COLLOQUIUM MATHEMATICUM

VOL. 99 2004 NO. 2

METRIC PROJECTIONS OF CLOSED SUBSPACES OF c_0 ONTO SUBSPACES OF FINITE CODIMENSION

BY

V. INDUMATHI (Pondicherry)

Abstract. Let X be a closed subspace of c_0. We show that the metric projection onto any proximinal subspace of finite codimension in X is Hausdorff metric continuous, which, in particular, implies that it is both lower and upper Hausdorff semicontinuous.

1. Proximinal subspaces of finite codimension. Let X be a real Banach space. Let $D \subseteq X$ and F be a map from D into a collection of non-empty subsets of X. If $x \in D$, the set-valued map F is *lower semicontinuous* at x if given $\varepsilon > 0$ and z in $F(x)$, there exists $\delta > 0$ such that for all y in D with $\|x-y\| < \delta$, there exists $w \in F(y)$ with $\|z-w\| < \varepsilon$. If δ can be chosen independent of z in $F(x)$ in the above definition, we say F is *lower Hausdorff semicontinuous* at x. The map F is said to be lower semicontinuous (resp. lower Hausdorff semicontinuous) on D if it is lower semicontinuous (resp. lower Hausdorff semicontinuous) at each point $x \in D$. A continuous map f defined on X, with $f(x)$ in $F(x)$ for each x in X, is called a *continuous selection* of the set-valued map F.

The set-valued map F is *upper Hausdorff semicontinuous* at x in D if given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(y) \subseteq F(x) + \varepsilon B_X$$

for all y in $D \cap B(x,\delta)$. The map F is said to be upper Hausdorff semicontinuous on D if it is upper Hausdorff semicontinuous at each $x \in D$. If $\mathbb{C}(Y)$ denotes the class of all bounded, closed convex subsets of Y, then $\mathbb{C}(Y)$ is a metric space with the *Hausdorff metric* given by

$$h(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

for A and B in $\mathbb{C}(Y)$. If $F(x)$ belongs to $\mathbb{C}(Y)$ for all x in $D \subseteq X$, we say F is *Hausdorff metric continuous* at x in D if the single-valued map F from D into the metric space $(\mathbb{C}(Y), h)$ is continuous. We say F is Hausdorff metric continuous on X if it is Hausdorff metric continuous at all x in X. We make

2000 *Mathematics Subject Classification*: 46B20, 41A50, 41A65.

Key words and phrases: proximinal, metric projection, lower semicontinuity.

an easy observation connecting the three semicontinuity concepts defined above.

REMARK 1.1. Let X and Y be Banach spaces and F a set-valued map from X into Y with $F(x)$ in $\mathbb{C}(Y)$ for all x in X. Then F is Hausdorff metric continuous at x in X if and only if F is both lower and upper Hausdorff semicontinuous at x.

Throughout, X denotes a real Banach space, B_X the closed unit ball of X, S_X the unit sphere of X, and $\operatorname{ext} B_X$ the set of extreme points of B_X. The class of all norm attaining functionals on X is denoted by $NA(X)$. For a subspace Y of X, let

$$Y^{\perp} = \{f \in X^* : f(x) = 0 \ \forall x \in Y\}$$

and if x is in X, $d(x,Y) = \inf\{\|x - y\| : y \in Y\}$. Further we set

$$D_Y = \{x \in X : d(x,Y) = 1\}.$$

All subspaces are assumed to be closed. Let Y be a subspace of X. For $x \in X$, let

$$P_Y(x) = \{y \in Y : \|x - y\| = d(x,Y)\}.$$

The subspace Y is said to be *proximinal* in X if for each $x \in X$, the set $P_Y(x)$ is non-empty. The set-valued map $P_Y : X \to 2^Y$ is called the *metric projection* onto Y. We set

$$Q_Y(x) = x - P_Y(x) \qquad \forall x \in X.$$

We note that an easy application of the duality formula

$$d(x,Y) = \max\{f(x) : f \in Y^{\perp},\ \|f\| = 1\}, \qquad x \in X,$$

implies that for any $x \in D_Y$,

$$Q_Y(x) = \{y \in S_X : f(x) = f(y)\ \forall f \in Y^{\perp}\}.$$

A finite-dimensional normed linear space X is called *polyhedral* if B_X has only a finite number of extreme points. In this case it can be shown that X^* is also polyhedral and every extreme point of B_X is, in fact, exposed. There are various notions of polyhedrality for infinite-dimensional Banach spaces (see [6]) and we use here the one given in [8] (see Definition 6.1 of [8]). We call an infinite-dimensional Banach space X *polyhedral* if every finite-dimensional subspace of X is polyhedral. We refer the reader to [6] and [8] for more details.

Proximinality and continuity properties of metric projections for subspaces of finite codimension have been studied for more than 40 years. Some sample references are [1], [2], [4], [5], [7], [9] and [12]–[21]. It is an easy consequence of one of Garkavi's earlier results [9] that if Y is a proximinal subspace of finite codimension in a normed linear space X, then $Y^{\perp}$ is contained in $NA(X)$. However, this condition is far from sufficient (see [12] or [18]).

It was observed in [12] that for a subspace Y of finite codimension in a Banach space X,

$$Y^{\perp} \subseteq NA(X) \text{ and } Y^{\perp} \text{ polyhedral } \Rightarrow Y \text{ is proximinal,}$$

and if X is a subspace of c_0, the above implication becomes an equivalence.

Fonf and Lindenstrauss [7] have considered spaces with property $(*)$ (see also Proposition 6.11 of [8] and Example 3.5 of [13]), defined as follows. A Banach space X has *property* $(*)$ if there exists a 1-norming subset B of S_{X^*} such that no weak* limit point of B of norm 1 attains its norm on B_X.

Property $(*)$ is hereditary and spaces with property $(*)$ are necessarily polyhedral. Also, it follows from the results of [10] that each polyhedral predual of l_1 (in particular c_0 and hence each subspace of c_0) has property $(*)$. In [7], the results of [12] for subspaces of c_0 are extended to Banach spaces with property $(*)$, and in particular it is shown that the above equivalences hold for Banach spaces with property $(*)$.

Easy examples are available to show that the above equivalences do not always hold. More sophisticated examples given in [7] show that there are polyhedral Banach spaces X such that $Y^{\perp} \subseteq NA(X)$ does not imply proximinality of Y, and proximinality of Y need not imply $Y^{\perp}$ is polyhedral, for a subspace Y of finite codimension in X.

By Michael's famous selection theorem, any lower semicontinuous map from a Banach space X into the class of all closed convex subsets of X has a continuous selection. However, examples are easily available (see for instance [3]) to show that lower semicontinuity is not necessary for the existence of a continuous selection.

In the rest of this section, we assume Y is a proximinal subspace of finite codimension of a Banach space X.

In a paper [13] subsequent to [12], it was shown that if $Y^{\perp}$ is polyhedral, then the metric projection P_Y has a continuous selection. This was done by constructing a lower semicontinuous submap of Q_Y and an application of Michael's selection theorem to this submap. It can easily be verified that this lower semicontinuous submap of Q_Y need not equal Q_Y, and this relatively short, simple proof (see Proposition 4.5 in [13]) for the existence of a continuous selection for P_Y does not seem adaptable to yield more, namely, the lower semicontinuity of P_Y.

We recall that if X has property $(*)$ then $Y^{\perp}$ is polyhedral. A natural question that arises in this context is whether P_Y is lower Hausdorff semicontinuous under suitable additional assumptions on X like having property $(*)$. This has been shown very recently by V. Fonf. In the special case when X

is a closed subspace of c_0, we prove the Hausdorff metric continuity of P_Y, which in particular implies the lower Hausdorff semicontinuity of P_Y (Theorem 4.3). We observe that, in this case, by the above quoted Proposition 4.5 of [13], the weaker conclusion that P_Y has a continuous selection is already known.

2. The set-valued map $Q_{f_1,\dots,f_k}$. We begin with some notation and remarks needed in what follows. If E is a normed linear space and $\{f_1,\dots,f_n\}$ is a finite subset of E^* and $x \in B_E$, we set

$$L_E(x, f_1, \dots, f_n) = \bigcap_{i=1}^{n} \{y \in B_E : f_i(y) = f_i(x)\}. \tag{1}$$

In the rest of this section, X denotes a Banach space and Y a subspace of finite codimension n in X. For $x \in D_Y$ and a finite set of functionals $f_1,\dots,f_k$ in $Y^\perp$, we define

$$Q_{f_1,\dots,f_k}(x) = \bigcap_{i=1}^{k} \{y \in B_X : f_i(y) = f_i(x)\}.$$

Remark 2.1. Note that $Q_{f_1,\dots,f_k}(x)$ always contains $Q_Y(x)$ and can be an empty set. However, if Y is proximinal, then $P_Y(x)$ and hence $Q_Y(x)$ is non-empty and so the sets $Q_{f_1,\dots,f_k}(x)$ are non-empty for any finite subset $f_1,\dots,f_k$ of $Y^\perp$. If $f_1,\dots,f_n$ is a basis of $Y^\perp$, then $Q_{f_1,\dots,f_n} = Q_Y$ irrespective of the basis $f_1,\dots,f_n$. ■

The following simple but useful remark is easily verified.

Remark 2.2. Let Y be proximinal. Then the following are equivalent.

(i) The metric projection P_Y is lower (resp. upper) Hausdorff semicontinuous on X.
(ii) The map Q_Y is lower (resp. upper) Hausdorff semicontinuous on D_Y.
(iii) For each $x \in D_Y$, there exists a basis $f_1,\dots,f_n$ of $Y^\perp$ such that the set-valued map $Q_{f_1,\dots,f_n}$, defined on the domain D_Y, is lower (resp. upper) Hausdorff semicontinuous at x. ■

We emphasize that the domain of the set-valued maps $Q_{f_1,\dots,f_k}$ will be assumed to be the set D_Y hereafter.

If $f_1,\dots,f_k$ is a linearly independent subset of $Y^\perp$, where $k > 1$ and Y is proximinal in X, we define numbers $\alpha_{x,k}$ and $\beta_{x,k}$, for x in D_Y, as follows:

$$\begin{aligned} \alpha_{x,k} &= \inf\{f_k(y) : y \in Q_{f_1,\dots,f_{k-1}}(x)\}, \\ \beta_{x,k} &= \sup\{f_k(y) : y \in Q_{f_1,\dots,f_{k-1}}(x)\}. \end{aligned} \tag{2}$$

We begin with a result on Hausdorff metric continuity of the maps $Q_{f_1,\dots,f_k}$. This is needed in the proof of the main theorem, Theorem 4.3. The proof uses arguments very similar to that of Theorem 2.5 in [13].

PROPOSITION 2.3. *Let X be a Banach space, Y be proximinal in X and $x \in D_Y$. Assume that there exists a finite subset $\{f_1, \dots, f_{k+1}\}$, $1 \le k < n$, of $Y^\perp$ such that the map $Q_{f_1,\dots,f_k}$ is Hausdorff metric continuous at x and further*

$$\alpha_{x,k+1} < f_{k+1}(x) < \beta_{x,k+1}.$$

Then $Q_{f_1,\dots,f_{k+1}}$ is Hausdorff metric continuous at x.

Proof. By Remark 1.1, we need to show that $Q_{f_1,\dots,f_{k+1}}$ is both lower and upper Hausdorff semicontinuous at x. Let

$$2\eta = \min\{\beta_{x,k+1} - f_{k+1}(x), f_{k+1}(x) - \alpha_{x,k+1}\}. \tag{3}$$

Then $\eta > 0$.

We first prove the lower Hausdorff semicontinuity. Since $Q_{f_1,\dots,f_k}$ is lower Hausdorff semicontinuous at x, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any z in $Q_{f_1,\dots,f_k}(x)$ and y in D_Y with $\|x - y\| < \delta$, there exists w in $Q_{f_1,\dots,f_k}(y)$ such that $\|z - w\| < \eta\varepsilon/8$. Without loss of generality we assume that $0 < \delta < \eta\varepsilon/8$, $0 < \varepsilon < 1$, and $\|f_i\| = 1$ for $1 \le i \le n$. Now, if $y \in D_Y$ and $\|x - y\| < \delta$, it follows easily that

$$\beta_{y,k+1} > \beta_{x,k+1} - \eta/8, \qquad \alpha_{y,k+1} < \alpha_{x,k+1} + \eta/8, \tag{4}$$

$$\alpha_{y,k+1} < f_{k+1}(y) < \beta_{y,k+1}. \tag{5}$$

Fix $z \in Q_{f_1,\dots,f_{k+1}}(x)$. We have to show that there exists v in $Q_{f_1,\dots,f_{k+1}}(y)$ such that $\|z - v\| < \varepsilon$.

Since $Q_{f_1,\dots,f_{k+1}}(x) \subseteq Q_{f_1,\dots,f_k}(x)$, there exists w in $Q_{f_1,\dots,f_k}(y)$ such that $\|z - w\| < \eta\varepsilon/8$. We have

$$f_{k+1}(z) = f_{k+1}(x), \qquad \|w - z\| < \eta/8, \qquad \|x - y\| < \eta\varepsilon/8 < \eta/8.$$

This together with (4) and (5) implies

$$\begin{aligned} \beta_{y,k+1} - f_{k+1}(w) &= \beta_{y,k+1} - \beta_{x,k+1} + \beta_{x,k+1} - f_{k+1}(x) \\ &\quad + f_{k+1}(x) - f_{k+1}(z) + f_{k+1}(z) - f_{k+1}(w) \\ &> 2\eta - (\eta/8 + \eta/8) > \eta. \end{aligned} \tag{6}$$

Similarly we can show that

$$f_{k+1}(w) - \alpha_{y,k+1} > \eta. \tag{7}$$

Also,

$$\begin{aligned} |f_{k+1}(y) - f_{k+1}(w)| &\le |f_{k+1}(w) - f_{k+1}(z)| + |f_{k+1}(z) - f_{k+1}(x)| \\ &\quad + |f_{k+1}(x) - f_{k+1}(y)| \\ &< \eta\varepsilon/8 + \eta\varepsilon/8 = \eta\varepsilon/4 < \eta/4. \end{aligned} \tag{8}$$

If $f_{k+1}(w) = f_{k+1}(y)$, then $w \in Q_{k+1}(y)$ and $\|w - z\| < \varepsilon$. Take $v = w$ in this case.

Otherwise, we slightly perturb w to get an element of $Q_{f_1,\dots,f_{k+1}}(y)$ as follows. Note that using (6)–(8), we can get w_1 in $Q_{f_1,\dots,f_k}(y)$ such that

$$|f_{k+1}(w_1) - f_{k+1}(w)| > \eta, \tag{9}$$

and $f_{k+1}(y)$ lies in between $f_{k+1}(w)$ and $f_{k+1}(w_1)$. Choose $0 < \lambda < 1$ such that

$$f_{k+1}(\lambda w + (1-\lambda)w_1) = f_{k+1}(y)$$

and take $v = \lambda w + (1-\lambda)w_1$. Since w and w_1 are in $Q_{f_1,\dots,f_k}(y)$, v is in $Q_{f_1,\dots,f_{k+1}}(y)$. Also,

$$(1-\lambda)[f_{k+1}(w_1) - f_{k+1}(w)] = f_{k+1}(y) - f_{k+1}(w).$$

This together with (8) and (9) gives

$$1 - \lambda < \frac{\eta\varepsilon}{4\eta} = \varepsilon/4.$$

Hence

$$\begin{aligned} \|w - v\| &= (1-\lambda)\|w - w_1\| \le 2(1-\lambda) < 2\varepsilon/4 = \varepsilon/2, \\ \|z - v\| &\le \|z - w\| + \|w - v\| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

We now prove the upper Hausdorff semicontinuity. Since $Q_{f_1,\dots,f_k}$ is upper Hausdorff semicontinuous at x, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any y in D_Y with $\|x - y\| < \delta$ and for any w in $Q_{f_1,\dots,f_k}(y)$, there exists z in $Q_{f_1,\dots,f_k}(x)$ such that $\|z - w\| < \eta\varepsilon/8$. Without loss of generality we assume that $0 < \delta < \eta\varepsilon/8$, $0 < \varepsilon < 1$, and $\|f_i\| = 1$ for $1 \le i \le n$. We have to show that there exists v in $Q_{f_1,\dots,f_{k+1}}(x)$ such that $\|z - v\| < \varepsilon$.

We have

$$f_{k+1}(w) = f_{k+1}(y), \qquad \|w - z\| < \eta/8, \qquad \|x - y\| < \eta\varepsilon/8 < \eta/8.$$

Also

$$\beta_{x,k+1} - f_{k+1}(y) = \beta_{x,k+1} - f_{k+1}(x) + f_{k+1}(x) - f_{k+1}(y) > 2\eta - \eta/8$$

and so

$$\begin{aligned} \beta_{x,k+1} - f_{k+1}(z) &= \beta_{x,k+1} - f_{k+1}(y) + f_{k+1}(y) - f_{k+1}(w) + f_{k+1}(w) - f_{k+1}(z) \\ &> 2\eta - \eta/8 - \eta/8 > \eta. \end{aligned}$$

Similarly we can show that

$$f_{k+1}(z) - \alpha_{x,k+1} > \eta.$$

Now

$$\begin{aligned} |f_{k+1}(x) - f_{k+1}(z)| &\le |f_{k+1}(x) - f_{k+1}(y)| + |f_{k+1}(y) - f_{k+1}(w)| \\ &\quad + |f_{k+1}(w) - f_{k+1}(z)| \\ &\le \varepsilon\eta/8 + \varepsilon\eta/8 = \varepsilon\eta/4 < \eta/4. \end{aligned}$$

Note that we have now obtained (6)–(8) with x and z in place of y and w respectively. Hence there exists v in $Q_{f_1,\dots,f_{k+1}}(x)$ satisfying $\|z - v\| < \varepsilon$. This completes the proof. ■

3. Hausdorff metric continuity of metric projections. In this section, we obtain a sufficient condition (Theorem 3.10) for Hausdorff metric continuity of the metric projection onto a proximinal subspace of finite codimension. We need some facts about finite-dimensional convex sets, gathered in the remarks and propositions below.

Let E be a Banach space and C be a closed convex subset of E. Any convex extremal subset of C is called a *face* of C. If $f \in E^*$, we set

$$J_E(f) = \{x \in S_E : f(x) = \|f\|\}.$$

If non-empty, the closed convex subset $J_E(f)$ is a face of B_E and is called an *exposed face* of B_E. A face need not be an exposed face.

If $f_1, \dots, f_k$ are in E^*, we define inductively, for $2 \le i \le k$, as in [13],

$$J_E(f_1, \dots, f_i) = \{x \in J_E(f_1, \dots, f_{i-1}) : f_i(x) = k_i\}, \tag{10}$$

where

$$k_i = \sup\{f_i(y) : y \in J_E(f_1, \dots, f_{i-1})\}.$$

Also, for any $f \in E^*$, $\ker f$ denotes the kernel of f and $\dim A$ denotes the dimension of the set A. The *relative interior* of a convex subset A of a normed linear space X is the interior of A when A is considered as a subset of the affine hull of A, and is denoted by $\operatorname{rel.int} A$.

REMARK 3.1. Let E be an n-dimensional normed linear space and x belong to S_E. Then it is known and easily shown that the minimal face of B_E containing x is a proper face of B_E and there exists a linearly independent subset $\{f_1, \dots, f_m\}$ of E^* such that $F = J_E(f_1, \dots, f_m)$. If x is in $\operatorname{ext} B_E$, or equivalently F is a singleton, then m can be taken to be n (see Lemma 1 in [15]). If x is not an extreme point of B_E, then $\dim F > 0$ and $x \in \operatorname{rel.int} F$ as F is the minimal face of B_E containing x. Now, if $F - x = A$, then $A \subseteq M = \bigcap_{i=1}^m \ker f_i$. If $\dim A < \dim M$, we can select $f_{m+1}, \dots, f_k$ such that $\{f_1, \dots, f_m, f_{m+1}, \dots, f_k\}$ is a linearly independent subset of E^* and $L = \bigcap_{i=1}^k \ker f_i$ is the subspace generated by the set A. Let $\Gamma_x = L + x$. Then $F = \Gamma_x \cap B_E$. We observe that zero is in the relative interior of A and the relative interior of F coincides with the interior of F with respect to the affine set Γ_x. Further,

$$F = J_E(f_1, \dots, f_m) = J_E(f_1, \dots, f_k).$$

In summary, if x is in S_E, then there exists a linearly independent subset $\{f_1, \dots, f_k\}$ of E^* such that the minimal face F of B_E containing x is $F =$

$J_E(f_1, \ldots, f_k)$ ($k = n$ if x is extreme) and x is in the interior of F with respect to the affine set Γ_x, as defined above.

The set $J_E(f_1, \ldots, f_k)$ in the above remark turns out to be a finite intersection of exposed faces of B_E when E is a polyhedral space.

Remark 3.2. Let E be an n-dimensional, polyhedral normed linear space. For x in S_E, we set

$$A_x = \{f \in S_{E^*} : f(x) = 1\}, \qquad C_x = \{f \in \operatorname{ext} S_{E^*} : f(x) = 1\}.$$

Since E is polyhedral, C_x is a finite set. Also,

$$\bigcap_{f \in A_x} J_E(f) = \bigcap_{f \in C_x} J_E(f).$$

Let $\{f_i : 1 \leq i \leq k\}$ be a maximal linearly independent subset of C_x. If

$$L = \bigcap_{f \in C_x} \ker f = \bigcap_{i=1}^{k} \ker f_i, \qquad \Gamma_x = L + x, \qquad \gamma_x = \Gamma_x \cap B_E,$$

then by Lemma I.5 of [7], x is in the interior of γ_x with respect to the affine set Γ_x, or equivalently, x is in the relative interior of the convex set γ_x. Since γ_x is an extremal subset of B_E, this implies $F = \gamma_x$, where F is the minimal face of B_E containing x. Clearly,

$$F = \gamma_x = \bigcap_{i=1}^{k} J_E(f_i). \tag{11}$$

We now make the following definition.

Definition 3.3. Let Y be a proximinal subspace of codimension n in a normed linear space X, and x an element of D_Y. We say x is a *k-corner point*, $1 \leq k \leq n$, with respect to a linearly independent set of functionals $f_1, \ldots, f_k$ in $Y^\perp$ if

$$Q_{f_1, \ldots, f_k}(x) = \bigcap_{i=1}^{k} J_X(f_i).$$

We need the following proposition (Proposition 2.4 in [13]). We present it with a minor correction in the statement.

Proposition 3.4. *Let E be an n-dimensional normed linear space, Φ be an element of $S_E \setminus \operatorname{ext} B_E$, and $F = J_E(f_1, \ldots, f_k)$ the minimal face to which Φ belongs, for suitable linearly independent functionals $f_1, \ldots, f_k$ in E^*. Then the set $\{f_1, \ldots, f_k\}$ can be expanded to a linearly independent set $\{f_1, \ldots, f_k, f_{k+1}, \ldots, f_l\}$ in E^* such that*

$$\inf\{f_i(\psi) : \psi \in L_E(\Phi, f_1, \ldots, f_{i-1})\} < f_i(\Phi) < \sup\{f_i(\psi) : \psi \in L_E(\Phi, f_1, \ldots, f_{i-1})\}$$

for all $k+1 \le i \le l$ *and for* $L_E(\Phi, f_1, \ldots, f_l) = \{\Phi\}$, *where the sets* $L_E(\Phi, f_1, \ldots, f_i)$ *are given by* (1).

The lemma below shows that if the functionals $f_1, \ldots, f_k$ are chosen as in Remark 3.1, then in the above proposition $l = n$ and $f_1, \ldots, f_n$ are, in fact, a basis of $Y^\perp$.

LEMMA 3.5. *Let* E *be an* n*-dimensional normed linear space*, x *be in* $S_E \backslash \operatorname{ext} B_E$, *and the set* F *and the functionals* $f_1, \ldots, f_k$ *be as in Remark* 3.1. *If* $\{f_{k+1}, \ldots, f_l\}$ *is a finite subset of* E^* *such that*

$$\{x\} = \bigcap_{i=1}^{l} \{z \in B_E : f_i(z) = f_i(x)\}$$

then the set $\{f_1, \ldots, f_l\}$ *is total over* E.

Proof. Since x is not an extreme point of B_E, $\dim F > 0$. Let Γ_x denote the affine set $x + \bigcap_{i=1}^k \ker f_i$. Then by Remark 3.1, there exists $\delta > 0$ such that if z in Γ_x satisfies $\|x - z\| < \delta$, then $z \in F$. Select any y in E such that $f_i(y) = 0$ for all $1 \le i \le l$. We will show that $y = 0$. We can assume $\|y\| < \delta$. Then $x + y \in F$ and hence $\|x + y\| \le 1$. Thus, $x + y \in \bigcap_{i=1}^l \{z \in B_E : f_i(z) = f_i(x)\}$ and by our assumption, y must be the zero element. ∎

The following proposition is an immediate consequence of Proposition 3.4 and the above lemma.

PROPOSITION 3.6. *Let* E *be an* n*-dimensional normed linear space*, Φ *be in* $S_E \setminus \operatorname{ext} B_E$, *and* $F = J_E(f_1, \ldots, f_k)$ *be the minimal face to which* Φ *belongs*, *for suitable linearly independent functionals* $f_1, \ldots, f_k$ *in* E^* *so that* Φ *is in the interior of* F *with respect to the affine set* $\Phi + \bigcap_{i=1}^k \ker f_i$. *Then the set* $\{f_1, \ldots, f_k\}$ *can be expanded to a basis* $\{f_1, \ldots, f_n\}$ *in* E^* *such that*

$$\inf\{f_i(\psi) : \psi \in L_E(\Phi, f_1, \ldots, f_{i-1})\} < f_i(\Phi) < \sup\{f_i(\psi) : \psi \in L_E(\Phi, f_1, \ldots, f_{i-1})\} \tag{12}$$

for all $k+1 \le i \le n$. ∎

Let $x \in X$. We denote by $\widehat{x}$ the image of x under the canonical embedding of X into X^{**}, and let $\widehat{x}_{|Y^\perp}$ denote the restriction of $\widehat{x}$ to $Y^\perp$.

Define a map $C_{Y^\perp} : X \to (Y^\perp)^*$ by $C_{Y^\perp}(x) = \widehat{x}_{|Y^\perp}$.

REMARK 3.7. Note that for any x in X and f in $Y^\perp$, we have

$$f(x) = (C_{Y^\perp}(x))(f), \qquad C_{Y^\perp}(D_Y) \subseteq S_{(Y^\perp)^*}.$$

An easily verified result of Garkavi, given in [9], says that

$$Y \text{ is proximinal in } X \Leftrightarrow C_{Y^\perp}(B_Y) = B_{(Y^\perp)^*}.$$

Let Y be a proximinal subspace of codimension n in X, x in D_Y, $\Phi = C_{Y^\perp}(x)$ and $\{f_1, \ldots, f_n\}$ a basis of $Y^\perp$. Considering $\{f_1, \ldots, f_n\}$ as a basis of $(Y^\perp)^{**}$, for any positive integer k with $1 < k \le n$, select any ψ in $L_{(Y^\perp)^*}(\Phi, f_1, \ldots, f_{k-1})$. Now Garkavi's condition shows that there exists z in B_X such that $C_{Y^\perp}(z) = \psi$. Clearly $z \in Q_{f_1,\ldots,f_{k-1}}(x)$. Note that we always have

$$C_{Y^\perp}(Q_{f_1,\ldots,f_{k-1}}(x)) \subseteq L_{(Y^\perp)^*}(\Phi, f_1, \ldots, f_{k-1}).$$

By proximinality of Y, it now follows that

$$C_{Y^\perp}(Q_{f_1,\ldots,f_{k-1}}(x)) = L_{(Y^\perp)^*}(\Phi, f_1, \ldots, f_{k-1}).$$

Hence, for $1 < k \le n$,

$$\alpha_{x,k} = \inf\{\psi(f_k) : \psi \in L_{(Y^\perp)^*}(\Phi, f_1, \ldots, f_{k-1})\},$$
$$\beta_{x,k} = \sup\{\psi(f_k) : \psi \in L_{(Y^\perp)^*}(\Phi, f_1, \ldots, f_{k-1})\},$$

where $\alpha_{x,k}$ and $\beta_{x,k}$ are given by (2).

We need the following characterization of proximinal subspaces of finite codimension.

PROPOSITION 3.8 ([16, Corollary 1.2]). *Let X be a normed linear space and Y be a subspace of finite codimension n in X. Then Y is proximinal in X if and only if for any basis $\{f_1, \ldots, f_n\}$ of $Y^\perp$,*

$$J_X(f_1, \ldots, f_k) \neq \emptyset$$

and

$$C_{Y^\perp}(J_X(f_1, \ldots, f_k)) = J_{(Y^\perp)^*}(f_1, \ldots, f_k) \quad \textit{for } 1 \le k \le n. \tag{13}$$

REMARK 3.9. Let X be a normed linear space and Y be a proximinal subspace of finite codimension n in X. Select any x in D_Y and let Φ be $C_{Y^\perp}(x)$. Then Φ is in $S_{(Y^\perp)^*}$.

Now assume Φ is in $\operatorname{ext} B_{(Y^\perp)^*}$. Taking $E = (Y^\perp)^*$ in Remark 3.1, we obtain a basis $\{f_1, \ldots, f_n\}$ of $Y^\perp$ such that

$$\{\Phi\} = J_{(Y^\perp)^*}(f_1, \ldots, f_n).$$

Then

$$L_{(Y^\perp)^*}(\Phi, f_1, \ldots, f_n) = J_{(Y^\perp)^*}(f_1, \ldots, f_n),$$

which together with (13) gives

$$Q_{f_1,\ldots,f_n}(x) = Q_Y(x) = J_X(f_1, \ldots, f_n).$$

If Φ is not in $\operatorname{ext} B_{(Y^\perp)^*}$ then taking $(Y^\perp)^*$ for E in Proposition 3.6, we get a basis $\{f_1, \ldots, f_n\}$ of $Y^\perp$ and a positive integer k with $1 \le k < n$ such that

$$\Phi \in J_{(Y^\perp)^*}(f_1, \ldots, f_k)$$

and (12) holds, with $(Y^\perp)^*$ in place of E. Clearly,

$$L_{(Y^\perp)^*}(\Phi, f_1, \ldots, f_k) = J_{(Y^\perp)^*}(f_1, \ldots, f_k)$$

and using (13) again, we have

$$J_X(f_1, \ldots, f_k) = Q_{f_1,\ldots,f_k}(x).$$

Now, by Remark 3.7,

$$C_{Y^\perp}(Q_{f_1,\ldots,f_i}(x)) = L_{(Y^\perp)^*}(\Phi, f_1, \ldots, f_i) \quad \text{for } k \le i \le n.$$

This together with (2) and also (12), with $(Y^\perp)^*$ in place of E, implies

$$\alpha_{x,i} < f_i(x) = \Phi(f_i) < \beta_{x,i} \quad \forall i \in \{k+1, \ldots, n\}. \tag{14}$$

Thus for each x in D_Y, there exists a basis $\{f_1, \ldots, f_n\}$ of $Y^\perp$ such that either

$$Q_{f_1,\ldots,f_n}(x) = J_X(f_1, \ldots, f_n)$$

or there exists a positive integer k with $1 \le k < n$ such that

$$Q_{f_1,\ldots,f_k}(x) = J_X(f_1, \ldots, f_k)$$

and (14) holds. *If further $Y^\perp$ is polyhedral, by Remark* 3.2, *the sets $J_X(f_1, \ldots, f_j)$ can be replaced by $\bigcap_{i=1}^{j} J_X(f_i)$, for j equal to n or k, in the above two equalities.*

Now we can prove the main result of this section.

THEOREM 3.10. *Let X be a Banach space and Y be a proximinal subspace of finite codimension n in X. Fix x in D_Y and a basis $\{f_1, \ldots, f_n\}$ of $Y^\perp$ as in Remark* 3.8. *Assume that the map $Q_{f_1,\ldots,f_k}$ is Hausdorff metric continuous at x if k is the largest integer, less than or equal to n, that satisfies*

$$Q_{f_1,\ldots,f_k}(x) = J_X(f_1, \ldots, f_k).$$

Then Q_Y, and hence the metric projection P_Y, is Hausdorff metric continuous at x.

Proof. By Remarks 1.1 and 2.2, it suffices to show that the map Q_Y, with domain D_Y, is Hausdorff metric continuous at x. If

$$Q_Y(x) = Q_{f_1,\ldots,f_n}(x) = J_X(f_1, \ldots, f_n)$$

there is nothing to prove. Otherwise, by Remark 3.9, there exists $1 \le k < n$ such that

$$Q_{f_1,\ldots,f_k}(x) = J_X(f_1, \ldots, f_k)$$

and (14) holds. Again by assumption, $Q_{f_1,\ldots,f_k}$ is Hausdorff metric continuous at x. Now, a repeated application of Proposition 2.4 using (14) shows that $Q_{f_1,\ldots,f_n} = Q_Y$ is Hausdorff metric continuous at x. ∎

It is now easily verified that Definition 3.3, Remark 3.9 and Theorem 3.10 yield

THEOREM 3.11. *Let X be a Banach space and Y be a proximinal subspace of finite codimension n in X with $Y^\perp$ polyhedral. Assume that, whenever x in D_Y is a k-corner point with respect to a set of linearly independent functionals $f_1, \ldots, f_k$ in $Y^\perp$, for some $1 \le k \le n$, then $Q_{f_1,\ldots,f_k}$ is Hausdorff metric continuous at x. Then the metric projection P_Y is Hausdorff metric continuous on X.*

4. Subspaces of $c_0(\mathbb{N})$. Let $\mathbb{N}$ denote the set of positive integers, and $c_0(\mathbb{N})$ the space of sequences of real scalars converging to zero with the usual sup norm, denoted by $\|\cdot\|_\infty$. Let X be a non-trivial subspace of $c_0(\mathbb{N})$, and Y a subspace of finite codimension n in X. By $Y^\perp$ we denote the annihilator of Y considered as a subspace of X, that is,

$$Y^\perp = \{F \in X^* : F(x) = 0 \ \forall x \in Y\}.$$

In this case, we have the following result, which is a corollary to Lemma 2 and Theorem 3 of [12] (see also Theorem III.5 in [11]).

PROPOSITION 4.1. *Let X be a subspace of $c_0(\mathbb{N})$ and Y be a subspace of finite codimension in X. Then*

$$Y^\perp \subseteq NA(X) \ \Leftrightarrow \ Y \text{ is proximinal and } Y^\perp \text{ is polyhedral.}$$

We can now state one of the main results of this paper.

THEOREM 4.2. *Let X be a subspace of $c_0(\mathbb{N})$ and Y be a proximinal subspace of finite codimension n in X. Assume x_0 in D_Y is a k-corner point for some $1 \le k \le n$ with respect to some linearly independent subset $\{F_1, \ldots, F_k\}$ of $Y^\perp$. Then the map $Q_{F_1,\ldots,F_k}$ is Hausdorff metric continuous at x_0.*

Before giving the rather long proof of Theorem 4.2, we observe that our main result, given below, follows immediately from Theorem 4.2, Proposition 4.1 and Theorem 3.11.

THEOREM 4.3. *Let X be a subspace of $c_0(\mathbb{N})$ and Y be a proximinal subspace of finite codimension in X. Then the metric projection P_Y is Hausdorff metric continuous on X.*

The above theorem is to be compared with Proposition 4.5 of [13], which says that if Y is a subspace of finite codimension in a Banach space X, with $Y^\perp$ polyhedral, then the metric projection P_Y has a continuous selection.

We now proceed to prove Theorem 4.2. The proof is split into a number of facts for clarity. We use the following notation in the proofs given below.

Let Λ denote a non-empty subset of $\mathbb{N}$. For $x = (x(n))_{n\geq 1}$ in $c_0(\mathbb{N})$, let x_Λ denote the element of $c_0(\mathbb{N})$ given by

$$x_\Lambda(n) = \begin{cases} x(n) & \text{if } n \in \Lambda, \\ 0 & \text{if } n \in \mathbb{N} \setminus \Lambda. \end{cases}$$

Similarly, if $f = (f(n))$ then f_Λ denotes the element of $l_1(\mathbb{N})$ given by

$$f_\Lambda(n) = \begin{cases} f(n) & \text{if } n \in \Lambda, \\ 0 & \text{if } n \in \mathbb{N} \setminus \Lambda. \end{cases}$$

We set $\Lambda^{\mathrm{c}} = \mathbb{N} \setminus \Lambda$.

Let X be a subspace of $c_0(\mathbb{N})$. We denote by X_Λ the subspace of $c_0(\mathbb{N})$ given by

$$X_\Lambda = \{x_\Lambda : x \in X\}.$$

If $X = c_0(\mathbb{N})$, we write $c_0(\Lambda)$ in place of X_Λ and set

$$l_1(\Lambda) = \{f = (f(n))_{n\geq 1} \in l_1(\mathbb{N}) : f(n) = 0 \ \forall n \in \Lambda^{\mathrm{c}}\}.$$

Also, we define a subspace of $l_1(\mathbb{N})$ by

$$X_\Lambda^\perp = \{f_\Lambda : f \in X^\perp\}.$$

We observe that the notation $X_\Lambda^\perp$ could have two legitimate meanings. However, we use it throughout to mean $(X^\perp)_\Lambda$, as in the definition above.

For convenience in notation, we denote by $X_{\Lambda^{\mathrm{c}}1}^\perp$ the closed unit ball of the subspace $X_{\Lambda^{\mathrm{c}}}^\perp$ of l_1. That is, we set

$$X_{\Lambda^{\mathrm{c}}1}^\perp = B_{X_{\Lambda^{\mathrm{c}}}^\perp} = \Big\{f \in X_{\Lambda^{\mathrm{c}}}^\perp : \|f\|_1 = \sum_{n=0}^{\infty} |f(n)| \leq 1\Big\},$$

where

$$X_{\Lambda^{\mathrm{c}}}^\perp = \{f_{\Lambda^{\mathrm{c}}} : f \in X^\perp\}.$$

Finally, c_0 denotes $c_0(\mathbb{N})$ and l_1 denotes $l_1(\mathbb{N})$.

Remark 4.4. Assume Λ is a finite subset of $\mathbb{N}$. Then X_Λ is a closed subspace of c_0. Also, it is easily verified that $X_{\Lambda^{\mathrm{c}}}^\perp$ is a weak* closed subspace of l_1. ■

If $x \in c_0$ and $f \in l_1$, we set

$$\langle x, f\rangle = \sum_{n=1}^{\infty} x(n)f(n), \qquad S(f) = \{n \in \mathbb{N} : f(n) \neq 0\}.$$

We recall that f is in $NA(c_0)$ if and only if $S(f)$ is a finite set.

The following remark is easy to verify.

Remark 4.5. Let X be c_0. Select any f in $NA(X)$ and x in $J_X(f)$. Let (y_n) be a sequence in B_X such that $\langle y_n, f\rangle \to 1$ as $n \to \infty$. Then, if $\Lambda = S(f)$,

$$\|(x-y_n)_\Lambda\|_\infty = \sup_{k\in\Lambda} |(x-y_n)(k)| \to 0 \quad \text{as } n\to\infty.$$

We now start proving Theorem 4.2 through a series of facts. *In the following results of this section, X denotes a subspace of c_0.*

FACT 4.6. *Let Λ be a finite subset of $\mathbb{N}$. Assume x in B_X, and (w_n) a sequence in B_X, are such that*

$$\|(x-w_n)_\Lambda\|_\infty = \sup_{k\in\Lambda} |(x-w_n)(k)| \to 0 \quad \text{as } n\to\infty.$$

Then

$$\lim_{n\to\infty} \sup\{\langle x-w_n, f\rangle : f\in X^\perp_{\Lambda^c 1}\} = 0.$$

Proof. Define a map T from $X^\perp$ into $X^\perp_{\Lambda^c}$ by

$$T(f) = f_{\Lambda^c}, \quad f\in X^\perp.$$

Then T is continuous, linear and onto. Since $X^\perp_{\Lambda^c}$ is a closed subspace of l_1, T is open. There exists an $M>0$ such that for any h in $X^\perp_{\Lambda^c 1}$, there exists an f in $X^\perp$ satisfying

$$\|f\|_1 \le M, \quad T(f) = f_{\Lambda^c} = h.$$

Note that in this case for any z in X we have

$$0 = \langle z, f\rangle = \langle z, f_\Lambda\rangle + \langle z, T(f)\rangle = \langle z, f_\Lambda\rangle + \langle z, h\rangle = \langle z_\Lambda, f_\Lambda\rangle + \langle z, h\rangle.$$

Thus

$$|\langle z, h\rangle| = |\langle z_\Lambda, f_\Lambda\rangle| \le M\|z_\Lambda\|_\infty$$

for any z in X and h in $X^\perp_{\Lambda^c 1}$.

Now, by assumption, $\lim_{n\to\infty} \|(x-w_n)_\Lambda\|_\infty = 0$, and $x-w_n$ is in X for all $n\ge 1$. By the above inequality, we have

$$\sup_{h\in X^\perp_{\Lambda^c 1}} \langle (x-w_n), h\rangle \le M\|(x-w_n)_\Lambda\|_\infty$$

and the required conclusion follows. ■

FACT 4.7. *Let $x\in B_X$, and Λ a finite subset of $\mathbb{N}$. Assume that*

$$\sup_{f\in X^\perp_{\Lambda^c 1}} \langle x, f\rangle = 1.$$

Then $A_x = \{f\in X^\perp_{\Lambda^c 1} : f(x) = 1\} \ne \emptyset$, and

$$\Lambda_1 = \bigcup\{S(f) : f\in A_x\}$$

is a finite subset of Λ^c. Further, if (w_n) is a sequence in B_X such that

$$\lim_{n\to\infty} \sup\{\langle x-w_n, f\rangle : f\in X^\perp_{\Lambda^c 1}\} = 0 \tag{15}$$

then

$$\lim_{n\to\infty} \|(x-w_n)_{\Lambda_1}\|_\infty = 0.$$

Proof. As mentioned in Remark 4.4, $X^\perp_{\Lambda^c 1}$ is a weak* compact subset of $l_1(\mathbb{N})$ and so A_x is non-empty. It is easily seen that $\Lambda_1 \subseteq \{k\in\mathbb{N} : |x(k)| = 1\}$ and since $x \in c_0$, Λ_1 is finite. Clearly, $\Lambda_1 \subseteq \Lambda^c$.

For any k in Λ_1, there exists f in $X^\perp_{\Lambda^c 1}$ such that $\langle x, f\rangle = 1$ and $k \in S(f)$. Now by (15) and Remark 4.5,

$$\lim_{n\to\infty} \|(x-w_n)_{S(f)}\|_\infty = 0.$$

As $k \in S(f)$, $\lim_{n\to\infty} |(x-w_n)(k)| = 0$. Since Λ_1 is a finite set, this implies $\lim_{n\to\infty} \|(x-w_n)_{\Lambda_1}\|_\infty = 0$. ∎

REMARK 4.8. Assume y in B_X satisfies $\|(x-y)_\Lambda\|_\infty = 0$, for Λ as in the above fact. Then by Fact 4.6 we conclude that

$$\sup\{\langle x-y, f\rangle : f \in X^\perp_{\Lambda^c 1}\} = 0.$$

Therefore, $A_x = A_y$. Further by Fact 4.7, $\|(x-y)_{\Lambda_1}\|_\infty = 0$, and

$$\lim_{n\to\infty} \|(y-w_n)_{\Lambda_1}\|_\infty = 0$$

if (w_n) is a sequence in B_X satisfying

$$\lim_{n\to\infty} \sup\{\langle x-w_n, f\rangle : f \in X^\perp_{\Lambda^c 1}\} = 0.$$

FACT 4.9. *Let $x \in B_X$, Λ_0 be a non-empty finite subset of* $\mathbb{N}$, *and*

$$A(x,\Lambda_0) = \{y \in B_X : \|(x-y)_{\Lambda_0}\|_\infty = 0\}.$$

Then there exists a finite subset Λ of $\mathbb{N}$ *containing Λ_0 and $\eta > 0$ such that*

$$\sup_{f\in X^\perp_{\Lambda^c 1}} \langle y, f\rangle = 1-\eta \quad \forall y \in A(x,\Lambda_0), \tag{16}$$

and for any sequence (w_n) in B_X satisfying $\lim_{n\to\infty} \|(x-w_n)_{\Lambda_0}\|_\infty = 0$, *we have*

$$\lim_{n\to\infty} \|(y-w_n)_\Lambda\|_\infty = 0, \quad \forall y \in A(x,\Lambda_0). \tag{17}$$

Proof. If (16) holds with $\Lambda = \Lambda_0$, we can take $\Lambda = \Lambda_0$ and there is nothing to prove. Otherwise, as by Fact 4.6,

$$\sup_{f\in X^\perp_{\Lambda_0^c 1}} \langle x-y, f\rangle = 0 \quad \forall y \in A(x,\Lambda_0),$$

we must have

$$\sup_{f\in X^\perp_{\Lambda_0^c 1}} \langle y, f\rangle = 1 \quad \forall y \in A(x,\Lambda_0).$$

We take $\Lambda = \Lambda_0$ in Fact 4.7 and using Remark 4.8 get a finite subset Λ_1 of Λ_0^c satisfying (17) for $\Lambda = \Lambda_1$. Let $\Lambda = \Lambda_0 \cup \Lambda_1$. Clearly (17) is satisfied for Λ and in particular,

$$\|(x-y)_{\Lambda}\|_{\infty}=0 \quad \forall y\in A(x,\Lambda_0),$$

which, in turn, implies (Fact 4.6)

$$\sup_{f\in X^{\perp}_{\Lambda^{c}1}}\langle x-y,f\rangle=0 \quad \forall y\in A(x,\Lambda_0).$$

If now (16) holds for Λ, we are done. Otherwise we must have

$$\sup_{f\in X^{\perp}_{\Lambda^{c}1}}\langle y,f\rangle=1 \quad \forall y\in A(x,\Lambda_0).$$

Now repeat the above argument with Λ_0 replaced by $\Lambda=\Lambda_0\cup\Lambda_1$ to get a finite subset Λ_2 of Λ^{c} satisfying (17) for $\Lambda=\Lambda_2$. Clearly (17) holds for $\Lambda=\bigcup_{i=0}^{2}\Lambda_i$ and if (16) also holds for Λ, then Λ is the required set.

We proceed inductively to get pairwise disjoint, finite sets Λ_i satisfying (17) for $\Lambda=\Lambda_i$. Note that, for each i, $\Lambda_i\subseteq\{n\in\mathbb{N}:|x(n)|=1\}$. Since $x\in c_0$, the inductive process must end at a finite stage, say l, with $\Lambda=\bigcup_{i=0}^{l}\Lambda_i$ satisfying (16). Clearly Λ also satisfies (17) and is the required set. ■

REMARK 4.10. It is clear from the above facts that the set Λ in the above fact and the constant η occurring in (16) are independent of the choice of the sequence (w_n).

FACT 4.11. *Let g in l_1 and x in S_X satisfy $g(x)=\|g\|_1$. Then there exists a finite subset Λ of* $\mathbb{N}$, *containing $\Lambda_0=S(g)$, and $\eta>0$ such that* (16) *is satisfied and also* (17), *for any sequence $(w_n)\subseteq B_X$ with* $\lim_{n\to\infty}\langle w_n,g\rangle=1$.

Proof. Note that Λ_0 is a finite set and $J_X(g)=A(x,\Lambda_0)$. Also, using Remark 4.5, we have

$$\lim_{n\to\infty}\|(x-w_n)_{\Lambda_0}\|_{\infty}=0$$

for any sequence (w_n) contained in B_X with $\lim_{n\to\infty}\langle w_n,g\rangle=1$. The required conclusion now follows from Fact 4.9. ■

FACT 4.12. *Let x be in B_X and assume there exists a finite subset Λ of* $\mathbb{N}$, $\eta>0$ *such that*

$$1-\sup_{f\in X^{\perp}_{\Lambda^{c}1}}\langle x,f\rangle=2\eta,$$

and w in B_X satisfying

$$\sup_{f\in X^{\perp}_{\Lambda^{c}1}}\langle x-w,f\rangle<\frac{\eta\varepsilon}{1-\varepsilon}$$

for some $0<\varepsilon<1/2$. Then there exists a t in c_0 such that

$$\|t\|_{\infty}\le 1, \quad \|x-t\|_{\infty}<3\varepsilon,$$
$$\langle t,f\rangle=\langle w,f\rangle \quad \forall f\in X^{\perp}_{\Lambda^{c}}.$$

Proof. By Remark 4.4, $X^{\perp}_{\Lambda^c}$ is a weak* closed subspace of l_1, and therefore,

$$X^{\perp}_{\Lambda^c} = M^{\perp} = \{f \in l_1 : \langle y, f\rangle = 0 \ \forall y \in M\},$$

where

$$M = (X^{\perp}_{\Lambda^c})_{\perp} = \{y \in c_0 : \langle y, f\rangle = 0 \ \forall f \in X^{\perp}_{\Lambda^c}\}.$$

We have $1 - 2\eta \geq 0,\ 0 < \varepsilon < 1/2$ and if

$$\frac{\eta\varepsilon}{1-\varepsilon} = \varepsilon',$$

then $\varepsilon' < \min\{\varepsilon, \eta\}$. Further, by assumption

$$\sup_{f \in X^{\perp}_{\Lambda^c 1}} \langle w, f\rangle < 1 - 2\eta + \varepsilon' \leq 1 - 2\eta + \eta = 1 - \eta.$$

Now, our assumption along with the above inequality and the duality formula implies that there exist y_1 and y_2 in M satisfying

$$\|w - y_1\|_\infty < 1 - \eta, \qquad \|x - w - y_2\|_\infty < \varepsilon'.$$

Let $s_1 = w - y_1$ and $s_2 = w + y_2$. Then

$$\langle s_i, f\rangle = \langle w, f\rangle \quad \forall f \in X^{\perp}_{\Lambda^c},\ i = 1, 2. \tag{18}$$

Also,

$$\|x - s_2\|_\infty < \varepsilon', \qquad \|s_1\|_\infty < 1 - \eta. \tag{19}$$

Note that

$$\|s_2\|_\infty \leq \|x\|_\infty + \varepsilon' \leq 1 + \varepsilon'$$

and $\lambda = \varepsilon$ satisfies the equation

$$\lambda(1 - \eta) + (1 - \lambda)(1 + \varepsilon') = 1.$$

Let $t = \lambda s_1 + (1 - \lambda)s_2$. Then $\|t\|_\infty \leq 1$. Also,

$$\begin{aligned}\|x - t\|_\infty &\leq \lambda\|x - s_1\|_\infty + (1 - \lambda)\|x - s_2\|_\infty \\ &\leq 2\lambda + \|x - s_2\|_\infty < 2\varepsilon + \varepsilon = 3\varepsilon\end{aligned}$$

by (19). Now using (18) we have

$$\langle t, f\rangle = \langle w, f\rangle \quad \forall f \in X^{\perp}_{\Lambda^c},$$

and this completes the proof. ∎

5. Hausdorff metric continuity of $Q_{f_1,\ldots,f_k}$. Having proved most of the required preliminary results in the previous section, we now prove Theorem 4.2. We begin with two observations. In the following, X stands for a subspace of c_0.

Remark 5.1. We have $X^* \simeq l_1/X^{\perp}$. Let T denote the quotient map from l_1 onto $l_1/X^{\perp}$. For $F \in X^*$, let

$$N(F) = T^{-1}(F) \cap \{f \in l_1 : \|f\|_1 = \|F\|\}.$$

If $f \in N(F)$ and $f_{|X}$ denotes f restricted to X then

$$f_{|X} = F, \qquad \sum_{n=1}^{\infty} |f(n)| = \|f\|_1 = \|f_{|X}\| = \|F\|.$$

Also, in this case we have

$$J_X(f) = J_{c_0}(f) \cap X = J_X(F).$$

Fact 5.2. *Let Y be a proximinal subspace of finite codimension n in X and x_0 in D_Y be a k-corner point, $1 \le k \le n$, with respect to a linearly independent subset $\{F_1, \dots, F_k\}$ of $S_{X^*} \cap Y^{\perp}$. Select any f_i in $N(F_i)$ for $1 \le i \le k$ and let g denote $k^{-1} \sum_{i=1}^{k} f_i$. Then*

$$S(g) = \bigcup_{i=1}^{k} S(f_i).$$

Proof. By the definition of a k-corner point with respect to $F_1, \dots, F_k$, we have

$$Q_{F_1,\dots,F_k}(x_0) = \bigcap_{i=1}^{k} \{x \in B_X : F_i(x) = F_i(x_0)\},$$

and by Remark 2.1, the set $Q_{F_1,\dots,F_k}(x_0)$ is non-empty. Now by the above remark,

$$Q_{F_1,\dots,F_k}(x_0) = \bigcap_{i=1}^{k} J_X(F_i) = \bigcap_{i=1}^{k} J_X(f_i) = \bigcap_{i=1}^{k} (J_{c_0}(f_i) \cap X) \ne \emptyset. \tag{20}$$

Clearly $S(g)$ is contained in $\bigcup_{i=1}^{k} S(f_i)$. Choose any m in $\bigcup_{i=1}^{k} S(f_i)$ and using (20), choose an element x in $\bigcap_{i=1}^{k} [J_{c_0}(f_i) \cap X]$. Let

$$A_m = \{i : 1 \le i \le k \text{ and } m \in S(f_i)\}.$$

For any real number α define

$$\operatorname{sgn} \alpha = \begin{cases} 1 & \text{if } \alpha > 0, \\ -1 & \text{if } \alpha < 0, \\ 0 & \text{if } \alpha = 0. \end{cases}$$

Clearly the set A_m is non-empty and

$$0 \ne x(m) = \operatorname{sgn} f_i(m) \quad \forall i \in A_m.$$

Therefore

$$0 \ne \operatorname{sgn} f_i(m) = \operatorname{sgn} f_j(m) \quad \forall i, j \text{ in } A_m.$$

This implies $g(m) \ne 0$ and $S(g) \supseteq \bigcup_{i=1}^{k} S(f_i)$. Hence

$$S(g) = \bigcup_{i=1}^{k} S(f_i). \ \blacksquare$$

Let x and Y be as in Fact 5.2. In the rest of this section, given a linearly independent subset $F_1, \dots, F_k$ of $Y^\perp$, $f_1, \dots, f_k$ and g are as defined in Fact 5.2.

We need the following fact in the proof of Theorem 4.2. We recall that the definition of k-corner point is given in Definition 3.3.

FACT 5.3. *Let X be a subspace of $c_0(\mathbb{N})$ and Y be a proximinal subspace of finite codimension n in X. Assume x_0 in D_Y is a k-corner point for some $1 \le k \le n$ with respect to some linearly independent subset $\{F_1, \dots, F_k\}$ of $Y^\perp$. Then there exists $\eta > 0$ and a finite subset Λ containing $S(g)$ such that*

$$\sup_{f \in X^\perp_{\Lambda^c 1}} \langle x, f \rangle = 1 - 2\eta, \qquad \forall x \in Q_{F_1,\dots,F_k}(x_0).$$

Further given $\varepsilon > 0$, there exists $\delta > 0$ (depending on ε and η) such that for any y in $D_Y \cap B(x_0, \delta)$, w in $Q_{F_1,\dots,F_k}(y)$ and x in $Q_{F_1,\dots,F_k}(x_0)$, we have

$$\|(x - w)_\Lambda\|_\infty < \varepsilon, \qquad \sup_{f \in X^\perp_{\Lambda^c 1}} \langle x - w, f \rangle < \frac{\eta\varepsilon}{1 - \varepsilon}.$$

Proof. Since x_0 is a k-corner point with respect to $F_1, \dots, F_k$, we have

$$Q_{F_1,\dots,F_k}(x_0) = \bigcap_{i=1}^{k} \{x \in B_X : F_i(x) = F_i(x_0) = \|F_i\|\} = \bigcap_{i=1}^{k} J_X(F_i).$$

Select any x in $Q_{F_1,\dots,F_k}(x_0)$. Then $\|x\|_\infty = 1$. We can assume $\|F_i\| = 1$ for $1 \le i \le k$. Clearly $\|g\|_1 = 1$. Also $g(x) = 1$, which implies $g \in NA(c_0)$ and therefore $S(g)$ is a finite set. Let Λ_0 denote the set $S(g)$. Note that

$$J_X(g) = \{y \in B_X : g(x) = \|g\|_1\} = A(x, \Lambda_0) = Q_{F_1,\dots,F_k}(x_0).$$

Consider any sequence (x_n) in D_Y that converges to x_0 in X. Choose any w_n in $Q_{F_1,\dots,F_k}(x_n)$. Then $w_n \in B_X$ for each $n \ge 1$. We have

$$\lim_{n\to\infty} F_i(w_n) = \lim_{n\to\infty} F_i(x_n) = F_i(x_0) = F_i(x) = 1 \quad \text{for } 1 \le i \le k$$

and so

$$\lim_{n\to\infty} f_i(w_n) = f_i(x) = 1 \quad \text{for } 1 \le i \le k.$$

This implies

$$\lim_{n\to\infty} g(w_n) = g(x) = 1,$$

and by Remark 4.5, we have

$$\lim_{n\to\infty} \|(x - w_n)_{\Lambda_0}\|_\infty = 0.$$

By Fact 4.9, there exists a finite subset Λ of $\mathbb{N}$ containing Λ_0 and $\eta > 0$ such that

$$\sup_{f \in X^\perp_{\Lambda^c 1}} \langle z, f \rangle = 1 - 2\eta, \qquad \forall z \in Q_{F_1,\dots,F_k}(x_0), \tag{21}$$

$$\lim_{n\to\infty} \|(z - w_n)_\Lambda\|_\infty = 0 \quad \forall z \in Q_{F_1,\dots,F_k}(x_0). \tag{22}$$

Now we apply Fact 4.6 to conclude that

$$\lim_{n\to\infty} \sup_{f\in X^{\perp}_{\Lambda^{c}1}} \langle z-w_n, f\rangle = 0 \quad \forall z\in Q_{F_1,\dots,F_k}(x_0). \tag{23}$$

It is clear from Remark 4.10 that η is independent of the choice of the sequence (w_n). Thus given $0<\varepsilon<1/2$, (22) and (23) imply that there exists $\delta>0$ such that if $y\in D_Y$ and $\|x_0-y\|<\delta$ then

$$\|(x-w)_\Lambda\|_\infty<\varepsilon, \qquad \sup_{f\in X^{\perp}_{\Lambda^{c}1}} \langle x-w, f\rangle < \frac{\eta\varepsilon}{1-\varepsilon}, \tag{24}$$

for any x in $Q_{F_1,\dots,F_k}(x_0)$ and w in $Q_{F_1,\dots,F_k}(y)$. This together with (21) completes the proof. ∎

We are now in a position to prove Theorem 4.2.

Proof of Theorem 4.2. Let x_0 in D_Y be a k-corner point for some $1\le k\le n$ with respect to some linearly independent subset $\{F_1,\dots,F_k\}$ of $Y^\perp$. If $0<\varepsilon<1/3$, use Fact 5.3 to get a finite subset Λ containing $S(g)$ and $\delta>0$ satisfying (24), where η is given by (21).

We first prove the lower Hausdorff semicontinuity of $Q_{F_1,\dots,F_k}$ at x_0. To this end, select any x in $Q_{F_1,\dots,F_k}(x_0)$, y in $D_Y\cap B(x_0,\delta)$ and w in $Q_{F_1,\dots,F_k}(y)$. We will construct v in $Q_{F_1,\dots,F_k}(y)$ such that $\|x-v\|_\infty<3\varepsilon$.

We apply Fact 4.12 to get $t\in c_0$ with $\|t\|_\infty\le 1$, $\|x-t\|_\infty<3\varepsilon$ and

$$\langle t,f\rangle=\langle w,f\rangle \quad \forall f\in X^{\perp}_{\Lambda^{c}}.$$

Define

$$v(m)=\begin{cases} w(m) & \text{if } m\in\Lambda,\\ t(m) & \text{if } m\in\Lambda^{c}.\end{cases}$$

We observe that at this point of the proof, we have made use of the special structure of c_0 in the construction of v and it is easily seen that v belongs to the unit ball of c_0. Also, by Fact 5.2,

$$S(g)=\bigcup_{i=1}^{k} S(f_i),$$

and since Λ contains Λ_0, which is $S(g)$, we have

$$\langle v,f_i\rangle=\langle w,f_i\rangle=\langle w,F_i\rangle=\langle y,F_i\rangle \quad \text{for } 1\le i\le k. \tag{25}$$

Further, if f is in $X^\perp$ then

$$\langle v,f\rangle=\langle v,f_\Lambda\rangle+\langle v,f_{\Lambda^c}\rangle=\langle w,f_\Lambda\rangle+\langle t,f_{\Lambda^c}\rangle=\langle w,f_\Lambda\rangle+\langle w,f_{\Lambda^c}\rangle=0$$

as w is in X. Hence $v\in X$ and so

$$\langle v,F_i\rangle=\langle v,f_i\rangle \quad \text{for } 1\le i\le k.$$

By (25), the above equality gives

$$\langle v,F_i\rangle=\langle y,F_i\rangle \quad \text{for } 1\le i\le k$$

and v is in $Q_{F_1,\ldots,F_k}(y)$. We have $\|x-t\|_\infty < 3\varepsilon$ and by (24) this implies

$$\|x-v\|_\infty < 3\varepsilon.$$

This proves the lower Hausdorff semicontinuity of $Q_{F_1,\ldots,F_k}$ at x_0.

Now we show the upper Hausdorff semicontinuity of $Q_{F_1,\ldots,F_k}$ at x_0. To this end, we select any w in $Q_{F_1,\ldots,F_k}(y)$, where $y \in D_Y \cap B(x_0,\delta)$. We will get v in $Q_{F_1,\ldots,F_k}(x_0)$ such that $\|w-v\|_\infty < 5\varepsilon$.

Select any x in $Q_{F_1,\ldots,F_k}(x_0)$. Note that since $\varepsilon < 1/3$, by (24),

$$\sup_{f\in X^\perp_{\Lambda^c 1}} \langle x-w,f\rangle < \frac{\eta\varepsilon}{1-\varepsilon} < \eta/2.$$

Since η satisfies (21), using the above inequality we have

$$1-2\alpha = \sup_{f\in X^\perp_{\Lambda^c 1}} \langle w,f\rangle < 1-2\eta+\eta/2 = 1-\frac{3}{2}\eta. \tag{26}$$

Hence $2\alpha > \eta$. Now

$$\sup_{f\in X^\perp_{\Lambda^c 1}} \langle x-w,f\rangle = \sup_{f\in X^\perp_{\Lambda^c 1}} \langle w-x,f\rangle < \frac{\eta\varepsilon}{1-\varepsilon} < \frac{2\alpha\varepsilon}{1-\varepsilon}. \tag{27}$$

Now it is easily verified, using (26), (27) and the proof of Fact 4.12, that we can get t in c_0 such that $\|t\|_\infty \le 1$, $\|w-t\|_\infty < 5\varepsilon$ and

$$\langle t,f\rangle = \langle x,f\rangle \quad \forall f \in X^\perp_{\Lambda^c}.$$

From this point onwards, we can follow the argument for lower Hausdorff semicontinuity, replacing y by x_0 and interchanging x and w, to get v in $Q_{F_1,\ldots,F_k}(x_0)$ satisfying $\|w-v\|_\infty < 5\varepsilon$. This completes the proof of Theorem 4.2. ∎

Acknowledgements. The author is glad to avail herself of this opportunity to express her thanks to Prof. Gilles Godefroy for numerous helpful discussions and comments, and in particular, for suggesting Fact 5.2. She would also like to thank Prof. Vladimir Fonf for indicating Lemma 3.5. She owes many thanks to the referee for a very patient, careful perusal of this paper and the useful comments which improved the draft considerably. The referee also corrected an argument in Fact 4.6. The initial draft of this paper was written during the visit of the author to Équipe d'Analyse, Université Paris 6, in October 2001. She expresses her thanks to this institution for the financial support and hospitality.

REFERENCES

[1] B. Brosowski and R. Wegmann, *On the lower semicontinuity of the set-valued metric projection*, J. Approx. Theory 8 (1973), 84–100.

[2] A. L. Brown, *A rotund reflexive space having a subspace of codimension two with a discontinuous metric projection*, Michigan Math. J. 21 (1974), 145–151.

[3] A. L. Brown, F. Deutsch, V. Indumathi and P. S. Kenderov, *Lower semicontinuity concepts, continuous selections and set valued metric projections*, J. Approx. Theory 115 (2002), 120–143.
[4] F. Centrone and A. Martellotti, *Proximinal subspaces of $C(Q)$ of finite codimension*, ibid. 101 (1999), 78–91.
[5] F. Deutsch and P. Kenderov, *When does the metric projection admit a continuous selection?*, in: Approximation Theory, III (Austin, TX, 1980), Academic Press, New York, 1980, 327–333.
[6] R. Durier and P. L. Papini, *Polyhedral norms and related properties in infinite-dimensional Banach spaces: a survey*, Atti Sem. Mat. Fis. Univ. Modena 40 (1992), 623–645.
[7] V. P. Fonf and J. Lindenstrauss, *On quotients of polyhedral spaces*, preprint.
[8] V. P. Fonf, J. Lindenstrauss and R. R. Phelps, *Infinite-dimensional convexity*, in: Handbook of the Geometry of Banach Spaces, W. B. Johnson and J. Lindenstrauss (eds.), Volume I, Elsevier, 2001, 599–670.
[9] A. L. Garkavi, *On best approximation by elements of infinite-dimensional subspaces of a certain class*, Mat. Sb. 62 (1963), 104–120 (in Russian).
[10] A. Gleit and R. McGuigan, *A note on polyhedral spaces*, Proc. Amer. Math. Soc. 33 (1972), 398–404.
[11] G. Godefroy, *The Banach space* c_0, Extracta Math. 16 (2001), 1–25.
[12] G. Godefroy and V. Indumathi, *Proximinality in subspaces of* c_0, J. Approx. Theory 101 (1999), 175–181.
[13] —, —, *Strong proximinality and polyhedral spaces*, Rev. Mat. Complut. 14 (2001), 105–125.
[14] G. Godefroy, V. Indumathi and F. Lust-Piquard, *Strong subdifferentiability of convex functionals and proximinality*, J. Approx. Theory 116 (2002), 397–415.
[15] V. Indumathi, *Proximinal subspaces of finite codimension in general normed linear spaces*, Proc. London Math. Soc. (3) 45 (1982), 435–455.
[16] —, *Proximinal subspaces of finite codimension in direct sum spaces*, Proc. Indian Acad. Sci. (Math. Sci.) 111 (2001), 229–239.
[17] P. D. Morris, *Metric projections onto subspaces of finite codimension*, Duke Math. J. 35 (1968), 799–808.
[18] R. R. Phelps, *Čebyšev subspaces of finite codimension in $C(X)$*, Pacific J. Math. 13 (1963), 647–655.
[19] I. Singer, *On best approximation in normed linear spaces by elements of subspaces of finite codimension*, Rev. Roumaine Math. Pures Appl. 17 (1972), 1245–1256.
[20] L. P. Vlasov, *Elements of best approximation relative to subspaces of finite codimension*, Mat. Zametki 32 (1982), 325–341 (in Russian).
[21] —, *Subspaces of finite codimension: Existence of elements of best approximation*, ibid. 37 (1985), 78–85 (in Russian).

Department of Mathematics
Pondicherry University, Kalapet
Pondicherry-605014, India
E-mail: pdy_indumath@sancharnet.in

Received 27 February 2002;
revised 26 March 2004 (4174)