, choose $x_0 \notin N \cup (M - y_0) \cup (M + y_0)$ to complete the proof. ■

However, we have not been able to prove (or disprove) the following strengthening of Lemma 3.1.

PROBLEM 3.4. *Is it true that for every perfect set $P \subseteq \mathbb{R}$ of positive measure and for every $M \in \mathcal{MGR}(P)$ and $N \in \mathcal{N}$ there exists $\langle x, y\rangle \in (P \setminus M)^2$ such that $\langle x, y\rangle \notin \tau[\mathcal{X}(N)]$?*

Moreover, we have not been able to solve the following version of the previous problem. Let $\mathcal{C}$ denote the Cantor ternary set.

PROBLEM 3.5. *Is it true that for every $M \in \mathcal{MGR}(\mathcal{C})$ and $N \in \mathcal{N}$ there exists $\langle x, y\rangle \in (\mathcal{C} \setminus M)^2$ such that $\langle x, y\rangle \notin \tau[\mathcal{X}(N)]$?*

Lemma 3.1 motivated us to introduce here the following notion:

DEFINITION 3.6. Let $\mathcal{F} \subseteq P(\mathbb{R})$ be an arbitrary collection of subsets of the real line (not necessarily an ideal). We say that $\mathcal{F}$ has the *STRIC* (*stripes covering*) *property* if for any sets $X, Y \in \mathcal{F}$ we have

$$\mathcal{X}(X) \cup \tau[\mathcal{X}(Y)] \neq \mathbb{R}^2.$$

Thus we can reformulate Lemma 3.1 as follows:

LEMMA 3.7. *The family $\mathcal{N} \cup \mathcal{MGR}$ has the STRIC property.*

Suppose that $\theta \in [0, \pi)$. By π_θ: $\mathbb{R}^2 \to \mathbb{R}$ we denote the projection on the x-axis in direction θ. If $\mathcal{F} \subseteq P(\mathbb{R})$ then we define the following cardinal coefficient:

$$\mathrm{STRIC}(\mathcal{F}) = \min\Big\{|\Theta| : \Theta \subseteq [0,\pi) \wedge \exists_{F:\Theta\to\mathcal{F}} F \text{ is 1-1} \wedge \bigcup_{\theta\in\Theta} \pi_\theta^{-1}(F(\theta)) = \mathbb{R}^2\Big\}.$$

Problem 3.8. *Suppose that* $\langle \theta_i, \mu_i \in [0,\pi) : i \in \omega \rangle$, *where* $\theta_i \neq \mu_i$, *and let* $M \in \mathcal{MGR}$ *and* $N \in \mathcal{N}$. *Is it true that*

$$\bigcup_{i\in\omega} \pi_{\theta_i}^{-1}(M) \cup \bigcup_{i\in\omega} \pi_{\mu_i}^{-1}(N) \neq \mathbb{R}^2?$$

If the answer is positive then we can ask what is $\mathrm{STRIC}(\mathcal{N} \cup \mathcal{MGR})$.

Theorem 3.9. *Assume CH. There exists a Sierpiński set* $S \subseteq \mathbb{R}$ *and a bijection* $p: S \to S$ *such that* $\pi'[p]$ *and* $\pi''[p]$ *are Lusin sets. Moreover,* $p^{-1} = p$.

Proof. Let $\langle M_\xi : \xi \in \omega_1 \rangle$ be an enumeration of all F_σ meager subsets of $\mathbb{R}$ and let $\langle N_\xi : \xi \in \omega_1 \rangle$ be an enumeration of all G_δ measure zero subsets of $\mathbb{R}$. At stage ξ we assume that all $x_\mu, y_\mu \in \mathbb{R}$ for $\mu < \xi$ have been chosen. Choose

$$\langle x_\xi, y_\xi \rangle \notin \mathcal{X}\Big(\bigcup_{\mu<\xi} N_\mu \cup \{x_\mu, y_\mu : \mu < \xi\}\Big) \cup \tau\bigg[\mathcal{X}\bigg(\bigcup_{\mu<\xi} \frac{M_\mu \cup -M_\mu}{2} \cup \{0\} \cup \bigg\{\frac{x_\mu}{2}, \frac{y_\mu}{2} : \mu < \xi\bigg\}\bigg)\bigg].$$

Notice that this is possible by Lemma 3.1. Finally, let $S = \{x_\xi, y_\xi : \xi < \omega_1\}$.

Now define a bijection $p: S \to S$ by

$$p(s) = \begin{cases} x_\xi & \text{if } s = y_\xi \text{ for some } \xi < \omega_1, \\ y_\xi & \text{if } s = x_\xi \text{ for some } \xi < \omega_1. \end{cases}$$

By construction, S is a Sierpiński set. To show that $\pi'[p]$ and $\pi''[p]$ are Lusin sets, it suffices to check that

$$\pi'(\langle x_\xi, y_\xi \rangle), \pi'(\langle y_\xi, x_\xi \rangle), \pi''(\langle x_\xi, y_\xi \rangle) \notin M_\mu \quad \text{for } \xi > \mu,$$

i.e. $x_\xi - y_\xi, y_\xi - x_\xi, x_\xi + y_\xi \notin M_\mu$ for $\xi > \mu$. But this follows immediately from the definition of $\langle x_\xi, y_\xi \rangle$, since $\langle x_\xi, y_\xi \rangle \notin \tau[\mathcal{X}((M_\mu \cup -M_\mu)/2)]$, i.e. $\tau^{-1}(x_\xi, y_\xi) \notin \mathcal{X}((M_\mu \cup -M_\mu)/2)$, and therefore $(y_\xi - x_\xi)/2 \notin M_\mu/2$, $(y_\xi - x_\xi)/2 \notin -M_\mu/2$ and $(x_\xi + y_\xi)/2 \notin M_\mu/2$. ∎

We will show the following theorem.

Theorem 3.10. *Assume CH. Let* $Y \subseteq \mathbb{R}$ *be such that* $Y \notin \mathcal{N}$. *Then there exists a Sierpiński set* S *and a function* $p: S \to Y$ *such that* $\pi'[p]$ *and* $\pi''[p]$ *are Lusin sets.*

Proof. Let $(M_\theta)_{\theta\in\omega_1}$ and $(N_\theta)_{\theta\in\omega_1}$ be enumerations of all F_σ meager subsets and of all G_δ, measure zero subsets of $\mathbb{R}$, respectively. From Lemma 3.3 it follows that there exist x_θ, y_θ such that

- $x_\theta + y_\theta, x_\theta - y_\theta \notin \bigcup_{\alpha<\theta} M_\alpha \cup \{x_\alpha + y_\alpha, x_\alpha - y_\alpha\}$,

- $x_\theta, y_\theta \notin \bigcup_{\alpha<\theta} N_\alpha \cup \{x_\alpha + y_\alpha, x_\alpha - y_\alpha\}$,
- $y_\theta \in Y$.

Set $S = \{x_\theta : \theta < \omega_1\}$. It is obvious that S is a Sierpiński set. Define $p = \{\langle x_\theta, y_\theta\rangle : \theta \in \omega_1\}$. Obviously p: $S \to Y$. Since $\pi'[p] = \{x_\theta - y_\theta : \theta \in \omega_1\}$ and $\pi''[p] = \{x_\theta + y_\theta : \theta \in \omega_1\}$, both are Lusin sets. ∎

Following [C] let us recall the following notions. We say that $H \subseteq \mathbb{R}^2$ is *symmetric* provided that $\forall_{\langle x,y\rangle \in H}\, \langle y, x\rangle \in H$. The lines $\{\langle x, x\rangle : x \in \mathbb{R}\}$ and $\{\langle x, 1-x\rangle : x \in \mathbb{R}\}$ are denoted by l_1 and l_2, respectively.

We denote by ϱ the projection onto l_2 defined by $\varrho(\langle x, y\rangle) = \langle x - y, 1 - x + y\rangle$. Notice that $\pi_x \circ \varrho = \pi'$.

Recall a lemma from [C] but relativised to a perfect set.

Lemma 3.11. *Let $P \subseteq \mathbb{R}$ be a perfect set. Suppose that μ is a finite measure on $\mathbb{R}^2$. Then there exists a symmetric subset H of $\mathbb{R}^2$ such that*

- *$\mu(H) = 0$.*
- *There exists a G_δ subset $h \subseteq \varrho(P \times \{0\})$, dense in $\varrho(P \times \{0\})$ and such that $\varrho^{-1}(h) \subseteq H$.*

Proof. Similar to the proof of the Lemma from [C]. ∎

In the next theorem we modify the construction of a Sierpiński set S and a function p: $S \to S$ having universal measure zero (given in the Theorem from [C]) to obtain the additional property that there is no function r: $S \to S$ such that $\pi'[r]$ and $\pi''[r]$ are Lusin sets.

Theorem 3.12. *Assume CH. Suppose that $M \in \mathcal{MGR}$ is such that $\mathbb{R} \setminus M \in \mathcal{N}$ and $0 \in M$. Then there exist a Sierpiński set $S \subseteq \mathbb{R}$ and a bijection p: $S \to S$ such that*

- *p is a universal measure zero set,*
- *$\tau[S^2] \subseteq M^2$.*

In particular, there is no function r: $S \to S$ such that $\pi'[r]$ and $\pi''[r]$ are Lusin sets. Moreover, we can assume that $p = p^{-1}$.

Proof. Let $P \subseteq M \cap (-M)$ be a perfect set such that $-P = P$. Let $\langle G_\alpha : \alpha < \omega_1\rangle$ be an enumeration of all G_δ measure zero subsets of $\mathbb{R}$. Inductively we construct (analogously to [C]) $\{\langle x_\alpha, y_\alpha\rangle : \alpha < \omega_1\} \subset \mathbb{R}^2$. Let $\langle \mu_\alpha : \alpha < \omega_1\rangle$ be an enumeration of all finite Borel measures on $\mathbb{R}^2$. For each $\alpha < \omega_1$ choose a symmetric subset $H_\alpha \subseteq \mathbb{R}^2$ as in Lemma 3.11. At stage θ we assume that all points $\{\langle x_\alpha, y_\alpha\rangle : \alpha < \theta\}$ have been chosen. Choose $\langle x_\theta, y_\theta\rangle$ such that

$$(1) \qquad \langle x_\theta, y_\theta\rangle \in \bigcap_{\alpha\le\theta} H_\alpha,$$

$$(2) \qquad \langle x_\theta, y_\theta\rangle \notin \pi_x^{-1}\Big(\bigcup_{\alpha\le\theta} G_\alpha \cup \{x_\alpha, y_\alpha : \alpha<\theta\}\Big) \cup \pi_y^{-1}\Big(\bigcup_{\alpha\le\theta} G_\alpha \cup \{x_\alpha, y_\alpha : \alpha<\theta\}\Big),$$

$$(3) \qquad \langle x_\theta, y_\theta\rangle \in (M_\theta^*)^2,$$

where $M_\theta^* = \bigcap_{\alpha<\theta}(M-x_\alpha)\cap(M-y_\alpha)\cap(x_\alpha-M)\cap(y_\alpha-M)\cap(M+x_\alpha)\cap(M+y_\alpha)$, and

$$(4) \qquad x_\theta, y_\theta \in M/2,$$

$$(5) \qquad \langle x_\theta, y_\theta\rangle \in H^*\cap H^{**},$$

where $H^* = \{\langle x,y\rangle : x+y\in M\}$ and $H^{**} = \{\langle x,y\rangle : x-y, y-x\in M\}$. To see that such a choice can be made, we first pick a line $k\subseteq \bigcap_{\alpha\le\theta} H_\alpha\cap H^{**}$ parallel to l_1, and then we pick a $\langle x_\theta, y_\theta\rangle$ on this line.

As in Theorem 3.9, set $S=\{x_\alpha, y_\alpha : \alpha<\omega_1\}$ and define a bijection $p\colon S\to S$ by

$$\begin{cases} p(x_\xi) = y_\xi, \\ p(y_\xi) = x_\xi. \end{cases}$$

Standard calculations show that $\tau[S^2]\subseteq M^2$. Similarly to [C] we can show that p is a universal measure zero set. By (2), S is a Sierpiński set. ∎

Unfortunately, the author has not been able to solve the following problem:

Problem 3.13. *Assume CH. Let M be as in our previous theorem. Does there exist a Sierpiński set $S\subseteq\mathbb{R}$ and a bijection $p\colon S\to S$ such that $\tau[S^2]\subseteq M^2$ and $\pi'[p]$, $\pi''[p]$ are universal measure zero sets?*

The following fact belongs to the set-theoretic folklore.

Lemma 3.14. *Suppose that $Q\subseteq\mathbb{R}$ is a perfect set and μ is a continuous Borel measure. Suppose that $p\colon X\to S$ (where $X\subseteq Q$ and S is a Sierpiński set) is a function with a* UMZ *graph. Then there exists a Borel set $B^*\subseteq\mathbb{R}^2$ such that*

$$p\subseteq B^*, \qquad \forall_{y\in\mathbb{R}}\, \mu[(B^*)^y\cap Q]=0.$$

Proof. Since $p\in$ UMZ, there exists a Borel set $B\subseteq\mathbb{R}^2$ such that $p\subseteq B$ and $(Q\times\mathbb{R})\cap B$ is of measure zero with respect to the product measure $\mu\otimes\lambda$, where λ is the Lebesgue measure on $\mathbb{R}$. By the Fubini theorem there exists a measure zero G_δ set $G\subseteq\mathbb{R}$ such that $\forall_{y\in\mathbb{R}\setminus G}\,\mu((B)^y\cap Q)=0$. Since S is a Sierpiński set, we have $|S\cap G|\le\omega$. Let $H\subseteq Q$ be such that $\forall_{y\in G\cap S}\,(p)^y\cap Q\subseteq H$, $\mu(H)=0$ and H is a G_δ set. Define $B^*=$

$[B \setminus (\mathbb{R} \times G)] \cup (H \times G)$. Obviously B^* is a Borel set. Moreover, $p \subseteq B^*$ and $\forall_{y\in\mathbb{R}}\, \mu[(B^*)^y \cap Q] = 0$. ∎

We will prove the following simple observation.

OBSERVATION 3.15. *Suppose that $S \subseteq \mathbb{R}$ is a Sierpiński set, $X \subseteq \mathbb{R}$ and there exists a* UMZ *function $p\colon X \to S$. Then X is a totally imperfect set, i.e. a set with no perfect subset.*

Proof. Suppose that there exists $p\colon X \to S$ such that $p \in$ UMZ. Let $Q \subseteq \mathbb{R}$ be any perfect set. Without loss of generality we may assume that Q is homeomorphic to the Cantor set. Let μ denote the Lebesgue measure defined on Q. By Lemma 3.14 we can find $B^* \subseteq \mathbb{R}^2$ such that $\forall_{y\in\mathbb{R}}\, \mu[(B^*)^{(y)} \cap Q] = 0$ and $p \subseteq B^*$.

Since S is a Sierpiński set, $S \in R^{\mathcal{N}}$. Therefore $\bigcup_{y\in S} (B^*)^y \cap Q \neq Q$, hence $Q \not\subseteq \bigcup_{y\in S} (B^*)^y$. Thus $Q \not\subseteq \bigcup_{y\in S} (p)^y$, hence $Q \not\subseteq X$. This proves that X is a totally imperfect set. ∎

On the other hand for some class of small sets (under CH) (in particular, for Sierpiński sets) there always exists a Sierpiński set such that no $p\colon S \to S$ is UMZ. Notice that Cox's argument to prove that (under CH) there exists a Sierpiński set S such that no $p\colon S \to S$ is UMZ involves Borel sets with all sections of measure zero. In the next theorem we modify Cox's argument using $R^{\mathcal{N}}$ sets.

THEOREM 3.16. *Assume CH. Suppose that S is a Sierpiński set. Then there exists a Sierpiński set S_1 such that $|S_1| = 2^\omega$ and no $p\colon S_1 \to S$ is* UMZ.

Proof. Suppose that $(B_\theta)_{\theta<\omega_1}$ is an enumeration of all Borel sets $B \subseteq \mathbb{R}^2$ such that $\forall_{y\in\mathbb{R}}\, (B)^y \in \mathcal{N}$. Let $(G_\theta)_{\theta\in\omega_1}$ be any enumeration of all G_δ measure zero subsets of $\mathbb{R}$. We will construct by transfinite induction a sequence $(x_\theta)_{\theta\in\omega_1}$. At stage θ we choose

$$x_\theta \notin \bigcup_{\alpha<\theta} G_\alpha \cup \{x_\alpha : \alpha < \theta\} \cup \Big(\bigcup_{\alpha<\theta} B_\alpha\Big)^{-1}[S],$$

where $B^{-1}[Y] = \{x : \exists_{y\in Y}\, \langle x, y\rangle \in B\}$. The choice of x_θ is possible, since $\forall_{y\in\mathbb{R}}\, (\bigcup_{\alpha<\theta} B_\alpha)^y \in \mathcal{N}$ and therefore $(\bigcup_{\alpha<\theta} B_\alpha)^{-1}[S] \notin$ co-$\mathcal{N}$, by the fact that $S \in R^{\mathcal{N}}$, since S is a Sierpiński set. Next, define $S_1 = \{x_\theta : \theta \in \omega_1\}$. It is easy to see that S_1 is a Sierpiński set. Moreover, suppose that $p\colon S_1 \to S$ has universal measure zero graph. By Lemma 3.14 (applied to $Q = \mathbb{R}$ and $\mu = \lambda$, the Lebesgue measure on $\mathbb{R}$) we can find a $\theta < \omega_1$ such that $p \subseteq B_\theta$. Next, $S_1 = p^{-1}[S] \subseteq B_\theta^{-1}[S]$, and therefore $x_\theta \notin S_1$, which is a contradiction. ∎

PROBLEM 3.17. *Can we assume in the previous theorem that S is only an $R^{\mathcal{N}}$ set?*

THEOREM 3.18. *Assume CH. Then there exists a Sierpiński set $S \subseteq \mathbb{R}$ such that there exists a* UMZ *function* $p: S \to S$ *but there is no function* $r: S \to S$ *such that* $\pi'[r], \pi''[r] \in \mathrm{UMZ}$.

Proof. We construct S by induction. Let $(\mu_\theta)_{\theta \in \omega}$ be an enumeration of all continuous measures defined on all Borel subsets of $\mathbb{R}^2$. For each $\theta \in \omega_1$ let H_θ be a dense G_δ subset of $\mathbb{R}$ such that $\mu_\theta((\pi')^{-1}[H_\theta]) = 0$ and $-H_\theta = H_\theta$. Such a set exists by Cox's Lemma from [C]. Let $(G_\theta)_{\theta<\omega_1}$ be an enumeration of all G_δ measure zero subsets of $\mathbb{R}$.

Assume that we have chosen x_μ, y_μ for $\mu < \theta$. First choose $t_\theta \in \bigcap_{\mu \le \theta} H_\mu \setminus \{0\}$. Next, choose x_θ such that

$$\{x_\theta, x_\theta - t_\theta\} \cap \{x_\mu, y_\mu : \mu < \theta\} = \emptyset, \tag{6}$$

$$\{x_\theta, x_\theta - t_\theta\} \cap \bigcup_{\mu<\theta} G_\mu = \emptyset, \tag{7}$$

$$(x_\theta + \{x_\mu, y_\mu : \mu < \theta\}) \cap G_\theta = \emptyset, \tag{8}$$

$$\{2x_\theta, 2x_\theta - t_\theta\} \cap G_\theta = \emptyset, \tag{9}$$

$$\{x_\theta, x_\theta - t_\theta\} \cap \bigcup_{\mu<\theta} (G_\mu - x_\mu) = \emptyset. \tag{10}$$

Define $y_\theta = x_\theta - t_\theta$ and $S = \{x_\theta, y_\theta : \theta < \omega_1\}$. It follows immediately from (6) and (7) that S is a Sierpiński set.

Define

$$p = \{\langle x_\theta, y_\theta\rangle : \theta < \omega_1\} \cup \{\langle y_\theta, x_\theta\rangle : \theta < \omega_1\}.$$

It is easy to observe that $p: S \to S$. Let $\theta \in \omega_1$. Then for each $\mu > \theta$ we have $x_\mu - y_\mu = t_\mu \in H_\theta$, hence $\langle x_\mu, y_\mu\rangle \in (\pi')^{-1}(H_\theta)$. Thus $\mu_\theta(\{\langle x_\mu, y_\mu\rangle : \mu > \theta\}) = 0$ and therefore $\mu_\theta(\{\langle x_\alpha, y_\alpha\rangle : \alpha < \omega_1\}) = 0$. Since $-t_\mu \in H_\theta$, the same argument shows that $\mu_\theta(\{\langle y_\alpha, x_\alpha\rangle : \alpha < \omega_1\}) = 0$, therefore $\mu_\theta(p) = 0$.

By way of contradiction suppose that there is $\widehat{p}: S \to S$ such that $\pi'[\widehat{p}], \pi''[\widehat{p}] \in \mathrm{UMZ}$. In particular, $\pi''[\widehat{p}] \in \mathcal{N}$, so $\pi''[\widehat{p}] \subseteq G_\theta$ for some $\theta < \omega_1$. Hence $x_\theta + \widehat{p}(x_\theta) \in G_\theta$. We will consider three cases.

CASE 1: $\widehat{p}(x_\theta) \in \{x_\mu, y_\mu\}$ for some $\mu < \theta$. Then $(x_\theta + \{x_\mu, y_\mu\}) \cap G_\theta \neq \emptyset$, contrary to (8).

CASE 2: $\widehat{p}(x_\theta) \in \{x_\theta, y_\theta\}$. Then $2x_\theta \in G_\theta$ or $2x_\theta - t_\theta \in G_\theta$, contrary to (9).

CASE 3: $\widehat{p}(x_\theta) \in \{x_\mu, y_\mu\}$ for some $\mu > \theta$. Then $(x_\theta + \{x_\mu, y_\mu\}) \cap G_\theta \neq \emptyset$, contrary to (10). Thus $\pi''[\widehat{p}] \notin \mathcal{N}$, which gives the required property of S. ■

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